A Hoare Logic for the State Monad

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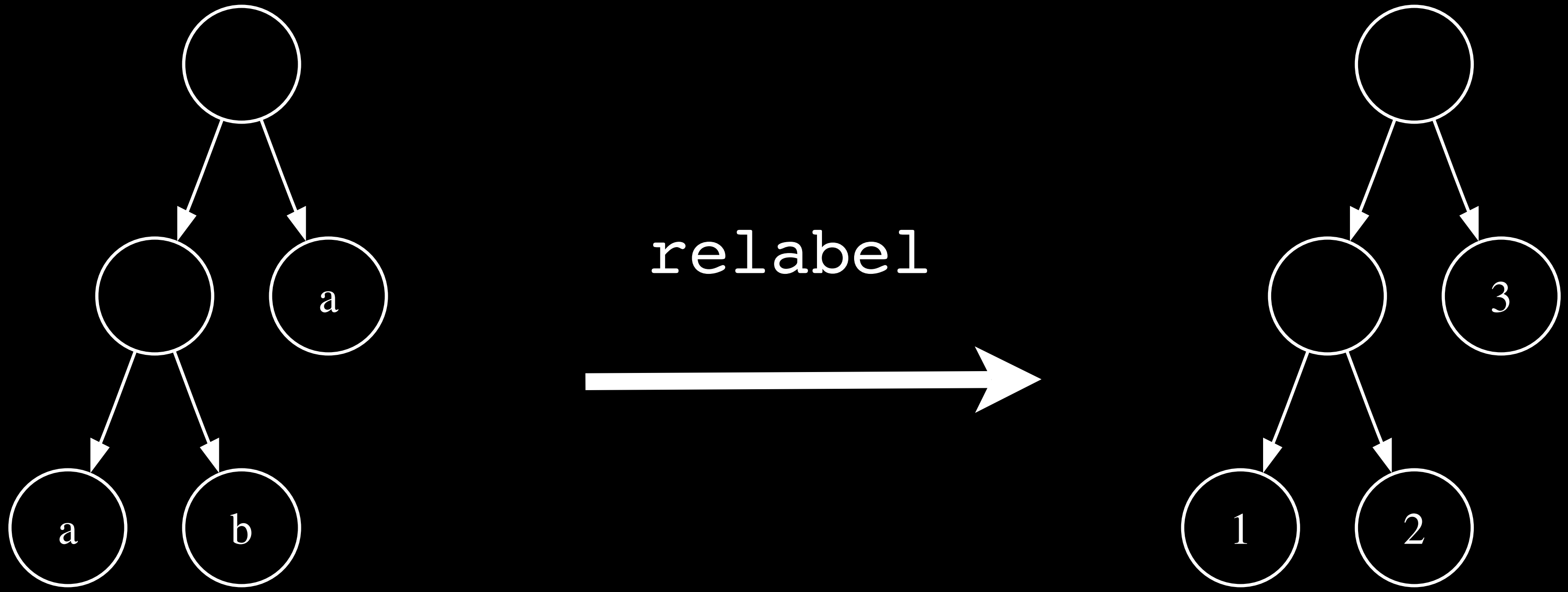
Relabelling a tree

```
Inductive Tree (a : Set) : Set :=
  | Leaf : a -> Tree a
  | Node : Tree a -> Tree a -> Tree a

relabel : forall a, Tree a -> Tree nat
```



`relabel`

Relabelling 1.0

```
Fixpoint relabel (a : Set)
 (t : Tree a) (s : nat) : Tree nat * nat
```

Relabelling 1.0

```
Fixpoint relabel (a : Set)
 (t : Tree a) (s : nat) : Tree nat * nat
 := match t with
      | Leaf _ => (Leaf s, s + 1)
      | Node l r =>
         let (l’, s’) := relabel l s
         in let (r’, s’’) := relabel r s’
         in (Node l’ r’, s’’)
    end
```

Recursive step

```
| Node l r =>
   let (l’, s’) := relabel l s
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Recursive step

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| Node l r =>
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```

Easy to make a mistake!

The state monad

```
(* For some fixed type s *)
Definition State (a : Set) : Type
   := s -> a * s

return : a -> State a
bind : State a
  -> (a -> State b)
  -> State b
```

Return

```
Definition State (a : Set) : Type
   := s -> a * s

Definition return (a : Set)
  : a -> State a :=
      fun x => fun s => (x, s)
```

Bind

```
Definition State (a : Set) : Type
   := s -> a * s

Definition bind (a b : Set)
(c1 : State a) (c2 : a -> State b) : State b
  := fun s => let (x, s’) := c1 s
              in c2 x s’
```

Bind

```
Definition State (a : Set) : Type
   := s -> a * s

Definition bind (a b : Set)
(c1 : State a) (c2 : a -> State b) : State b
  := fun s => let (x, s’) := c1 s
              in c2 x s’
```

I’ll use an infix operator, `>>=` , instead of `bind`

Relabelling 2.0

```
| Node l r =>
   relabel l >>= fun l’ =>
   relabel r >>= fun r’ =>
   return (Node l’ r’)
```

Relabelling 2.0

```
| Node l r =>
   relabel l >>= fun l’ =>
   relabel r >>= fun r’ =>
   return (Node l’ r’)
```

No more passing around the state explicitly!

Challenge:
verify the relabelling function, without expanding the definitions of return and bind.

Strong specifications

- Strong specifications:
    - define a value;
    - together with a proof that that value satisfies the spec.
- Notation in Coq:

```
{n : nat | n > 7}
```

Program

- Coq's Program framework for working with strong specifications
  - let's you define functions manipulating strongly specified values,
  - and collects assumptions and obligations.
- You need to prove any proof obligations (using tactics) before Program generates a complete Coq term.

Idea:
Decorate the state monad with pre- and postconditions.

Pre- and postconditions

Define the following types:

```
Pre := s -> Prop

Post (a : Set) := s -> a -> s -> Prop
```

The Hoare State Type

Define the following type:

```
  HoareState P a Q :=
    {i : s | P i} ->
      {(x,f) : a * s | Q i x f}
```

Remaining questions

- How can we define return?
- How can we define bind?
- How can we use these functions to verify our relabelling function?

Return

```
Definition return (x : a) :
  HoareState
    (fun i => True)
    a
    (fun i y f => i = f /\ x = y)
:= fun i => (x,i)
```

Return

```
Definition return (x : a) :
  HoareState
    (fun i => True)
    a
    (fun i y f => i = f /\ x = y)
:= fun i => (x,i)
```

Need to complete one trivial proof.

Bind - I

```
HoareState P1 A Q1 ->
(A -> HoareState P2 B Q2) ->
HoareState ... B ...
```

What should the pre- and postconditions be?

Bind - II

```
HoareState P1 A Q1 ->
((x:A) -> HoareState (P2 x) B (Q2 x)) ->
HoareState ... B ...
```

What should the pre- and postconditions be?

Bind’s precondition

```
\s1 -> P1 s1
   /\ forall x s2, Q1 s1 x s2 -> P2 x s2
```

The initial state must satisfy
the first computation’s precondition

Bind’s precondition

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The intermediate state satisfies
the second computation's precondition.

Bind’s precondition

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   /\ forall x s2, Q1 s1 x s2 -> P2 x s2
```

The intermediate state satisfies
the second computation’s precondition.

Bind’s postcondition

```
\s1 y s3 -> exists x, exists s2,
    Q1 s1 x s2 /\ Q2 x s2 y s3
```

There is an intermediate result and an intermediate state relating the two computations.

Implementing bind

- The definition of bind is **exactly the same** as for the state monad...
- ...but we need to fulfill one or two proof obligations.

Using the Hoare State Monad

To verify programs in the state monad, all we need to do is change the type signature, that is, choose the pre- and postconditions.

The program remains unchanged.

Relabelling revisited

```
HoareState
  (fun i => True)
  (Tree nat)
  (fun i t f =>
     flatten t = [i .. i + size t])
```

Relabelling revisited

```
HoareState
  (fun i => True)
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     /\ f = i + size t)
```

The proof

- The definition gives rise to two proof obligations, one for every case branch.
  - We’ve automated away all work involved in keeping track of the state;
  - The proof for the recursive case is only about 5 lines long (but uses some fancy Program tactics).

Discussion

- Other choices for pre- and postconditions?
- Is the HoareState type a monad?
- Further automation using Ltac.
- Coq script available from my homepage.

Relabelling revisited

- For our relabelling function:
  - the precondition is trivial;
  - for the postcondition we choose:

```
fun i t f =>

   flatten t = [i .. i + size t]
```

Relabelling revisited

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fun i t f =>
   flatten t = [i .. i + size t]
```

Postcondition not strong enough!

Relabelling, revisited

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  - the precondition is trivial;
  - for the postcondition we choose:

```
fun i t f =>
  flatten t = [i .. i + size t]
  /\ f = i + size t
```