Purely Functional, Fully in-Place Programming

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data List a = Nil | Cons a (List a)

reverse :: List a → List a
reverse xs = reverseAcc xs Nil

where
reverseAcc :: List a → List a → List a
reverseAcc Nil acc = acc
reverseAcc (Cons x xs) acc = reverseAcc xs (Cons x acc)
reverseProperty :: List → Bool
reverseProperty xs = reverse (reverse xs) == xs

Property-based testing libraries like QuickCheck generate inputs for our reverse function, trying to falsify the property we have formulated:

> quickCheck reverseProperty
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reverseProperty xs = reverse (reverse xs) == xs

Property-based testing libraries like QuickCheck generate inputs for our reverse function, trying to falsify the property we have formulated:

> quickCheck reverseProperty

OK, passed 100 tests.
Reasoning about functional programs

**Theorem** \( \forall \text{xs}, \text{reverse (reverse xs)} = \text{xs} \)

**Proof** Base case (when \( \text{xs} \) is Nil)

\[
\text{reverse (reverse Nil)}
\]

\[
\text{by definition of reverse}
\]

\[
= \text{reverseAcc (reverseAcc Nil Nil) Nil}
\]

\[
\text{by definition of reverseAcc}
\]

\[
= \text{reverseAcc Nil Nil}
\]

\[
\text{by definition of reverseAcc}
\]

\[
= \text{Nil}
\]

Inductive case (when \( \text{xs} \) is of the form Cons \( y \) \( \text{ys} \))...
Proof assistants

We can even formalise such proofs in a proof assistant (such as Coq, Agda, Lean, or others) that checks our reasoning:

Lemma rev_involutive : forall xs, reverse (reverse xs) = xs.

Proof.

induction l as [ | a l IHl].

simpl in |- *; auto.

simpl in |- *.

rewrite (rev_unit (rev l) a).

rewrite IHl; auto.

Qed.
Pure functional programming

*Composition*al programs - each assembled from other functions that can be tested and verified independently.

There are a variety of different techniques for verifying that a program is correct:

- testing automatically;
- writing pen and paper proofs;
- developing a formal proofs using proof assistants.
Pure functional programming

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- writing pen and paper proofs;
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How does this compare to list reversal in an imperative language?
list_t* reverse( list_t* curr ) {
    list_t* prev = NULL;
    while( curr != NULL ) {
        list_t* next = curr->tail;
        curr->tail = prev;
        prev = curr;
        curr = next;
    }
    return prev; }

Each cell in the list stores a value and a pointer to the remaining lists.

To reverse the list, we update the pointer in each cell to point to the previous element, until there are no cells left.
In-place execution
In-place execution
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In-place execution
Correctness?

Giving a *formal proof* that such a program is correct is **very hard**.

- Hoare logic is not suitable for reasoning about this kind of imperative code: what if the list structure in memory has cycles? Or if the two pointers map to lists sharing the same memory locations?
- Extensions of Hoare logic, notably separation logic, have had a lot of success - but these methods are certainly not elementary.
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- Hoare logic is not suitable for reasoning about this kind of imperative code: what if the list structure in memory has cycles? Or if the two pointers map to lists sharing the same memory locations?
- Extensions of Hoare logic, notably separation logic, have had a lot of success - but these methods are certainly not elementary.

But the C algorithm has an important property: it is executed in-place - it does not need to allocate new memory or deallocate unused memory.
Functional reverse

reverseAcc :: List a → List a → List a

reverseAcc Nil acc = acc
reverseAcc (Cons x xs) acc = reverseAcc xs (Cons x acc)
Functional reverse

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Despite the apparent virtues of functional programming, *memory management matters*. 
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Now reversing a linked list is not that exciting...

But many algorithms and datastructures – such as *splay trees* or *quicksort* – use careful pointer management to avoid (de)allocating memory.

Overly liberal allocation & garbage creation has a real performance impact.
Let’s revisit our reversal function.

```
reverseAcc :: List a → List a → List a
reverseAcc Nil acc = acc
reverseAcc (Cons x xs) acc = reverseAcc xs (Cons x acc)
```

Here we see that we are both allocating a new Cons cell and creating a new ‘garbage’ Cons cell in each recursive step.
reverseAcc :: List a → List a → List a
reverseAcc Nil acc = acc
reverseAcc (Cons x xs) acc = reverseAcc xs (Cons x acc)

The reverseAcc function has a few important properties:

• we can see that we are matching on one Cons cell on the left;
• and allocating one new Cons cell on the right.
• all other variables (like x or acc) are used linearly, i.e. they are not copied or discarded.

We will call such programs *fully in-place* – or fip for short.
Making this more precise..

\[
\begin{align*}
\Gamma & := \emptyset \mid \Gamma, x \mid \Gamma, \phi_k & \text{(owned environment)} \\
\Delta & := \emptyset \mid \Delta, y & \text{(borrowed environment)} \\
\frac{}{\Delta \mid x \vdash x} & \text{VAR} \\
\frac{\Delta \mid \Gamma_i \vdash v_i}{\Delta \mid \Gamma_1, \ldots, \Gamma_n \vdash (v_1, \ldots, v_n)} & \text{TUPLE} \\
\frac{\bar{y} \in \Delta, \text{dom}(\Sigma) \quad \Delta \mid \Gamma \vdash e}{\Delta \mid \Gamma \vdash f(\bar{y}; e)} & \text{CALL} \\
\frac{y \in \Delta \quad \Delta \mid \Gamma \vdash y \; e}{\Delta \mid \Gamma \vdash y \; e} & \text{RAPP} \\
\frac{\Delta \mid \Gamma \vdash e}{\Delta \mid \Gamma, \phi_0 \vdash e} & \text{EMPTY} \\
\frac{}{\vdash \emptyset} & \text{DEFBASE} \\
\end{align*}
\]

\[\frac{\Delta \mid \emptyset \vdash C}{\Delta \mid \emptyset \vdash C} \quad \text{ATOM}\]
\[\frac{\Delta \mid \Gamma_i \vdash v_i}{\Delta \mid \Gamma_1, \ldots, \Gamma_n \vdash \phi_k \mid C^k v_1 \ldots v_k} \quad \text{REUSE}\]
\[\frac{\Delta, \Gamma_2 \mid \Gamma_1 \vdash e_1 \quad \Delta, \Gamma_2, \Gamma_3, \bar{x} \vdash e_2 \quad \bar{x} \notin \Delta, \Gamma_2, \Gamma_3}{\Delta \mid \Gamma_1, \Gamma_2, \Gamma_3 \vdash \text{let } \bar{x} = e_1 \text{ in } e_2} \quad \text{LET}\]
\[\frac{y \in \Delta \quad \Delta, \bar{x}_i \mid \Gamma \vdash e_i \quad \bar{x}_i \notin \Delta, \Gamma}{\Delta \mid \Gamma \vdash \text{match } y \{ C_i \bar{x}_i \mapsto e_i \}} \quad \text{BMATCH}\]
\[\frac{\Delta \mid \Gamma, \bar{x}_i, \phi_k \vdash e_i \quad k = |\bar{x}_i| \quad \bar{x}_i \notin \Delta, \Gamma}{\Delta \mid \Gamma \vdash \text{match! } x \{ C_i \bar{x}_i \mapsto e_i \}} \quad \text{DMATCH!}\]
\[\frac{\vdash \Sigma' \quad \bar{y} \mid \bar{x} \vdash e}{\vdash \Sigma', f(\bar{y}; \bar{x}) = e} \quad \text{DEFUN}\]

Fig. 4. Well-formed FIP expressions, where the multiplicity of each variable in \( \Gamma \) is 1.
In-place reverse in Koka

These rules (and corresponding memory reuse) have been implemented in the Koka compiler.

Re-implementing the reverse function in Koka becomes:

```koka
fip fun reverse-acc( xs : list<a>, acc : list<a> ) : list<a>
  match xs
    Cons(x,xx) -> reverse-acc( xx, Cons(x,acc) )
    Nil       -> acc
```

The `fip` keyword indicates that a function can be executed fully in place.

The compiler checks that each FIP function can be executed in constant stack space, without allocating or deallocating memory.

This gives great performance, while still writing purely functional programs.
List reversal is not so interesting - what about algorithms for trees?

```plaintext
type tree
    Node( left : tree, key : int, right : tree )
    Leaf

Let's look at algorithms on binary search trees.

In particular, restructuring binary search trees, where accessing an element restructures the tree.
fun rotateRight (t : tree) : tree

match t
    (Node (Node (t2, a, t3), f, t1) -> Node (t2, a, Node (t3, f, t1)))
Allen & Munroe suggest *repeated* rotations, ensuring the key being looked up is moved to the root of the new binary tree.

In this fashion, frequently accessed elements naturally ‘bubble up’ to the root of the tree.

Can we give a purely functional—yet in place—algorithm?
fun lookup (k : int, t : tree)
    match t with
    Node (l, x, y) -> if k == x
        then x
        else if k < x
            then lookup (k, l)
            else lookup (k, r)
fun lookup (k : int, t : tree)

match t with
    Node (l, x, y) -> if k == x
    then x
    else if k < x
    then lookup (k, l)
    else lookup (k, r)

Key idea
Search through the tree recursively, accumulating the unvisited subtrees on a ‘stack’.

Once we find the element, unwind the ‘stack’ to rebuild the new tree, rotating as we move back up.
A simple stack of subtrees will not work – to reconstruct a binary search tree we need to record if we went left or right!

```go
type zipper

    Done

        // we went left; the tree stores bigger elements than the key being accessed
    Left(up : zipper, x : int, right : tree )

        // we went right; the tree stores smaller elements than the key being accessed
    Right(left : tree, x : int, up : zipper )
```
A simple stack of subtrees will not work – to reconstruct a binary search tree we need to record if we went left or right!

```go
type zipper
    Done
    // we went left; the tree stores bigger elements than the key being accessed
    Left(up : zipper, x : int, right : tree )
    // we went right; the tree stores smaller elements than the key being accessed
    Right(left : tree, x : int, up : zipper )
```

Using these zippers we can construct a pair of functions:

```go
// search through the tree for the given key
fun access (key : int, t : tree, z : zipper): (tree, zipper)

// rebuild the entire tree, rotating as necessary
fun rebuild (t : tree, z : zipper): tree
```
The access function has the same structure as the (more familiar) `lookup` function.

If the key is already present, the tree is restructured; otherwise the new key is inserted (at the root).

The function is in-place, but may do at most one allocation.
In-place reuse

\texttt{fip fun access (...)}

\texttt{Node(l,x,r) \rightarrow if \ldots then access(l, k, Left(z,x,r))}

Why is this in-place? We match on a node, but extend our zipper?
In-place reuse

\textbf{fun} \textit{access} (...) \\
\textit{Node}(l,x,r) \rightarrow \textbf{if} \ldots \textbf{then} \textit{access}(l, k, \textit{Left}(z,x,r))

Why is this in-place? We match on a node, but extend our zipper?

Memory re-use is \textit{not} restricted to the same constructors or even the same types!

Instead, we only check that the variables are not duplicated or discarded;

And that the \textit{sizes} of deallocations and allocations line up.
Reconstructing the tree

```fip
fun rebuild(z : zipper, t : tree)
  match z
  | Done        → t
  | Right(l,x,z) → match t // we went right looking for k
  |           → rebuild(z, Node( Node(l,x,s), k, b))
  | Left(z,x,r) → match t // we went left looking for k
  |           → rebuild(z, Node( s, k, Node(b,x,r)))
```

Now we rebuild the entire tree, popping elements off the zipper.

In each case, we rotate the tree as required to ensure the result remains a binary search tree.

From this definition, it is easy to see that the key k stays at the root – just as we wanted!
The original zipper paper describes a functional approach to navigating through a tree (Huet 1997).

A pair of a zipper and tree, allows you to ‘move focus’ to a child, without loss of information:

```haskell
fip fun left (t : tree, z : zipper) : (tree, zipper)
  match t
    Node (l, x, r) ⇒ (l, Left(z, x, r))
```
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\[
\text{fip fun left (t : tree, z : zipper) : (tree, zipper)}
\]

\[
\text{match t}
\]

\[
\text{Node (l, x, r) \rightarrow (l, Left(z, x, r))}
\]

Huet (1997) writes:

\text{Efficient destructive algorithms on binary trees may be programmed with these completely applicative primitives, which all use constant time, since they all reduce to local pointer manipulation.}

Using our fip calculus we can make this precise – and check this property statically!

Zipper-based traversal turn out to be very useful.
Schorr-Waite-Deutsch style map on trees

**fip fun load** (t : tree, f : int \(\rightarrow\) int, z : zipper) : tree

**match** t

Node (l, x, r) \(\rightarrow\) load(l, f, Left(z, f(x), r))

Leaf \(\rightarrow\) unload(z, f, Leaf)

**fip fun unload** (z : zipper, f : int \(\rightarrow\) int, t : tree) : tree

**match** z

Left (z, x, r) \(\rightarrow\) load(r, f, Right(t, x, z))

Right (l, x, z) \(\rightarrow\) unload(z, f, Node(l, x, t))

Done \(\rightarrow\) t

This traversal does not allocate any heap or stack space – yet works for any* data type!
In-place sorting

Sorting algorithms - like quicksort or mergesort - tend to perform poorly in functional languages compared with their in-place imperative counterparts.

Writing *in-place* versions of these algorithms is not easy...
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Mergesort usually uses three steps:

1. Turn a list of numbers into a list of lists `map \( \langle x \rightarrow [x] \rangle \)`
2. Repeatedly merge adjacent pairs of lists...
3. Until we have a single sorted list left over.

Step 1) allocates new memory; step 2) repeatedly deallocates memory.

Does an in-place algorithm exist?
Instead of sorting a simple list, we move to a different data representation:

```plaintext
type plist
    Cons(hd : int, tl : plist)
    Nil
    Sorted(hd : list<int>, tl : plist)
```
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```

1. Turn a list of numbers into a plist built from Cons cells;
2. Repeatedly merge adjacent pairs of lists
3. Until we have a single sorted list left over.
In-place merge

```haskell
fun merge (xs : plist, ys : plist) : plist

match xs, ys
  Cons(x,xx),Cons(y,yy) ->
    if x <= y then Sorted(...,merge(xx,yy)) else Sorted(...,merge(xx,yy))
```

We can reuse one Cons cell for the outermost Sorted cell.

But how to complete the picture? We need to store two elements (x and y) - but we only have one memory cell to reuse!
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But how to complete the picture? We need to store two elements (x and y) - but we only have one memory cell to reuse!

Idea: we know that the sorted lists always have at least length two.

Let’s revise our data representation again.
In-place merging

Instead of sorting a simple list, we move to a different data representation:

```go
type plist
    Cons(hd : int, tl : plist )
    Nil
    Sorted(hd : list2, tl : plist)

type list2
    Nil2 (x : int, y : int)
    Cons2 (x : int, xs : int)
```

Using these types, we have defined an in-place merge operation!

Repeatedly merging eventually yields a single Sorted list.
In-place merging

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type plist
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  Cons2 (x : int, xs : int)
```

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Different data representations support different in place operations.
What about the imperative pseudocode?
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```plaintext
Definition heap_mtr_insert_td : val :=
fun (name, root) {
  var: left_dummy := #0 in
  var: right_dummy := #0 in
  var: node := root in
  var: left_hook := &left_dummy in
  var: right_hook := &right_dummy in
  while: ( true ) ( node != #0 ) {
    if: ( node->value == name ) {
      <left_hook = node->left;
      <right_hook = node->right;
      root = node;
      break;
    } else {
      if: ( node->value > name ) {
        <right_hook = node->right;
        right_hook = &(node->left);
        node = node->left
      } else {
        <left_hook = node;
        left_hook = &(node->right);
        node = node->right
      } 
    }
  }
}
```

node := root;
left_hook := addr(left(dummy));
right_hook := addr(right(dummy));

while: node != #null do
  if: value(node) == name then
    begin
      0(left_hook) := left(node);
      0(right_hook) := right(node);
      root := node;
      go to bottom
    end;
  if: value(node) > name then
    begin
      0(right_hook) := node;
      right_hook := addr(left(node));
      node := left(node)
    end
  else 
    begin
      0(left_hook) := node;
      left_hook := addr(right(node));
      node := right(node)
    end;
  0(left_hook) := null;
  0(right_hook) := null;
  root := new_node ( );
  value(root) := name;

bottom:
  left(root) := left(dummy);
  right(root) := right(dummy)
```

Fig. 1. The move-to-root top-down algorithm formalized in AddressC on the left, versus a screenshot of Stephenson’s published algorithm on the right
Verifying imperative version

Using a proof assistant, we have given a formal proof of correctness of both top-down and bottom-up move to root trees.

That is, we can prove a Hoare triple of (roughly) the following form:

**Lemma** heap_mtr_insert_td_correct (k : key) (p : ptr) (t : tree):

{ is_tree t p }
heap_mtr_insert_td k p
{ is_tree (mtr_insert_td k t) p }.

In this way, we prove that the functional version (mtr_insert_td) coincides precisely with the (published) imperative algorithms (heap_mtr_insert_td).
• These proofs are non-trivial! If you've ever tried to write out a formal proof in Hoare logic for any program longer than 5 lines, you know there is a lot of bookkeeping involved.
Verifying the imperative version

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• Theorem proving technology (Coq and Iris) are fairly impressive - the proof is about 50 loc, half of which is formulating the loop invariant.
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• The functional implementation captures the key parts of the specification – it is essential for spelling out the loop invariant.
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- The functional implementation captures the key parts of the specification – it is essential for spelling out the loop invariant.

- This is (to the best of our knowledge) the first formal proof of these algorithms.
Move-to-root trees guarantee that a freshly accessed key always moves to the root.

But it does not do any *rebalancing* along the way...
Move-to-root trees guarantee that a freshly accessed key always moves to the root.

But it does not do any rebalancing along the way...

The more established splay trees (Sleator and Tarjan 1985) address precisely this issue.

The key difference is in the rebuild function that tries to rebalance the resulting tree.
Rebuilding splay trees

Self-Adjusting Binary Search Trees

DANIEL DOMINIC SLEATOR AND ROBERT ENDORE TARJAN

AT&T Bell Laboratories, Murray Hill, NJ

(a) Zig

(b) Zigzig

(c) Zigzag
Rebuilding splay trees

```haskell
fip fun rebuild(z : zipper, t : tree) : tree

match tree

  Node(tl,tx,tr) -> match z

  Done -> Node(tl,tx,tr)

  Right(rl,rx,Done) -> Node(Node(rl,rx,tl),tx,tr)        -- zig

  Left(Done,lx,lr) -> Node(tl,tx,Node(tr,lx,lr))

  Right(rl,rx,Right(1,x,up)) -> rebuild(up, Node(Node(Node(1,x,rl),rx,tl),tx,tr)) -- zigzig

  Left(Right(1,x,up),lx,lr) -> rebuild(up, Node(Node(1,x,tl),tx,Node(tr,lx,lr))))

  Right(rl,rx,Left(up,x,r)) -> rebuild(up, Node(Node(rl,rx,tl),tx,Node(tr,x,r)))  -- zigzag

  Left(Left(up,x,r),lx,lr) -> rebuild(up, Node(tl,tx,Node(tr,lx,Node(lr,x,r)))))
```
Comparing the same algorithm across different languages is always going to be unfair.

- Koka, Haskell, and OCaml have automatic memory management - as opposed to C;
- Haskell and OCaml use mark-and-sweep garbage collectors; Koka uses reference counting.
- Koka uses arbitrary precision integers for keys and all comparisons and arithmetic operations include branches for the case where big integer arithmetic is required;
- Haskell is lazy, most other functional languages are not.
Benchmarks

10M pseudorandom insertions of a key between 0 and 100,000 on an initially empty tree.
Beyond splay trees

• We have similar results for red-black trees and the more recent *ziptrees* (Tarjan, Levy & Timmel 2021) - in some cases, this leads to (slightly) improved algorithms over the best known implementations.

• Okasaki’s famous book on *Purely functional datastructures* is chock-full of fip algorithms...
Conclusions & trolling

• Best-of-both worlds approach: purely functional programs with low memory usage.
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• But if you need to write the functional version to verify imperative code – why write imperative programs to begin with?
Conclusions & trolling

- Best-of-both worlds approach: purely functional programs with low memory usage.

- Elementary verification techniques only! Structural induction on trees suffices.

- Modern proof assistants can relate the functional and imperative versions - the functional algorithm captures the essence of the loop invariant.

- But if you need to write the functional version to verify imperative code – why write imperative programs to begin with?

- And do we need separation logic at all?
Questions?
When to execute in place?

Even if a function is fip, it is not *always* safe to execute it in place.

Consider the following example:

```plaintext
fun makePalindrome( xs : list<a>) : list<a>
    return (append(xs, reverse(xs))
```

Clearly this call to reverse should not run in place!
When to execute in place?

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Consider the following example:

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fun makePalindrome( xs : list<a>) : list<a>
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```

Clearly this call to `reverse` should not run in place!

When can we tell it is safe to execute fip functions in place?
Koka uses a *reference counted garbage collection*. As a result, we know at execution time how many references exist to any given value. We can check if a reference is unique at run-time and re-use existing memory locations when possible:

```plaintext
fip fun reverse-acc( xs : list<a>, acc : list<a> ) : list<a>
  match xs
    Cons(x,xx) ->
    val addr = if is-unique(xs) then &xs else { dup(x); dup(xx); decref(xs); alloc(2) }
    reverse-acc( xx, Cons@addr(x,acc) )...
```
So what’s our the papers

- In place versions of:
  - Splay trees;
  - Schorr Waite traversal of trees;
  - Generic map over any algebraic datatype;
  - Red black tree insertion;
  - Mergesort;
  - Finger tree insertions;
  - ...

- Top-down & bottom-up implementations of:
  - move to root trees;
  - splay trees;
  - zip trees;
  - proofs that they ‘are all equal’;
  - proofs that the functional versions coincide with published imperative ones.

- Lots of metatheory, showing that it is safe to execute fip programs in place.