The Problem of the Dutch National Flag

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AIM X

There is a row of buckets numbered from 1 to n. It is given that:

- each bucket contains one pebble
- each pebble is either red, white, or blue.

A mini-computer is placed in front of this row of buckets and has to be programmed in such a way that it will rearrange (if necessary) the pebbles in the order of the Dutch national flag.

A Discipline of Programming, E.W. Dijkstra

Specification

- The mini-computer supports two commands:
  - swap (i,j) exchanges the pebbles in buckets numbered i and j for $1 \leq i,j \leq n$;
  - read (i) returns the colour of the pebble in bucket number i for $1 \leq i \leq n$.
- Solution should use one pass only and constant memory.

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The Problem of the ~~Dutch~~ Polish National Flag

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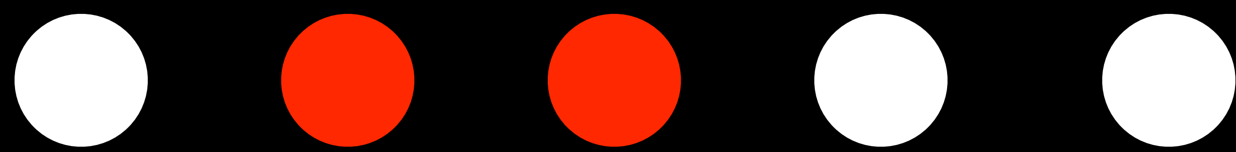
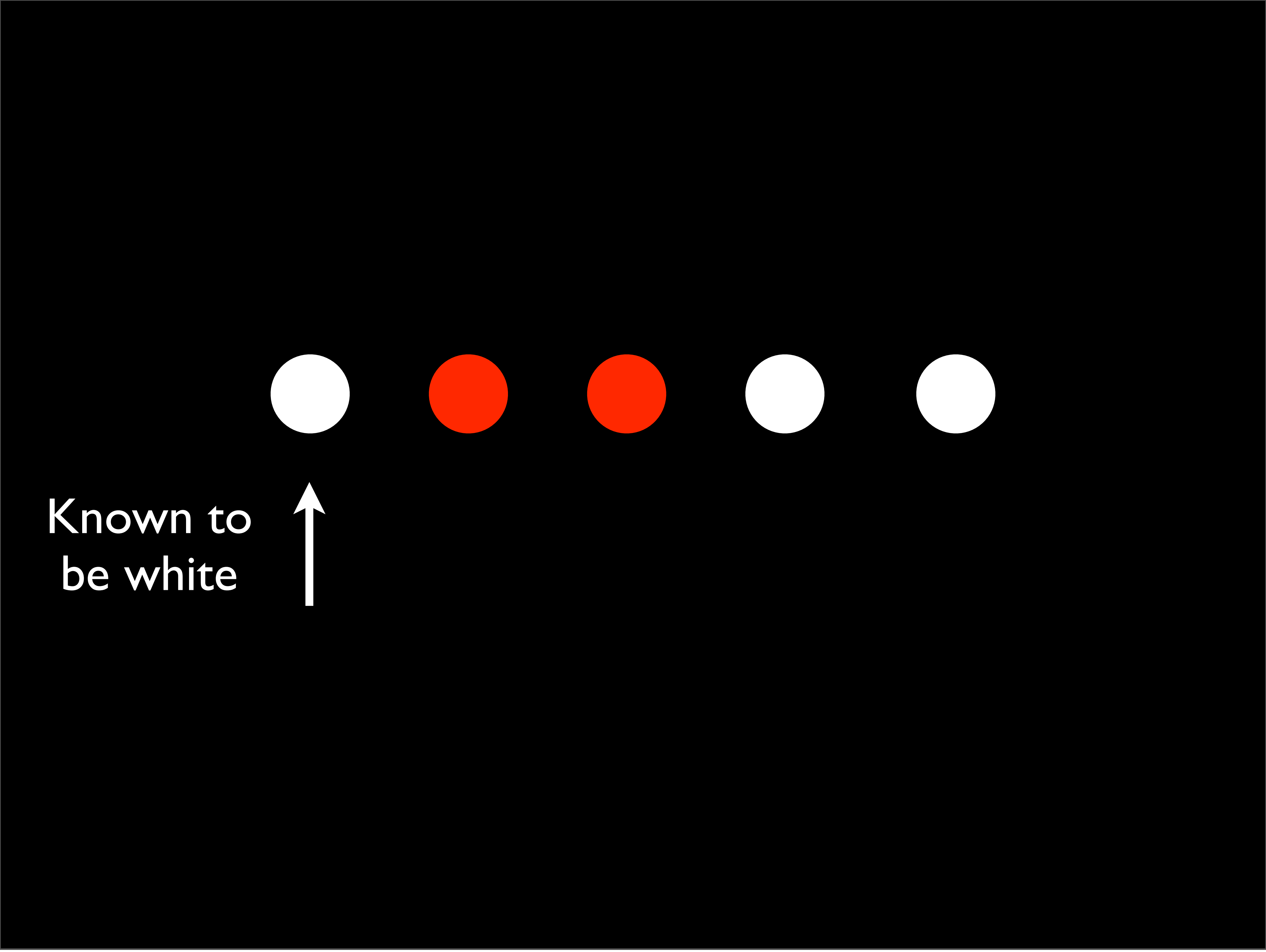
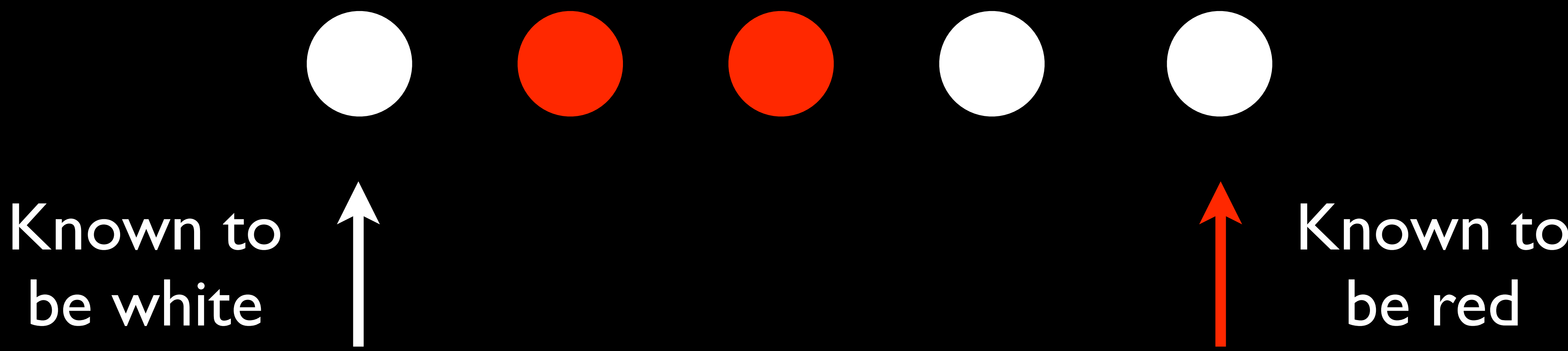
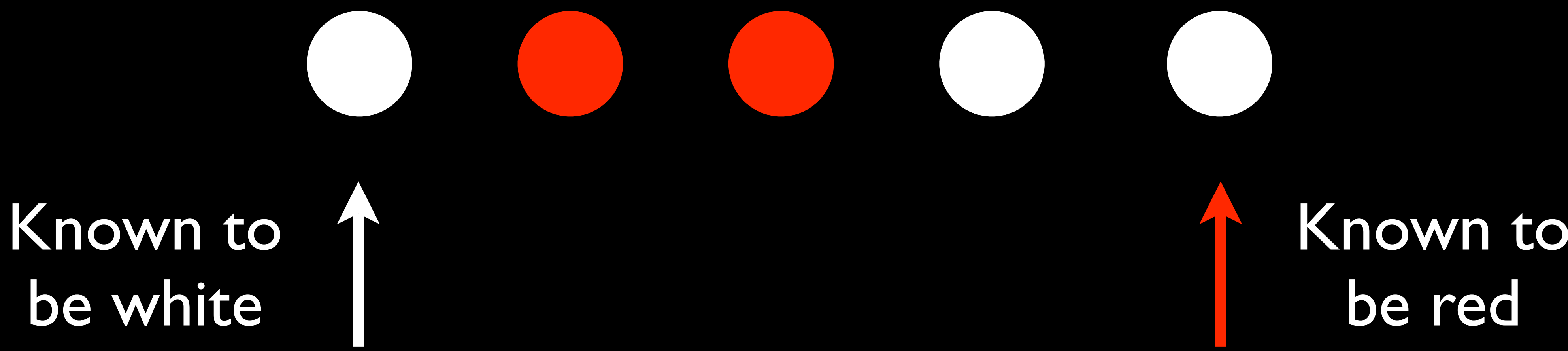



Known to
be white


Known to be white
Known to be red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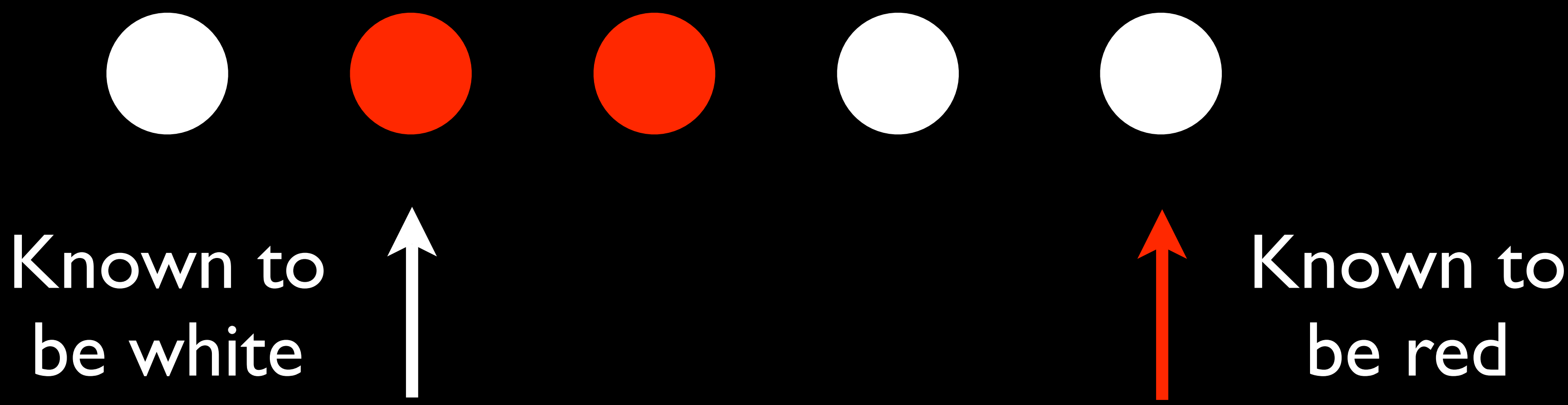
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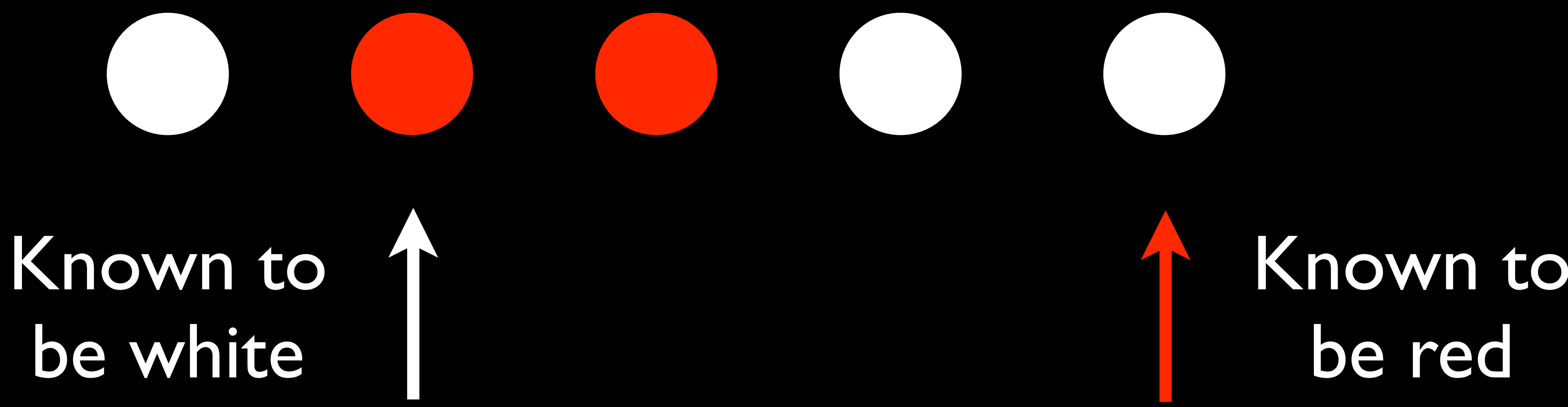
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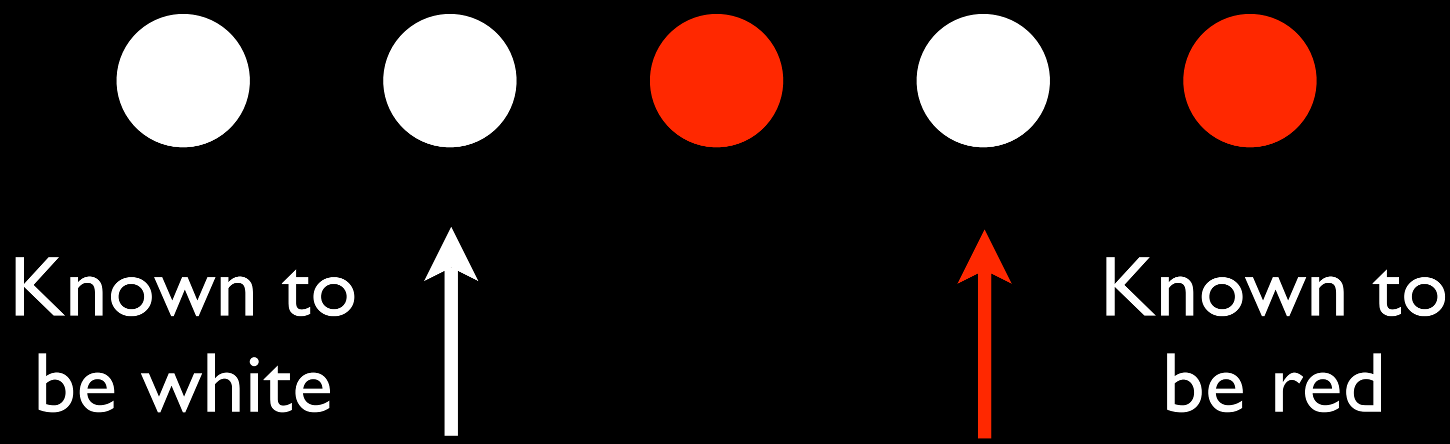
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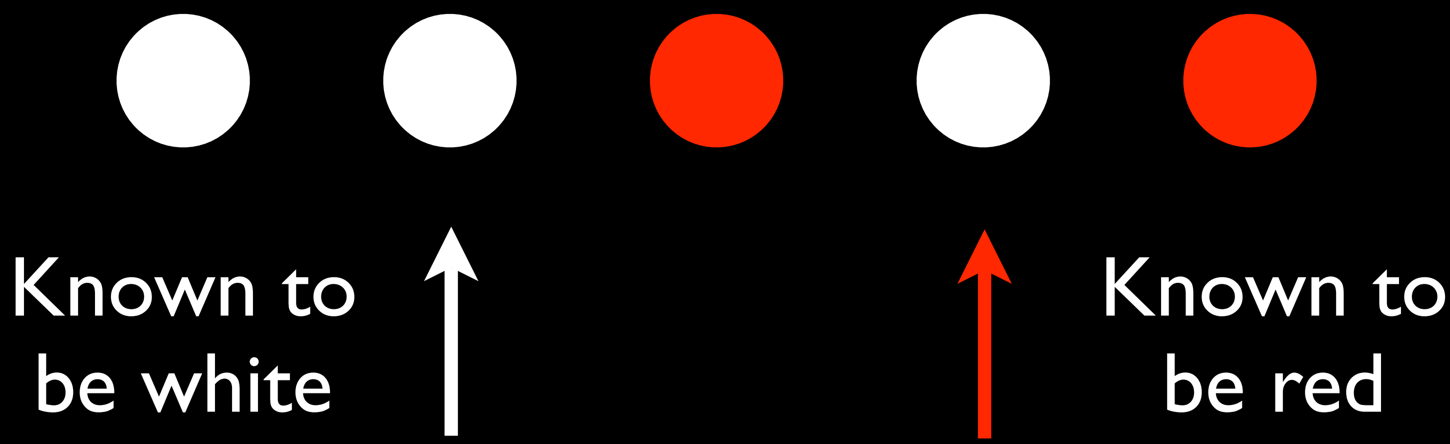
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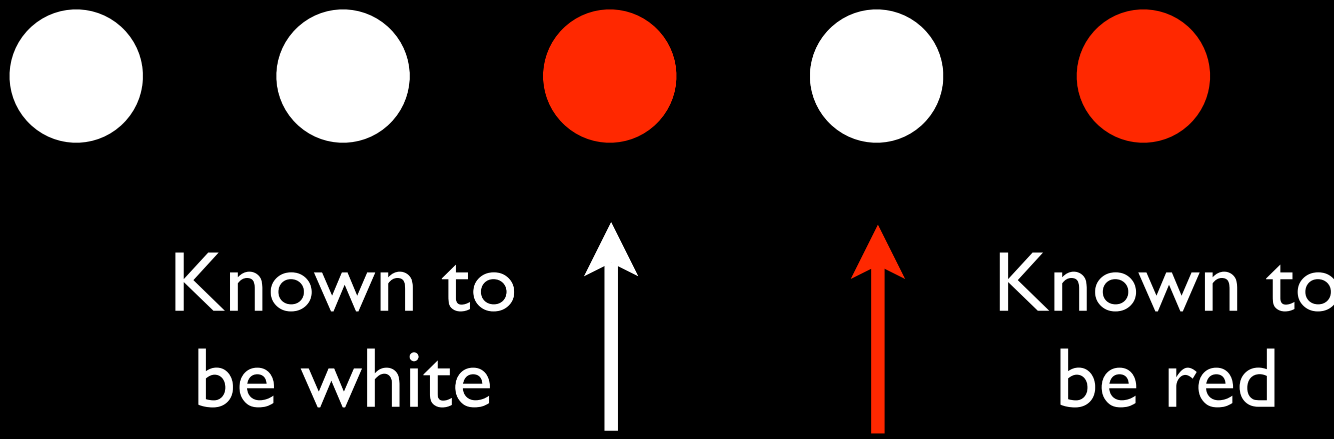
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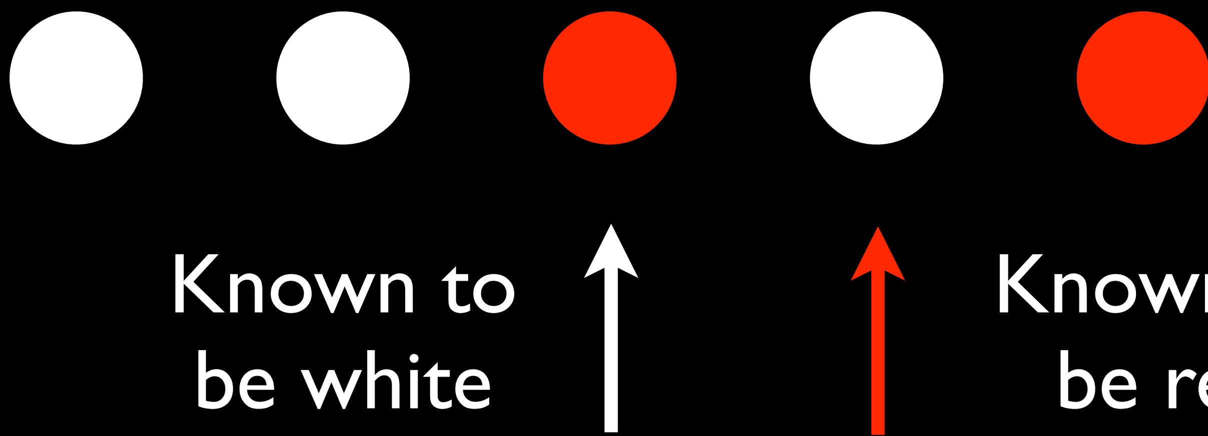
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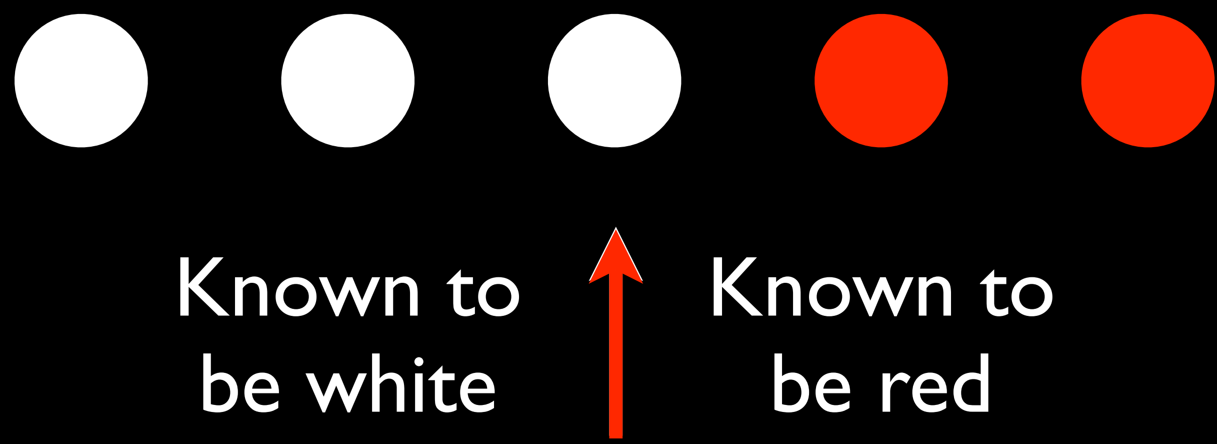
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Plan of attack

- Implement the mini-computer in Agda;
- Write a solution for the Problem of the Dutch National Flag;
- Verify our solution is correct.

Pebbles and Buckets

```
data Pebble : Set where
  Red : Colour
  White : Colour

data Buckets : Nat -> Set where
  Nil : Buckets Zero
  Cons : Pebble -> Buckets n ->
           Buckets (Succ n)
```

Indices

```
data Fin : Nat -> Set where
  Fz : Fin (Succ n)
  Fs : Fin n -> Fin (Succ n)
```

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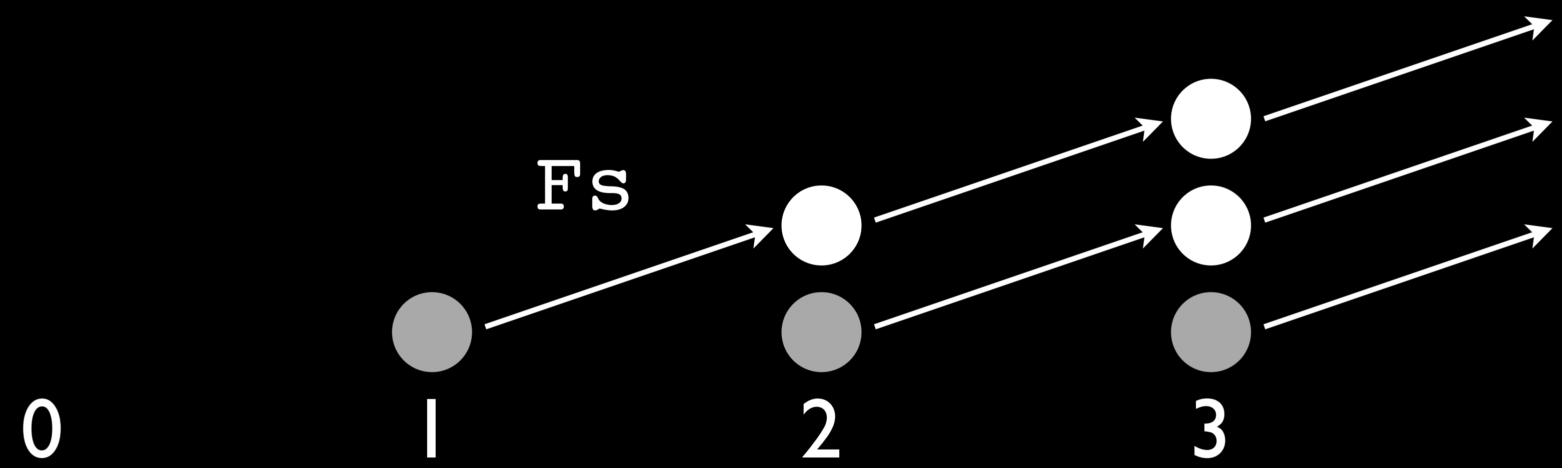

The state monad

```
State : Nat -> Set -> Set
State n a =
  Buckets n
    -> Pair a (Buckets n)
```

Reading

```
read : Fin n -> State Pebble
read i bs = (bs ! i , bs)
  where
  (Cons p _) ! Fz = p
  (Cons _ ps) ! (Fs i) =
    ps ! i
```

Swap

```
swap : Fin n -> Fin n
       -> State n Unit
swap i j =
  read i >>= \pi ->
  read j >>= \pj ->
  write i pj >>
  write j pi
```

Back to the problem

An approximation

```
sort :: Int -> Int -> IO ()
sort w r =
  if w == r then return ()
  else case read w of
    White -> sort (w + 1) r
    Red  -> swap w r >>
            sort w (r - 1)
```

An approximation

```
sort :: Int -> Int -> IO ()
sort w r =
  if w == r then return ()
  else case read w of
    White -> sort (w + 1) r
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           sort w (r - 1)
```

Why does this terminate?

An approximation

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    White -> sort (w + 1) r
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             sort w (r - 1)
```

Only terminates if $w \leq r$

Manipulating `Fin n`

```
sort :: Int -> Int -> IO ()
sort r w =
  if r == w then return ()
  else case read r of
    White -> sort (w + 1) w
    Red -> swap r w >>
           sort r (r - 1)
```

Two problems

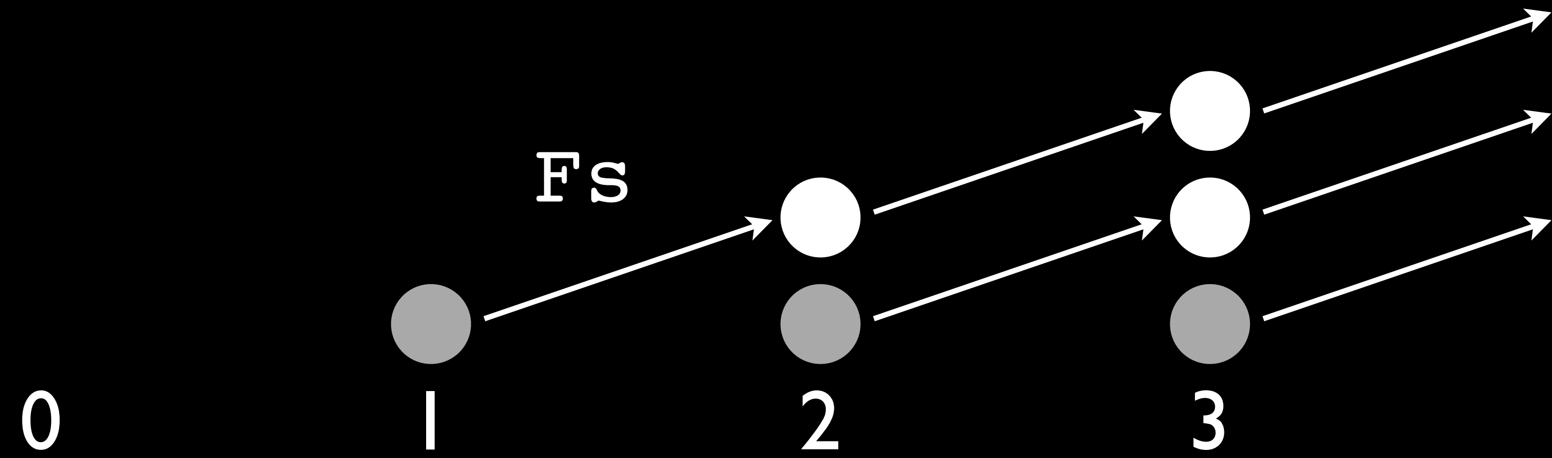
- We need to increment and decrement inhabitants of `Fin n`;
- We need to prove that our algorithm terminates.

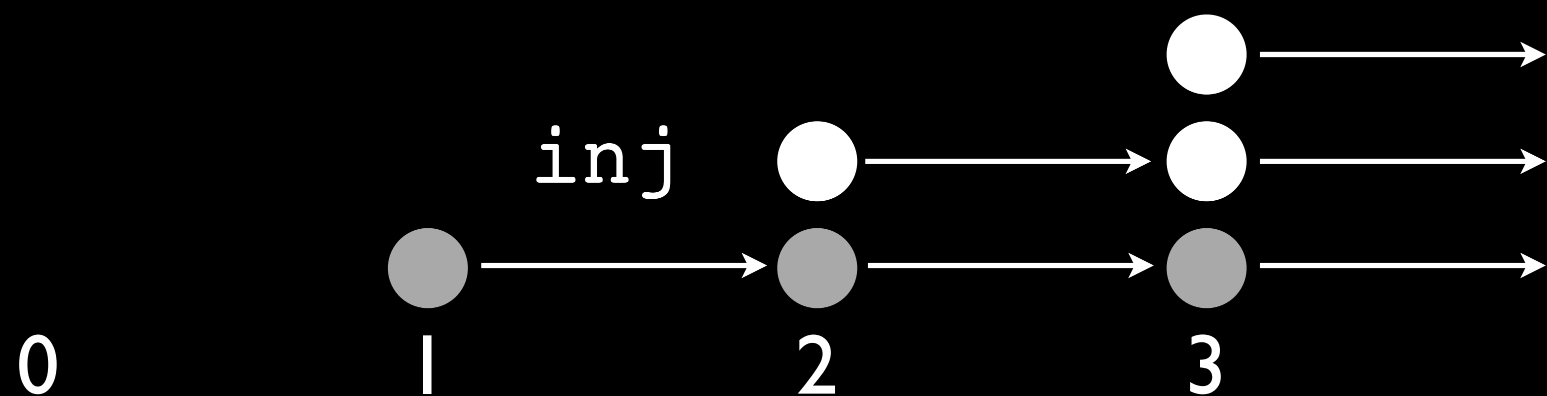
```
Fs : Fin n -> Fin (Succ n)
```

Injection

```
inj : Fin n -> Fin (Succ n)
inj Fz = Fz
inj (Fs i) = Fs (inj i)
```

`Fs` or `inj`





Idea

- Only increment the image of `inj`;
- Only decrement the image of `Fs`.

Less than or equal

```
data _<=_ : (i j : Fin n) -> Set where
  Base : (i : Fin (Succ n) -> Fz <= i
  Step : (i j : Fin n) ->
     (i <= j) -> (Fs i <= Fs j)
```

Difference

```
data Diff : (i j : Fin n) -> Set where
  Base : (i : Fin (Succ n) -> Diff i i
  Step : (i j : Fin n) ->
    Diff i j -> Diff (inj i) (Fs j)
```

Sort – Base case

```
sort : (w r : Fin n) ->
       Diff w r ->
       State n Unit
sort i .i Base = return unit
```

Sort – Base case

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sort : (w r : Fin n) ->
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```

```
sort : (w r : Fin n) ->
       Diff w r ->
       State n Unit
sort .(inj w) .(Fs r) (Step w r p)
  = read (inj w) >>= \p ->
    case p of
      White -> sort (Fs w) (Fs r) ?
      Red ->
        swap (inj w) (Fs r) >>
        sort (inj w) (inj r) ?
```

Lemmas

- We need to prove a few useful lemmas:
  - `Diff i j -> Diff (Fs i) (Fs j)`
  - `Diff i j -> Diff (inj i) (inj j)`
- Actually, we need to choose
  - `Diff : Nat -> (i j : Fin n) -> Set`

Verification

the easy part

Correctness Theorem

```
(h : Buckets n) (w r : Fin n)
(p : Diff w r)
(forall i -> i < w -> h ! i == White) ->
(forall i -> r < i -> h ! i == Red) ->
let h' = exec (sort w r p) h
in Sigma (Fin n) (\m ->
  forall i -> i < m -> h' ! i == White
  /\ forall i -> m < i -> h' ! i == Red)
```

Proof sketch

- Proof proceeds by induction on `Diff`
- Distinguish three cases:
  - Base case (trivial);
  - No swap happens (not too hard);
  - Swap happens (a bit trickier).
- In the latter two cases, we establish the invariant holds and make a recursive call.

Conclusions

- It is possible to reason about “impure” functions using Agda;
- It is not entirely trivial.
- A simple algorithm leads to simple proofs.