The Problem of the Dutch National Flag

Wouter Swierstra

IFIP WG 2.1 #66

Jeremy’s Problem

The State Monad

```
State s a := s -> a * s

return : a -> State s a
(>>=) : State s a
  -> (a -> State s b)
  -> State s b
```

```
relabel : State nat (Tree nat)
relabel t = match t with
 | Leaf _ =>
    get >>= fun c =>
    put (c + 1) >>=
    return (Leaf c)
 | Node l r =>
    relabel l >>= fun l' =>
    relabel r >>= fun r' =>
    return (Node l' r')
 end
```

Idea:
Decorate the state monad with pre- and postconditions.

Pre- and postconditions

Define the following types:

```
Pre := s -> Prop

Post (a : Set) := s -> a -> s -> Prop
```

The Hoare State Type

Define the following type:

```
HoareState s P a Q :=
  {i : s | P i} ->
    {(x,f) : a * s | Q i x f}
```

Plan

- Define return and bind with a fancy HoareState type.
- Choose a suitable type for our relabelling function.

Relabelling revisited

The type of relabel becomes:

```
HoareState
  (fun i => True)
  (Tree nat)
  (fun i t f =>
     flatten t = [i .. i + size t])
```

Relabelling revisited

The type of relabel becomes:

```
HoareState
  (fun i => True)
  (Tree nat)
  (fun i t f =>
     flatten t = [i .. i + size t]
     /\ f = i + size t)
```

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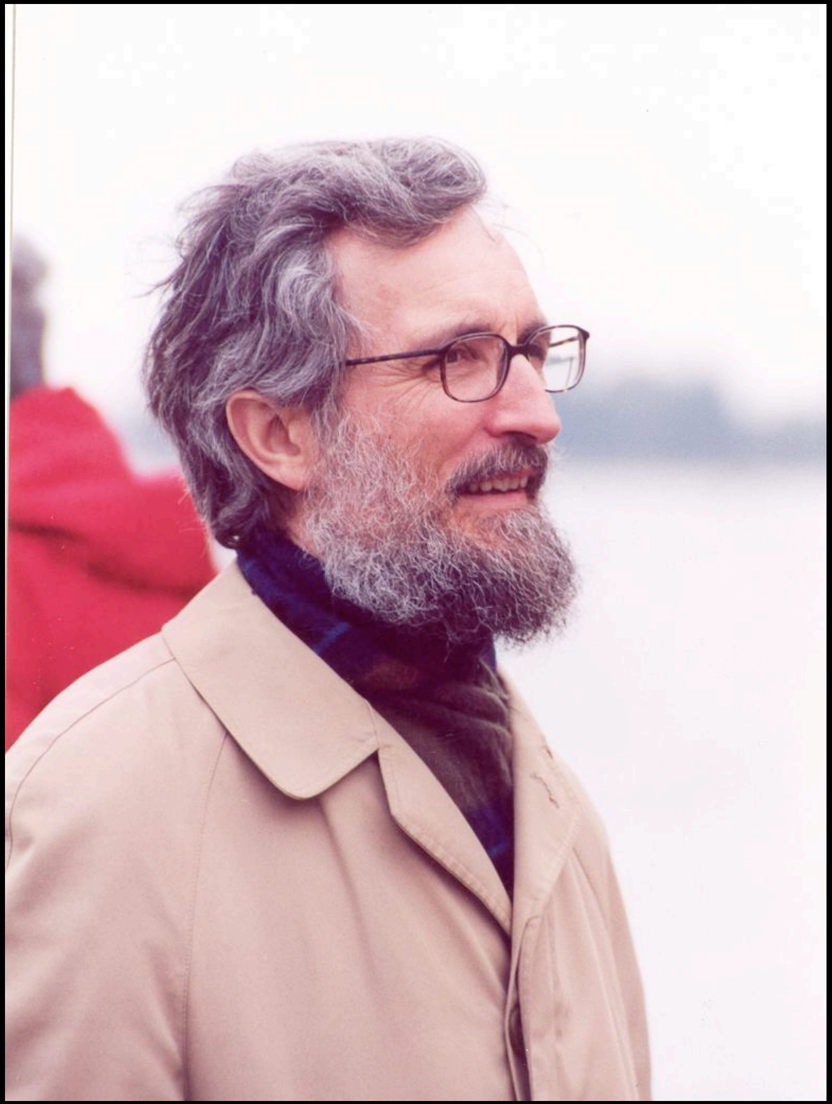
Type Theory
Per Martin-Löf

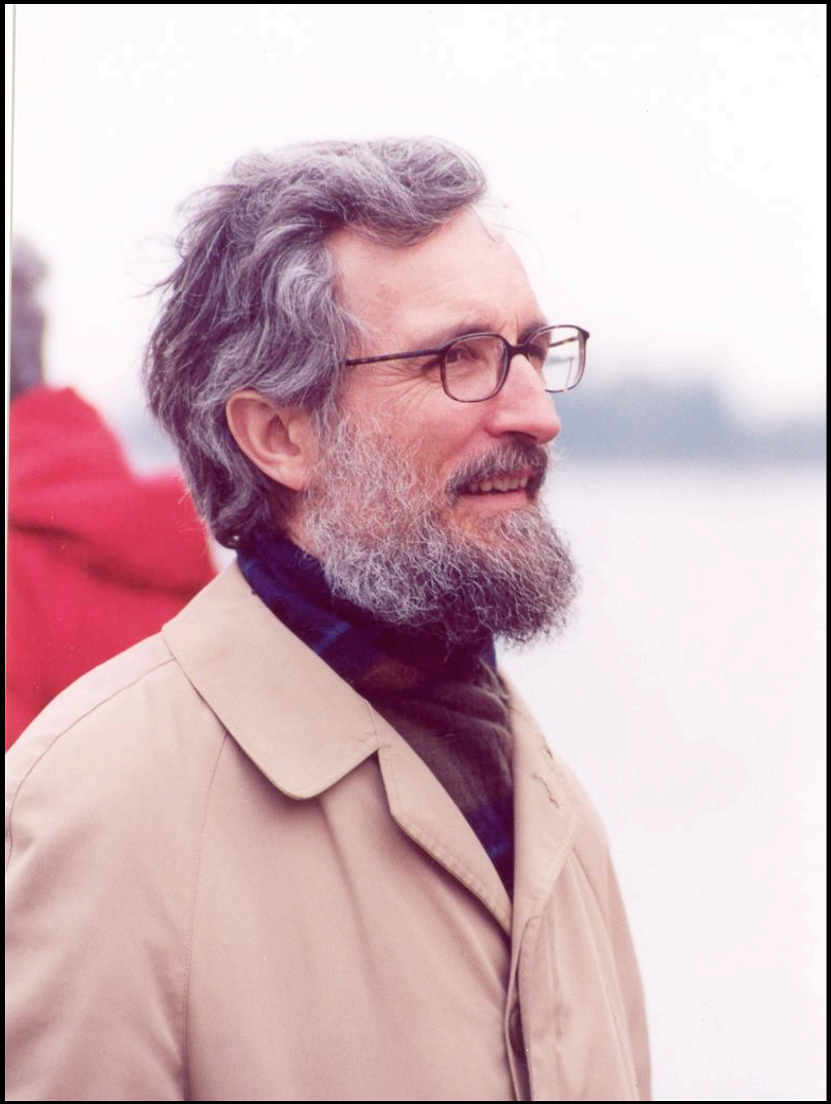
- A foundation of constructive mathematics;
- a functional programming language.



Type Theory
Per Martin-Löf

- A foundation of constructive mathematics;
- a functional programming language.

Really?



What about...

- mutable references?
- arrays?
- exceptions?
- concurrency?
- a GUI?
- a foreign function interface?
- network communication?
- a compiler?
- general recursion?
- file manipulation?
- random numbers?
- ...

There is a row of buckets numbered from 1 to n. It is given that:

- each bucket contains one pebble
- each pebble is either red, white, or blue.

A mini-computer is placed in front of this row of buckets and has to be programmed in such a way that it will rearrange (if necessary) the pebbles in the order of the Dutch national flag.

A Discipline of Programming, E.W. Dijkstra

Specification

- The mini-computer supports two commands:
  - swap (i,j) exchanges the pebbles in buckets numbered i and j for $1 \leq i,j \leq n$;
  - read (i) returns the colour of the pebble in bucket number i for $1 \leq i \leq n$.
- Solution should use one pass only and constant memory.

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The Problem of the ~~Dutch~~ Polish National Flag

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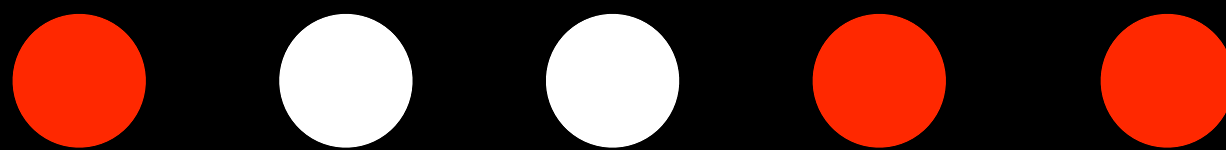
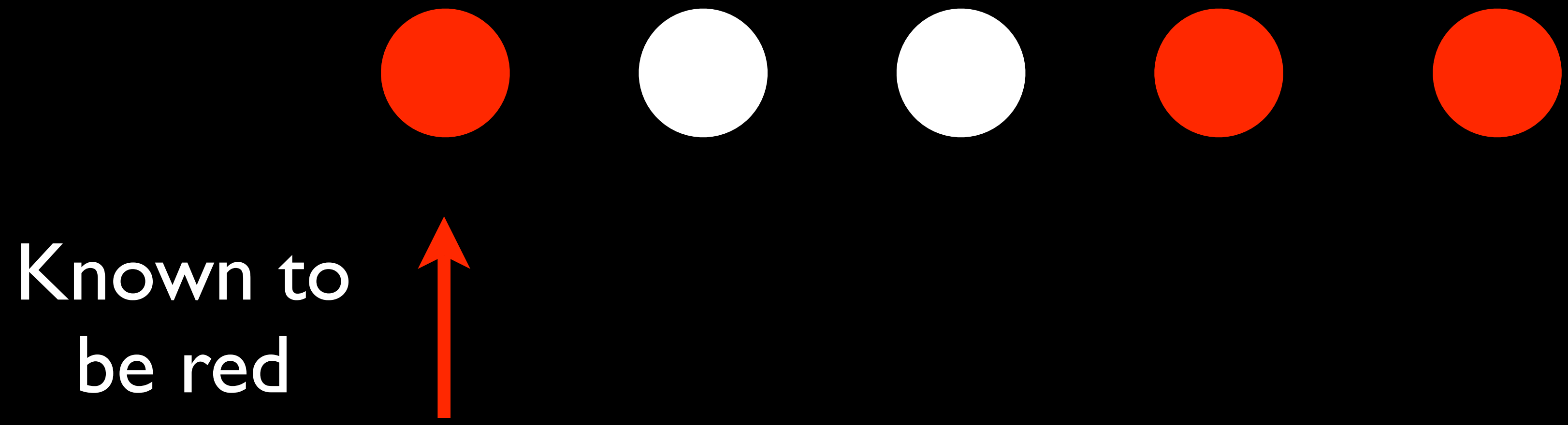
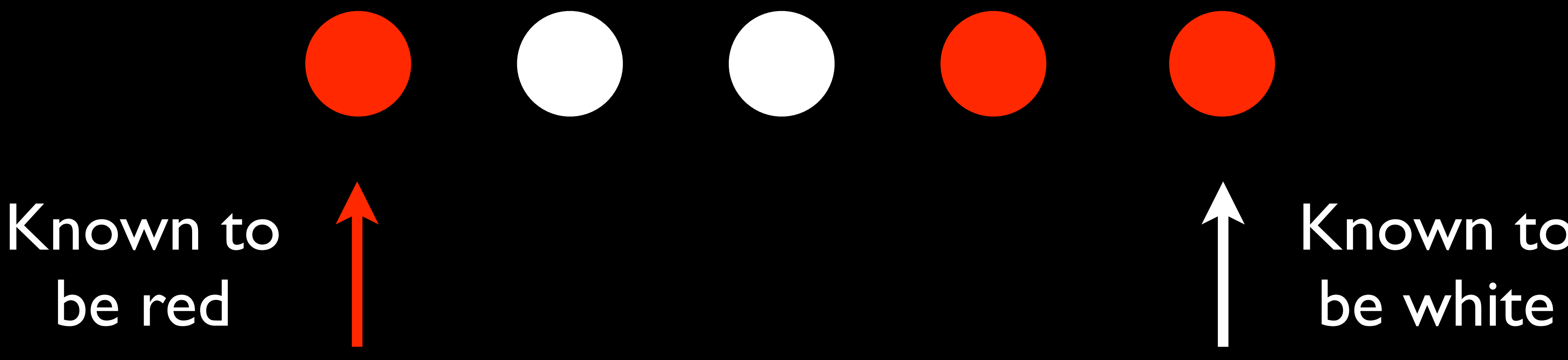



Known to
be red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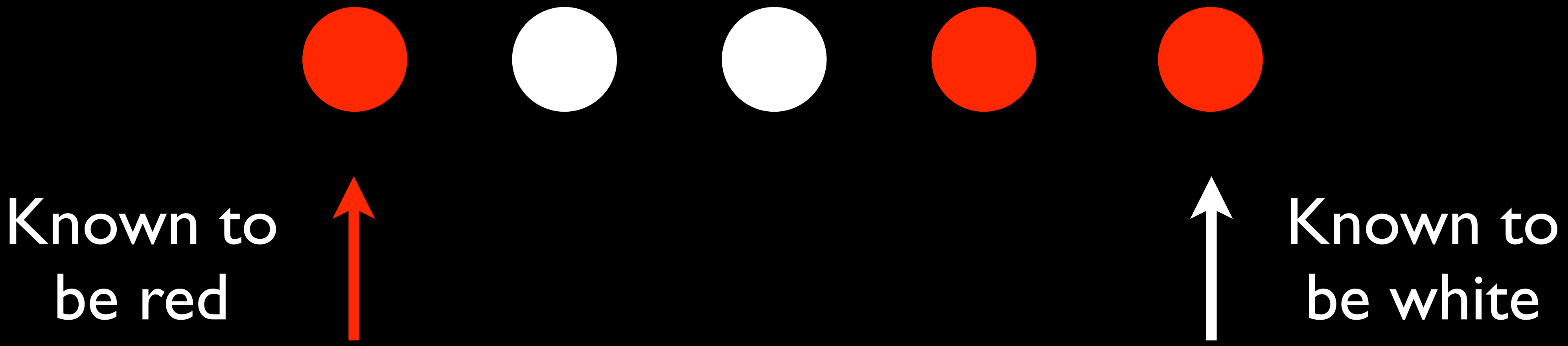
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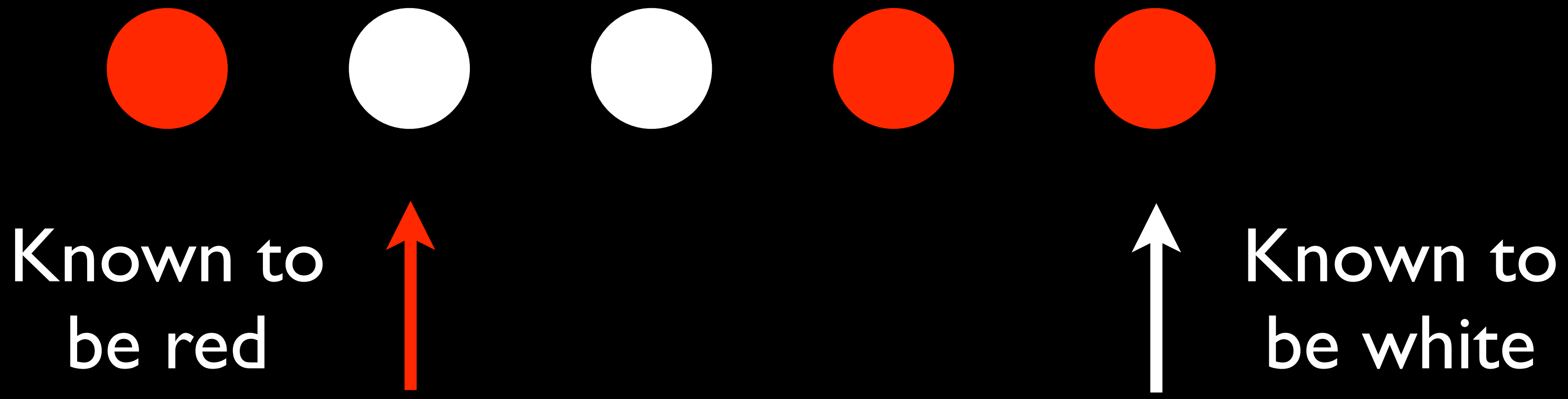
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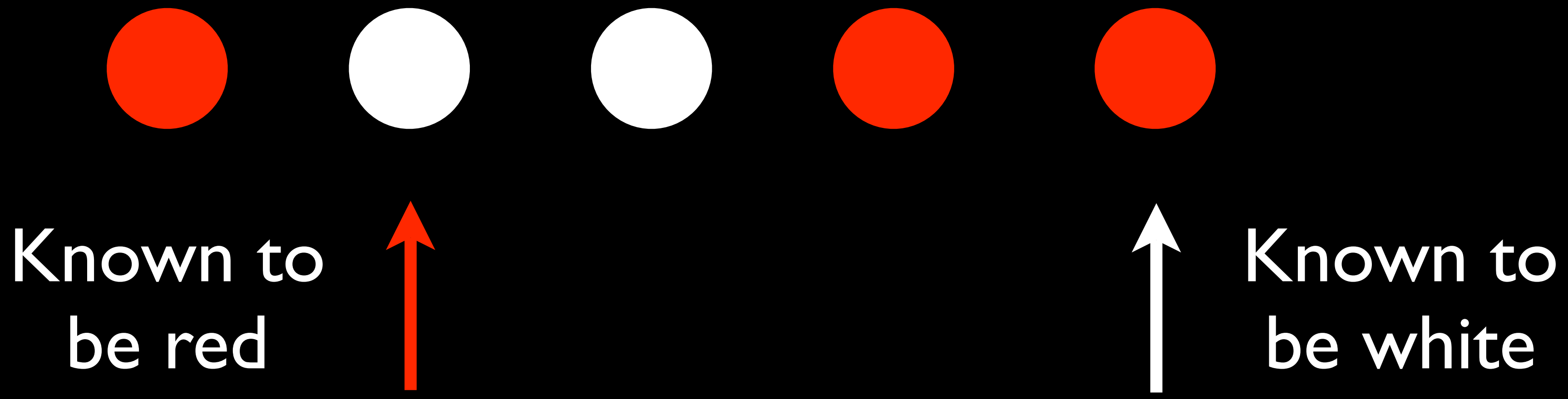
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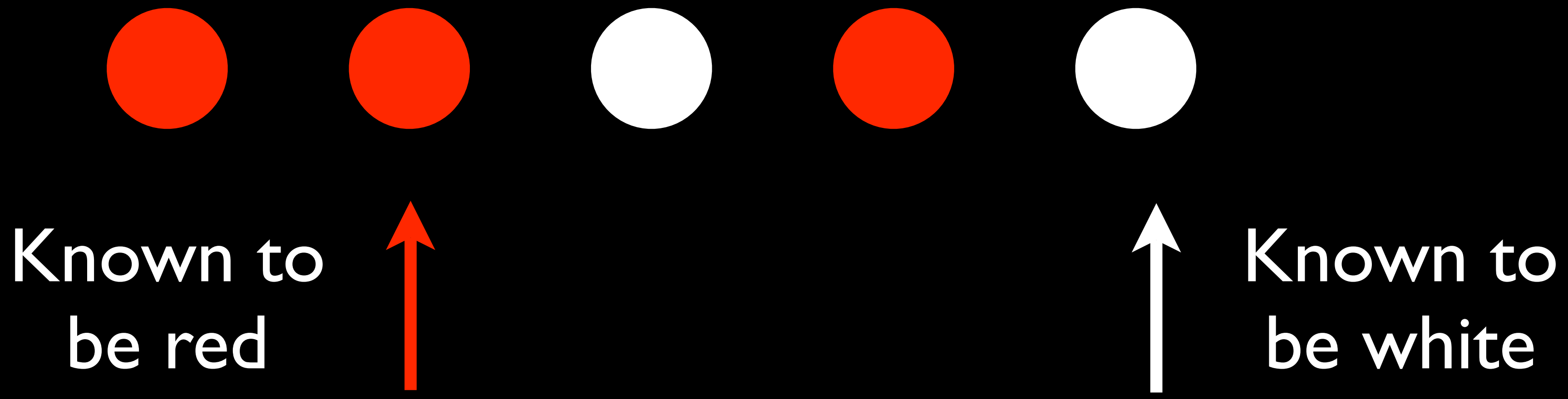
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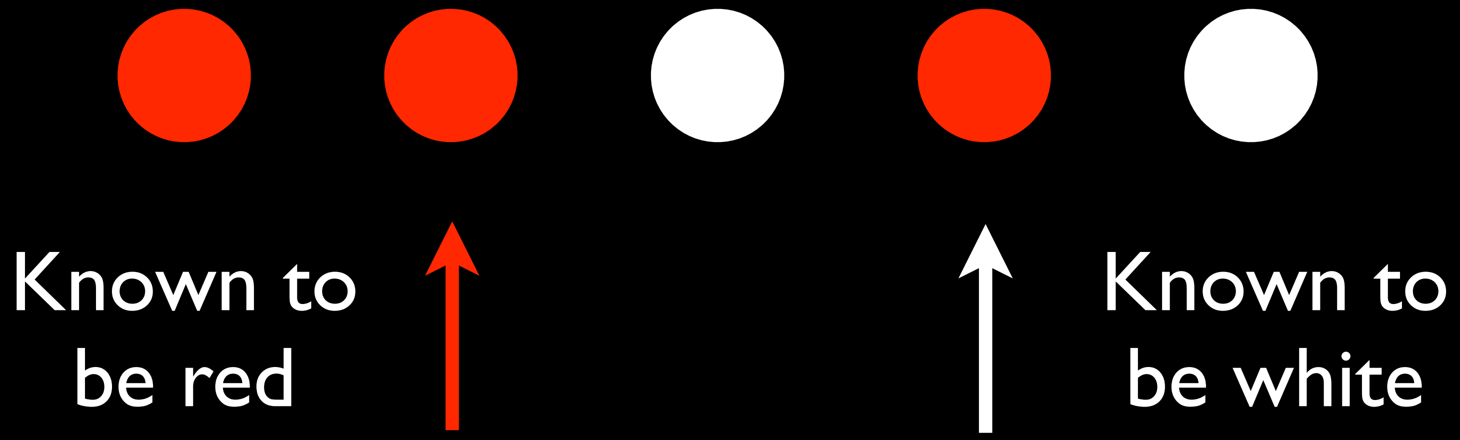
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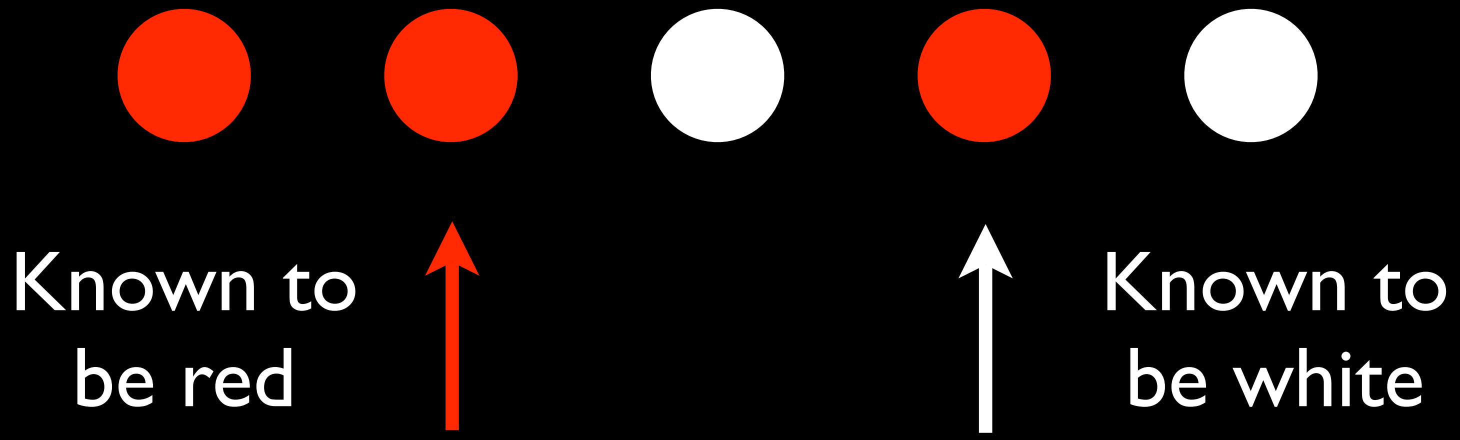
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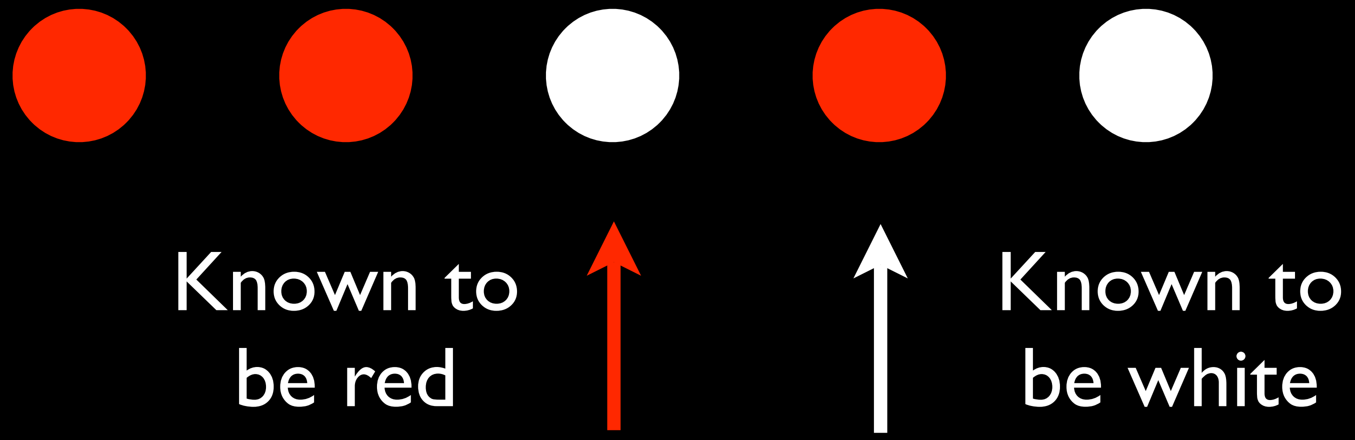
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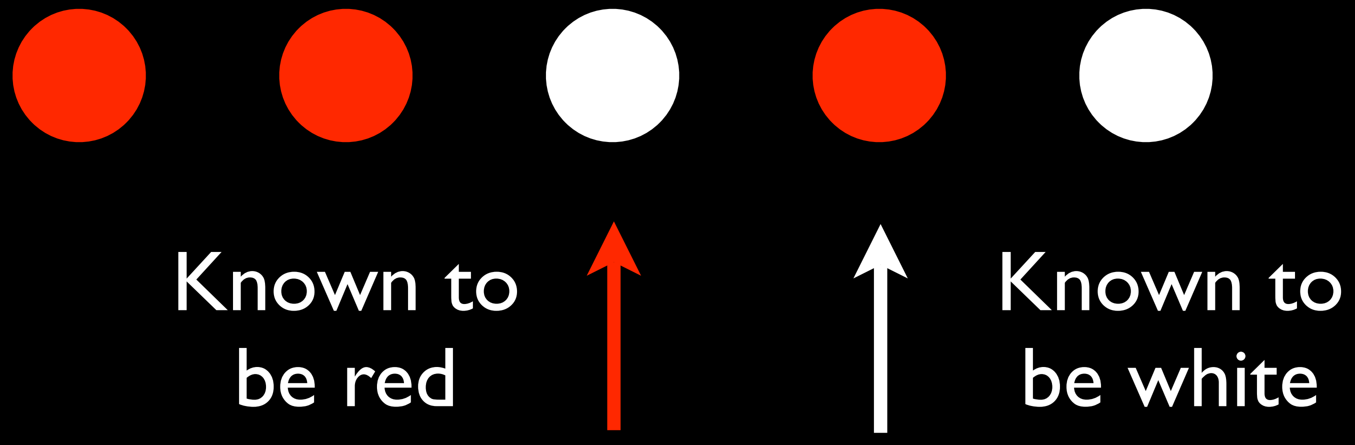
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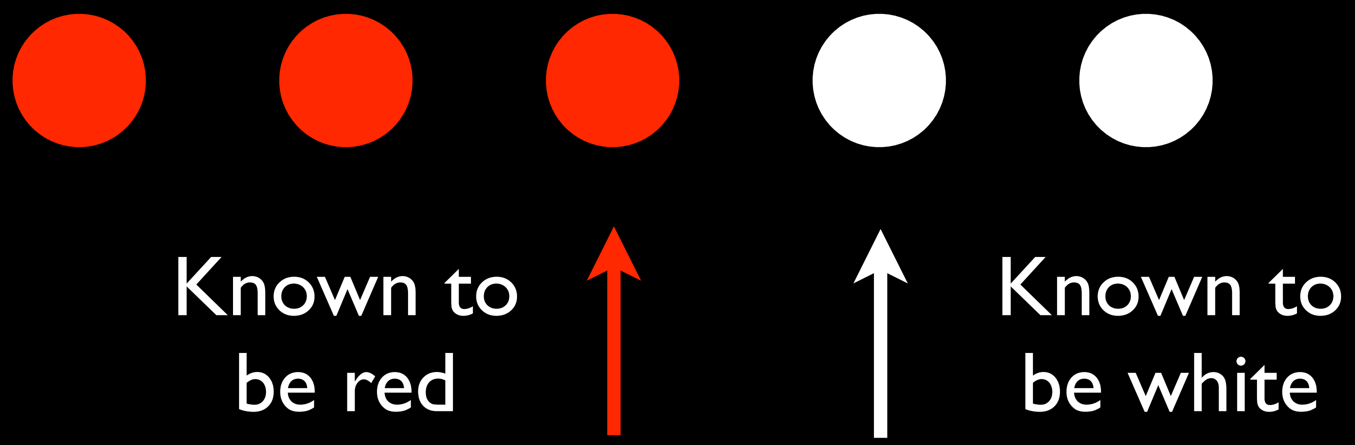
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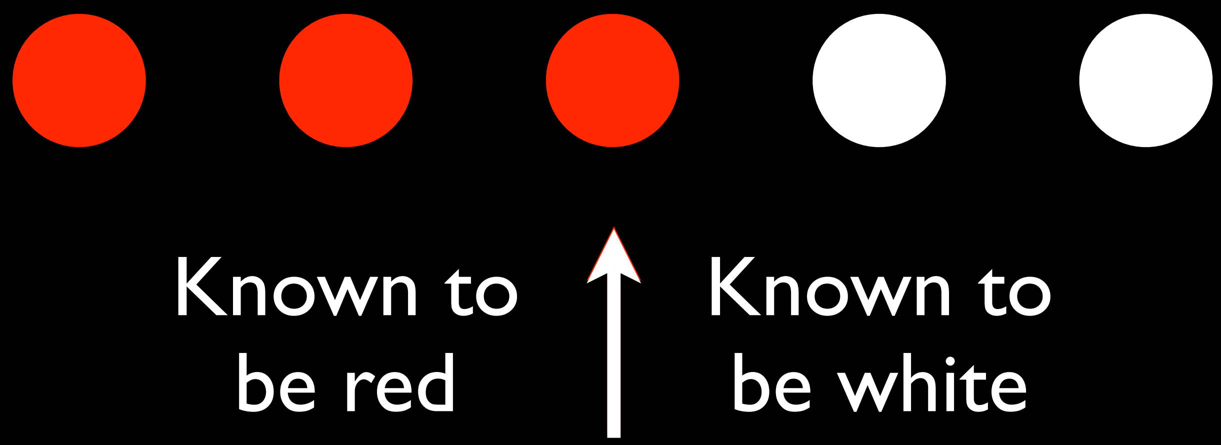
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Can we find a solution:

- that terminates on all inputs;
- satisfies the specification;
- and has machine verified proofs of both these properties.

Plan of attack

- Use the dependently typed programming language Agda to:
    - implement the mini-computer;
    - write an algorithm that sorts the pebbles;
    - prove the algorithm correct.

The Mini-Computer

Pebbles

```
data Pebble : Set where
  Red : Colour
  White : Colour
```

Natural numbers

```
data Nat : Set where
  Zero : Nat
  Succ : Nat -> Nat
```

Buckets

```
data Buckets : Nat -> Set where
  Nil : Buckets Zero
  Cons : Pebble -> Buckets n ->
         Buckets (Succ n)
```

The state monad

```
State : Nat -> Set -> Set
State n a =
  Buckets n -> Pair a (Buckets n)

return : a -> State n a
_>>=_ : State n a ->
  (a -> State n b) -> State n b
```

Indices

```
data Index : Nat -> Set where
  One : Index (Succ n)
  Next : Index n ->
         Index (Succ n)
```

Indices

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data Index : Nat -> Set where
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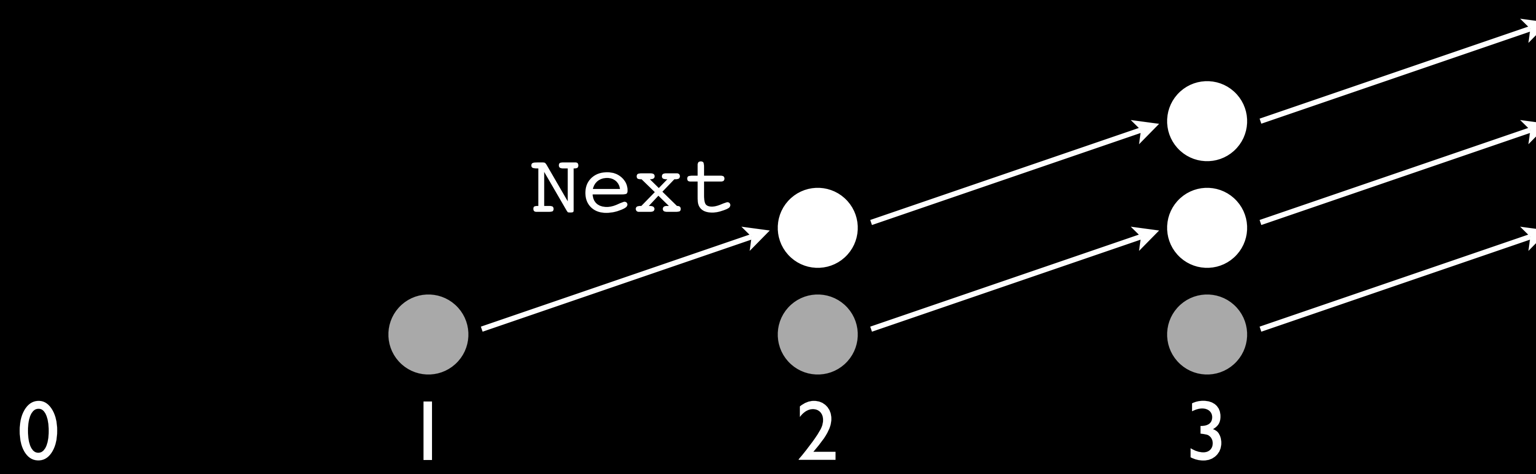


Reading

```
read : Index n -> State Pebble
read i bs = (bs ! i , bs)
  where
  _!_ : Buckets n -> Index n
        -> Pebble
  (Cons p _) ! One = p
  (Cons _ ps) ! (Next i) = ps ! i
```

Swap

```
swap : Index n -> Index n
       -> State n Unit
swap i j =
  read i >>= \pi ->
  read j >>= \pj ->
  write i pj >>
  write j pi
```

Back to the problem

An approximation

```
sort :: Index n -> Index n
     -> State n Unit
sort r w =
  if w == r then return unit
  else case read r of
    Red   -> sort (r + 1) w
    White -> swap r w >>
             sort r (w - 1)
```

An approximation

```
sort :: Index n -> Index n
     -> State n Unit
sort r w =
  if w == r then return unit
  else case read r of
    Red   -> sort (r + 1) w
    White -> swap r w >>
             sort r (w - 1)
```

Why does this terminate?

An approximation

```
sort :: Index n-> Index n
      -> State n Unit
sort r w =
  if r == w then return unit
  else case read r of
    White ->  sort (r + 1) w
    Red ->   swap r w >>
             sort r (w - 1)
```

An approximation

```
sort :: Index n-> Index n
    -> State n Unit
sort r w =
  if r == w then return unit
  else case read r of
    White ->  sort (r + 1) w
    Red ->  swap r w >>
              sort r (w - 1)
```

Only terminates

if $r \leq w$

Manipulating Indices

```
sort :: Index n-> Index n
      -> State n Unit
sort r w =
  if r == w then return unit
  else case read r of
    White -> sort (r + 1) w
    Red ->  swap r w >>
            sort r (w - 1)
```

Two problems

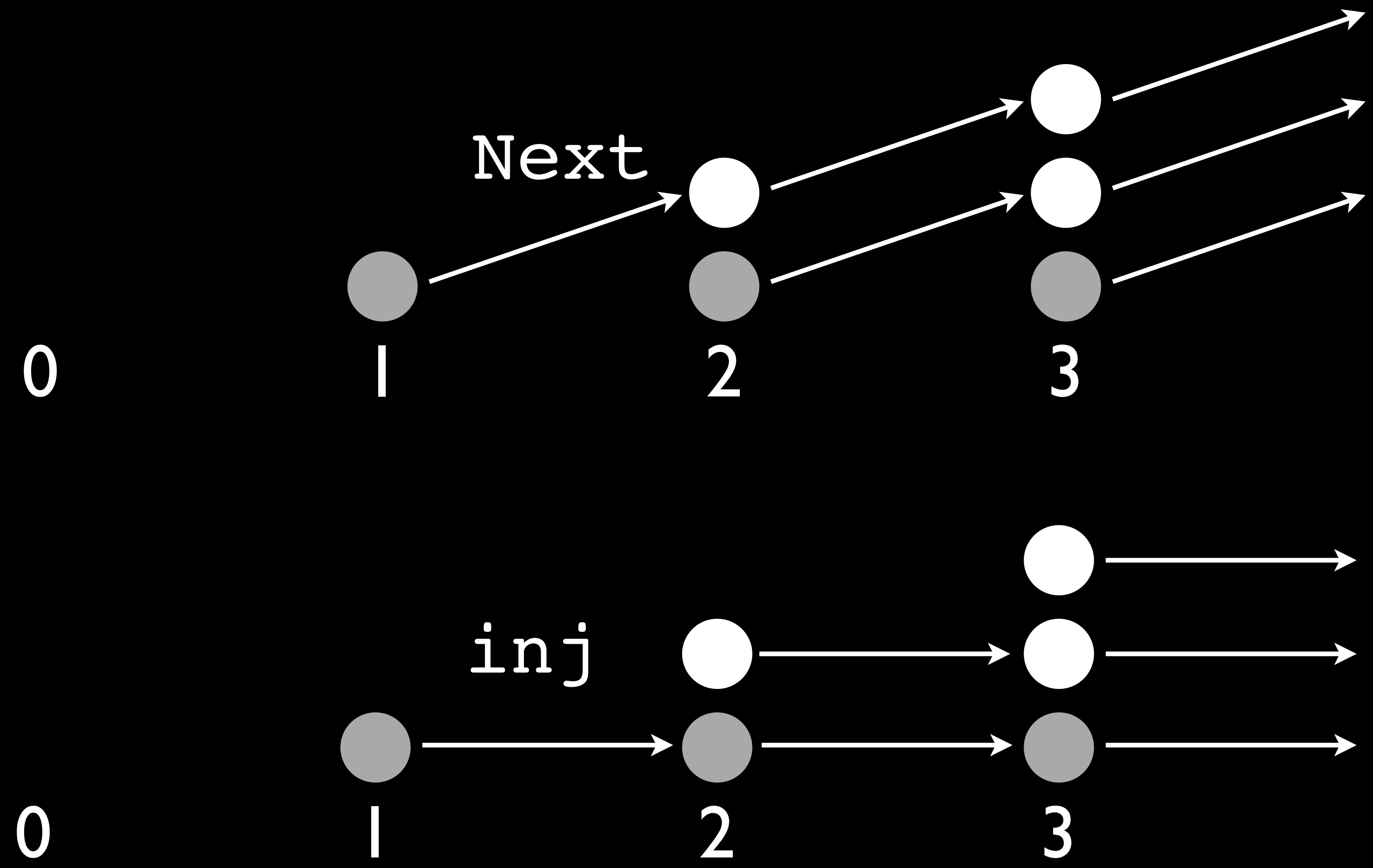
- We need to increment and decrement inhabitants of `Index n` ;
- We need to prove that our algorithm terminates.

```
Next : Index n -> Index (Succ n)
```

Injection

```
inj : Index n -> Index (Succ n)
inj One = One
inj (Next i) = Next (inj i)
```

Next or inj



Idea

- Only increment the image of `inj`;
- Only decrement the image of `Next`.

Less than or equal

```
data _<=_ : (i j : Index n) -> Set where
  Base : (i : Index (Succ n)) -> One <= i
  Step : (i j : Index n) ->
     (i <= j) -> (Next i <= Next j)
```

Difference

```
data Diff : (i j : Index n) -> Set where
  Base : (i : Index n) -> Diff i i
  Step : (i j : Index n) ->
     Diff i j -> Diff (inj i) (Next j)
```

Sort

```
sort : (r w : Index n) ->
       Diff r w ->
       State n Unit
```

Sort – Base case

```
sort : (r w : Index n) ->
       Diff r w ->
       State n Unit
sort .i .i (Base i) = return unit
```

```
sort : (r w : Index n) ->
       Diff r w ->
       State n Unit
```

```
sort : (r w : Index n) ->
       Diff r w ->
       State n Unit
sort .(inj i) .(Next j) (Step i j d) =
```

```
sort : (r w : Index n) ->
       Diff r w ->
       State n Unit
sort .(inj i) .(Next j) (Step i j d) =
    read (inj i) >>= \p ->
    case p of
      Red ->
      White ->
```

```
sort : (r w : Index n) ->
       Diff r w ->
       State n Unit
sort .(inj i) .(Next j) (Step i j d) =
    read (inj i) >>= \p ->
    case p of
      Red -> sort (Next i) (Next j) ?
      White ->
```

```
sort : (r w : Index n) ->
       Diff r w ->
       State n Unit
sort .(inj i) .(Next j) (Step i j d) =
    read (inj i) >>= \p ->
    case p of
      Red -> sort (Next i) (Next j) ?
      White ->
        swap (inj i) (Next j) >>
        sort (inj i) (inj j) ?
```

Lemmas

- We need to prove a few useful lemmas:
  - `Diff i j -> Diff (Next i) (Next j)`
  - `Diff i j -> Diff (inj i) (inj j)`

Lemmas

- We need to prove a few useful lemmas:
  - `Diff i j -> Diff (Next i) (Next j)`
  - `Diff i j -> Diff (inj i) (inj j)`

...but even then the algorithm is not *structurally* recursive.

Difference, revisited

```
data Diff : (i j : Index n) -> Set where
 Base : (i : Index n) -> Diff i i
 Step : (i j : Index n) ->
    Diff (inj i) (inj j) ->
    Diff (Next i) (Next j) ->
    Diff (inj i) (Next j)
```

Verification

Verification

the easy part

Formalizing the Invariant

```
Invariant : (r w : Index n)
  -> Buckets n -> Set
Invariant r w bs =
 (∀ i -> w < i -> bs ! i = White)
 && (∀ i -> i < r -> bs ! i = Red)
```

Correctness Theorem

```
∀ r w bs,
Invariant r w bs ->
  ∃ m : Index n,
  Invariant m m (sort r w bs)
```

Proof sketch

- Proof proceeds by induction on `Diff`
- Distinguish three cases:
  - Base case (trivial);
  - No swap happens (not too hard);
  - Swap happens (a bit trickier).
- In the latter two cases, we establish the invariant holds and make a recursive call.

The Dutch National Flag

- The *structure* of the algorithm stays the same.
  - similar invariant;
  - similar termination proof.
- Program does more case analysis...
- ... and so do the proofs.
- Messier but no harder.

Conclusions

- You need a PhD to verify a four line C program in Agda.
- ... but it is possible to verify non-structurally recursive, 'impure' functions in type theory.