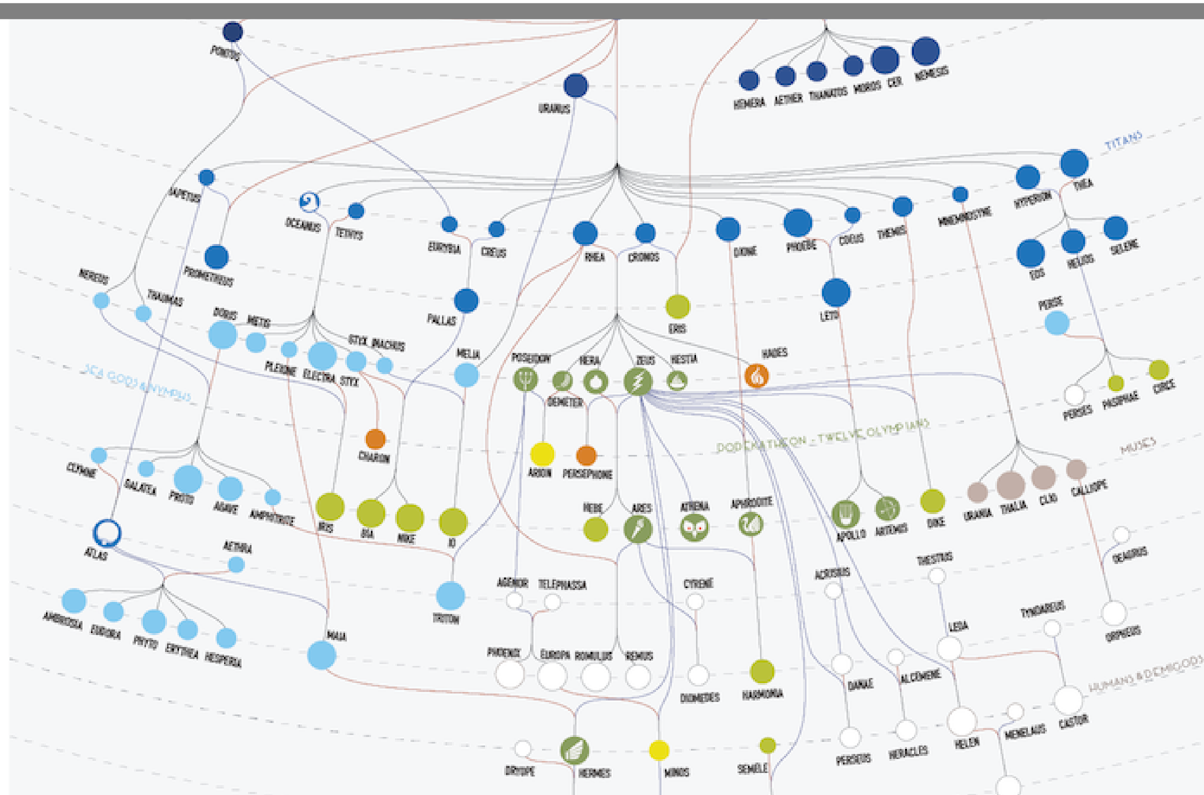
Algorithm for
Visualization of Directed Graphs

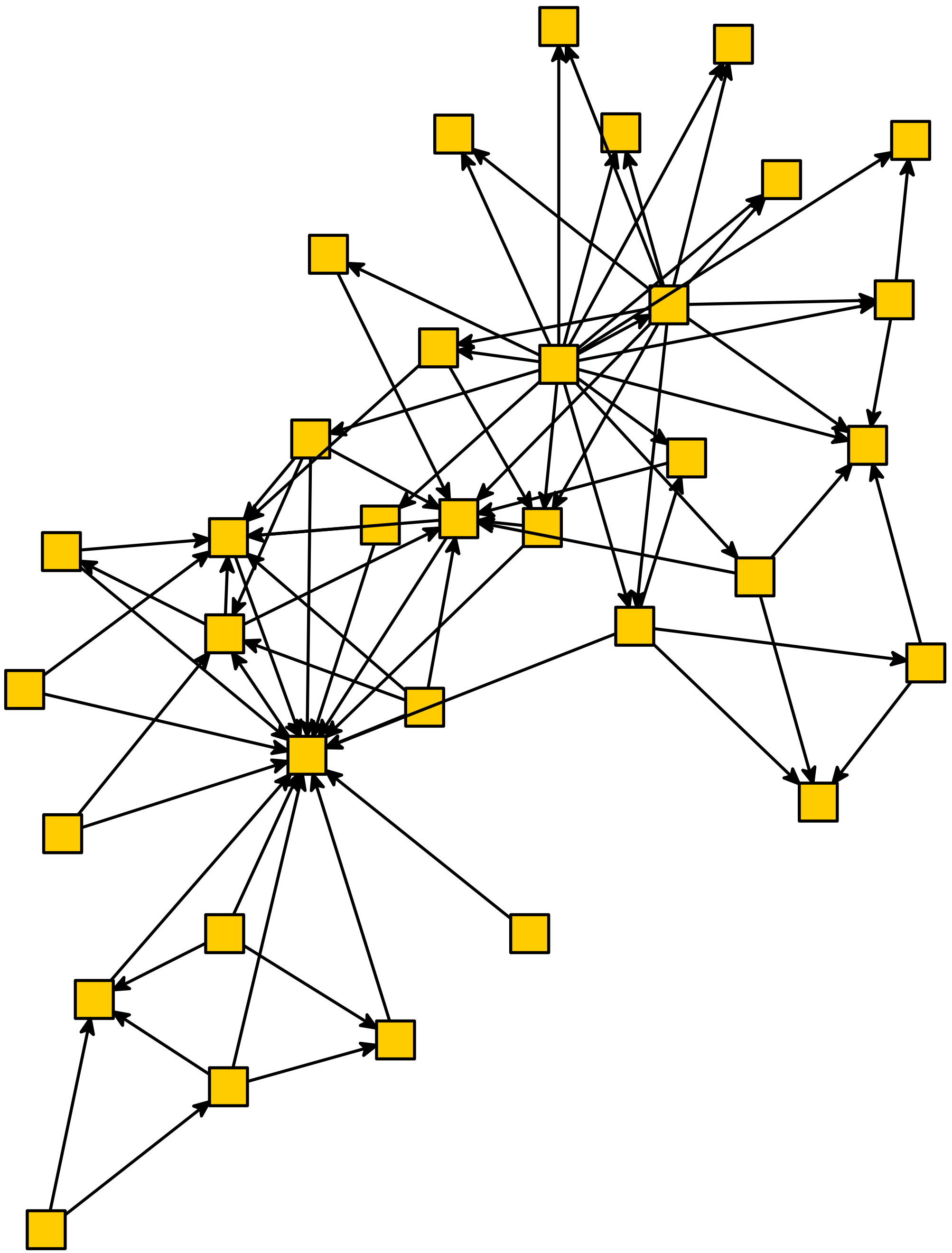
Course : Data Visualization
Lecturer : Tamara Mchedlidze
Utrecht University, Dept. of Information and Computing Sciences



Lecture Overview

- How to draw a directed graph
- Examples
- Sugiyama framework

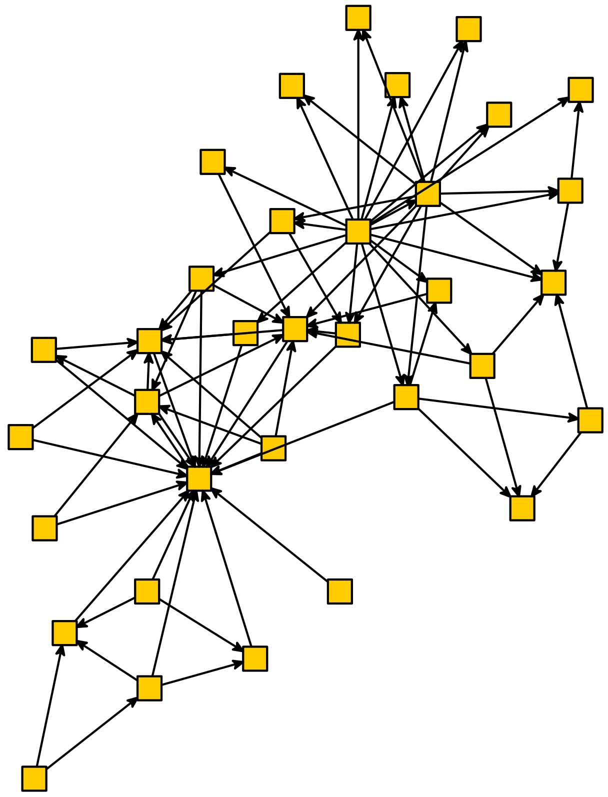
Example

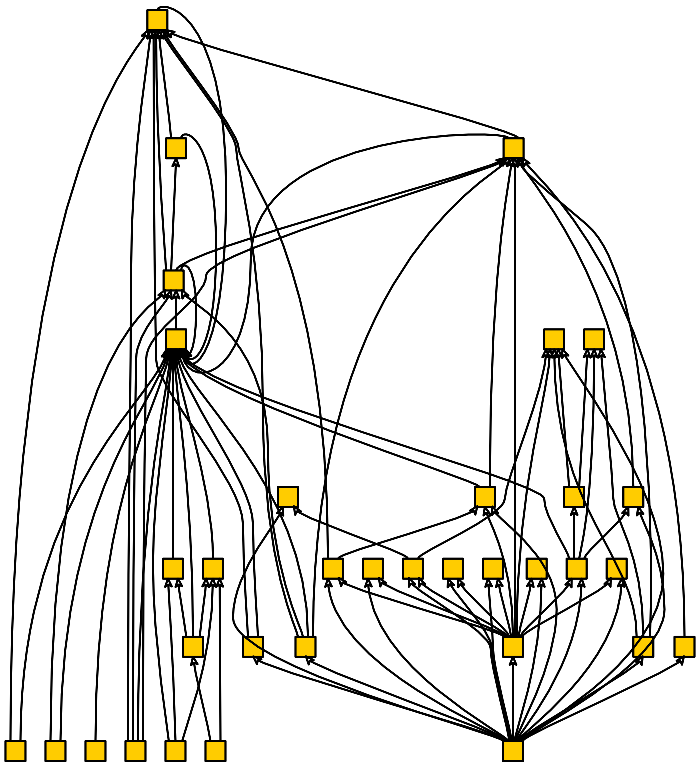


Layered Layout

Given: directed graph $D = (V, A)$

Find: drawing of D that emphasized the hierarchy





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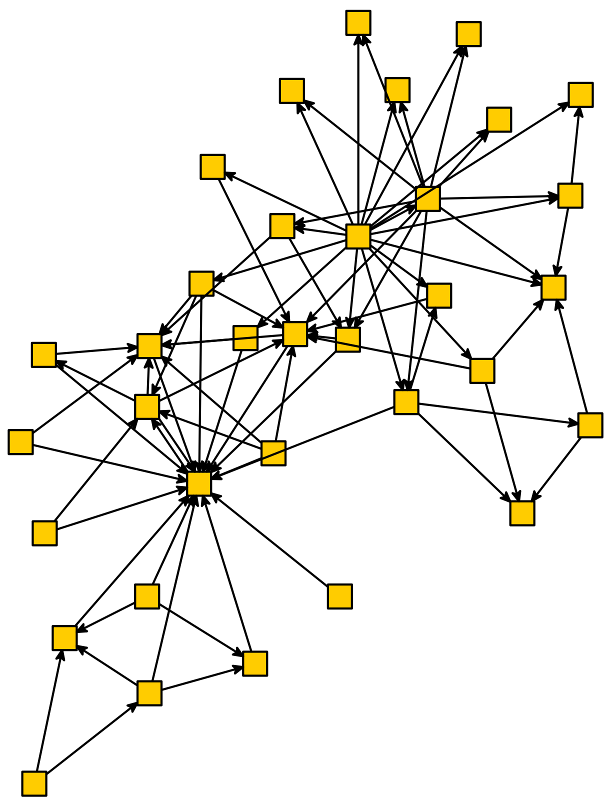

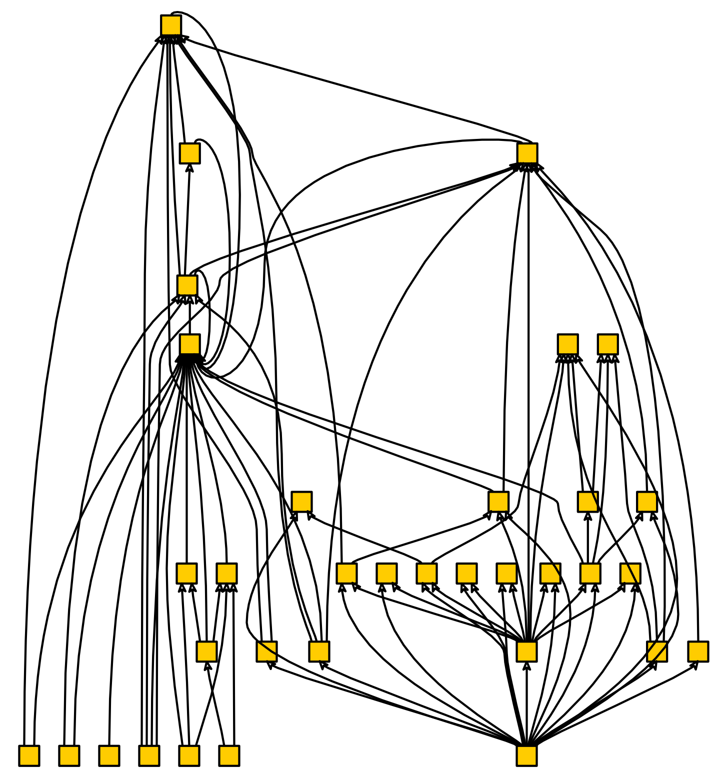

- many edges pointing to the same direction
- nodes lie on (few) horizontal lines

Layered Layout

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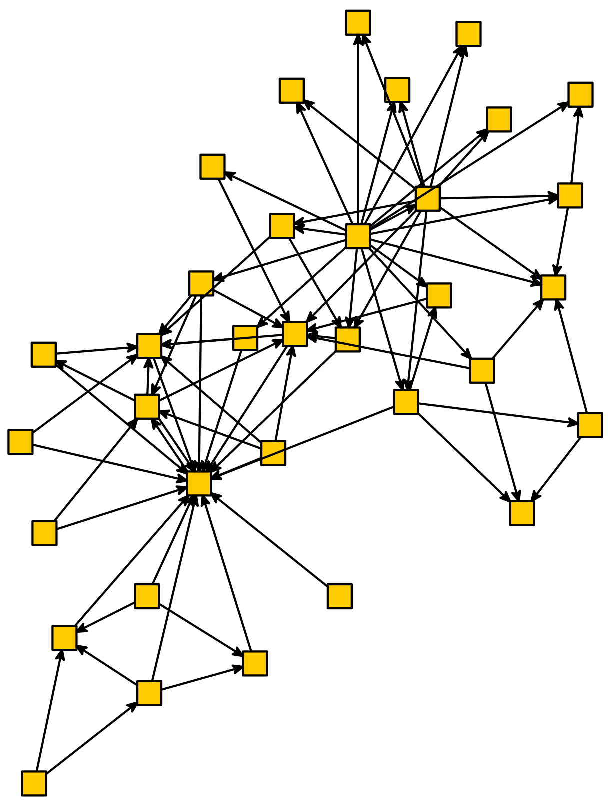
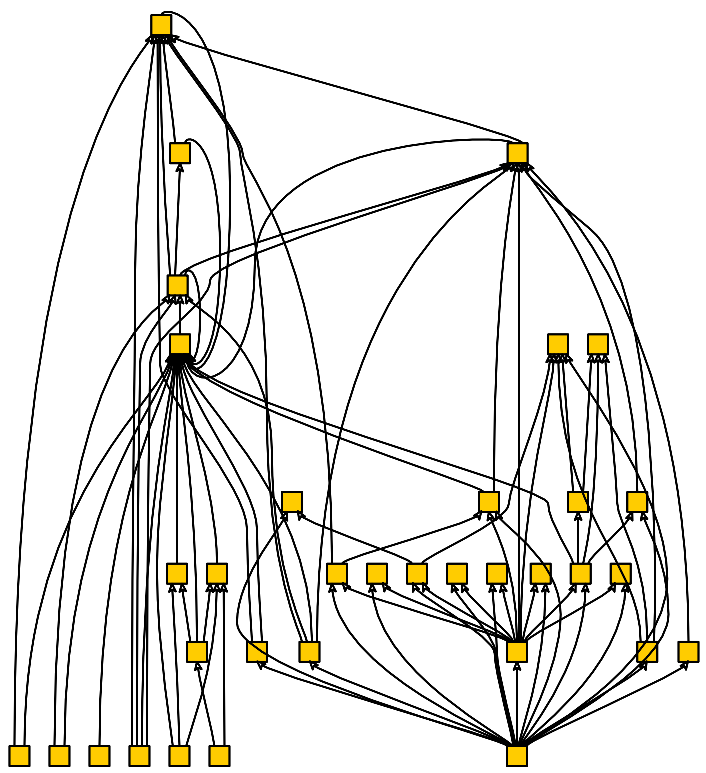
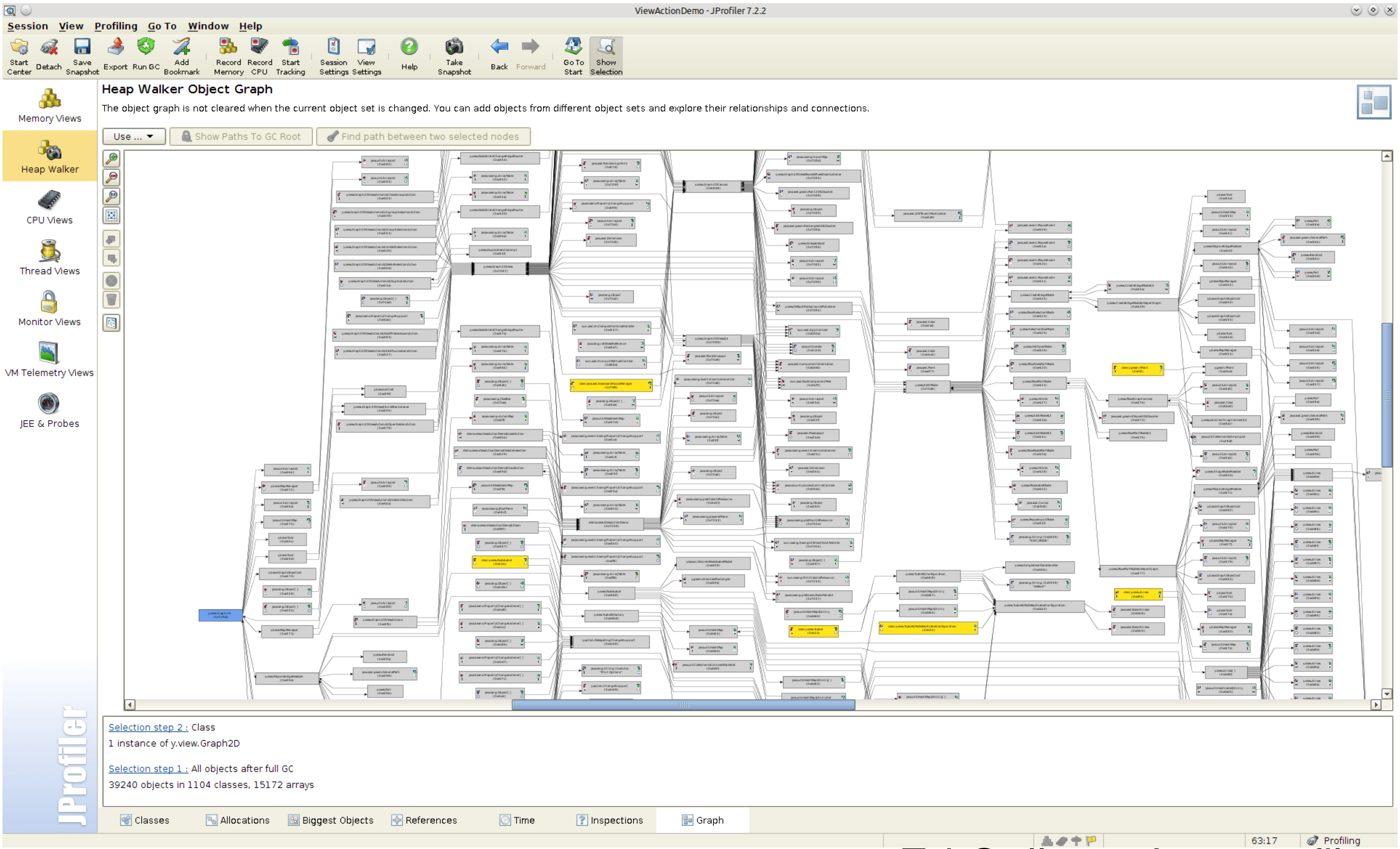




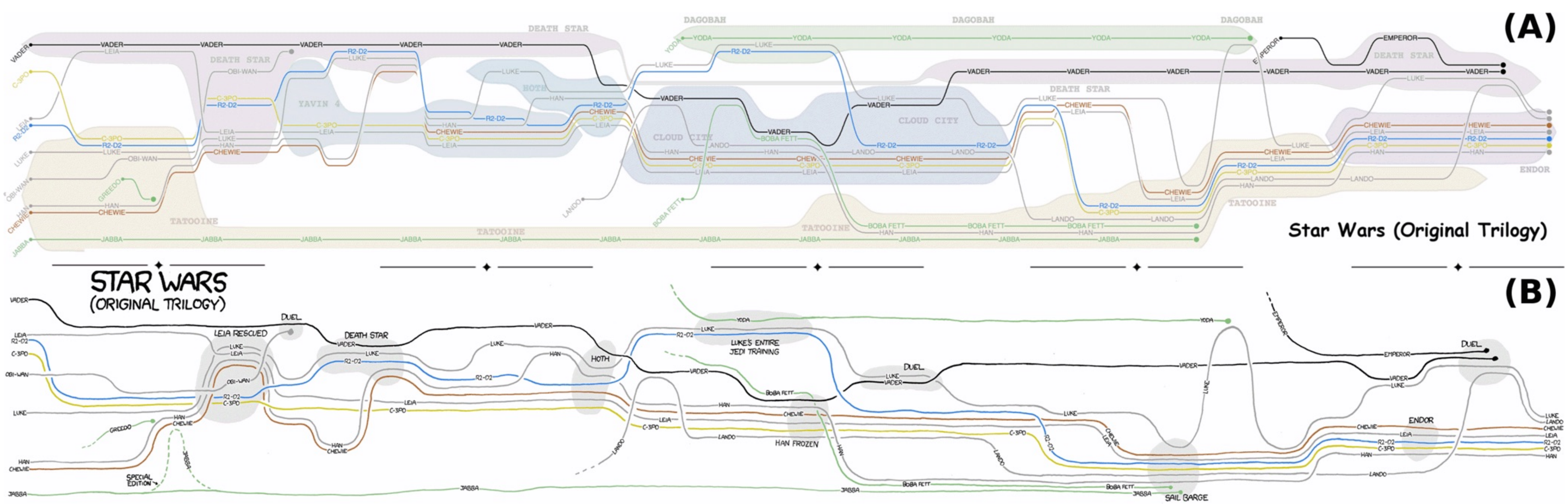
- edges as straight as possible and short
- few edge crossings
- nodes distributed evenly

Application: Java Profiler



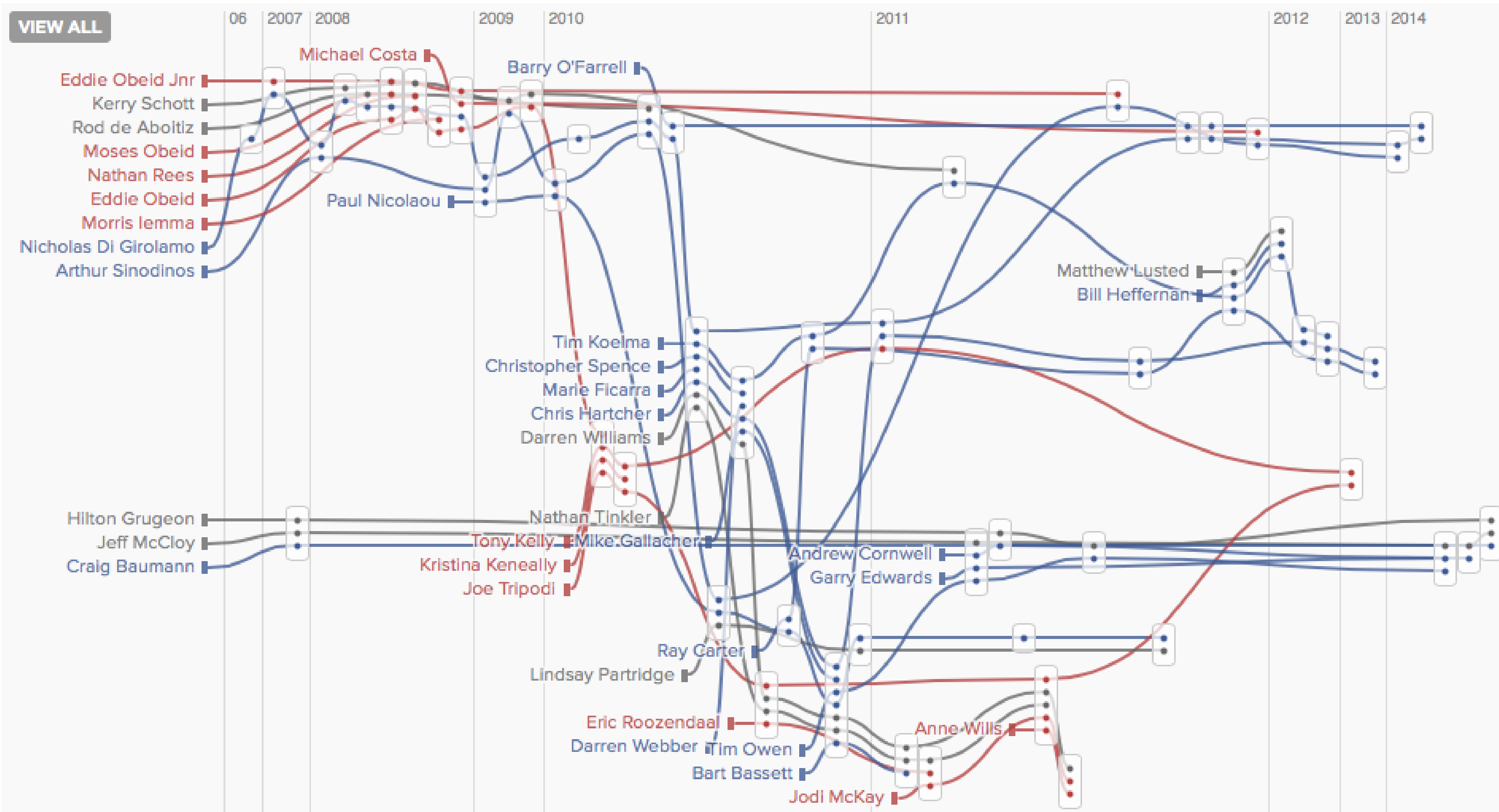
yEd Gallery: Java profiler
JProfiler using yFiles

Application: Storylines



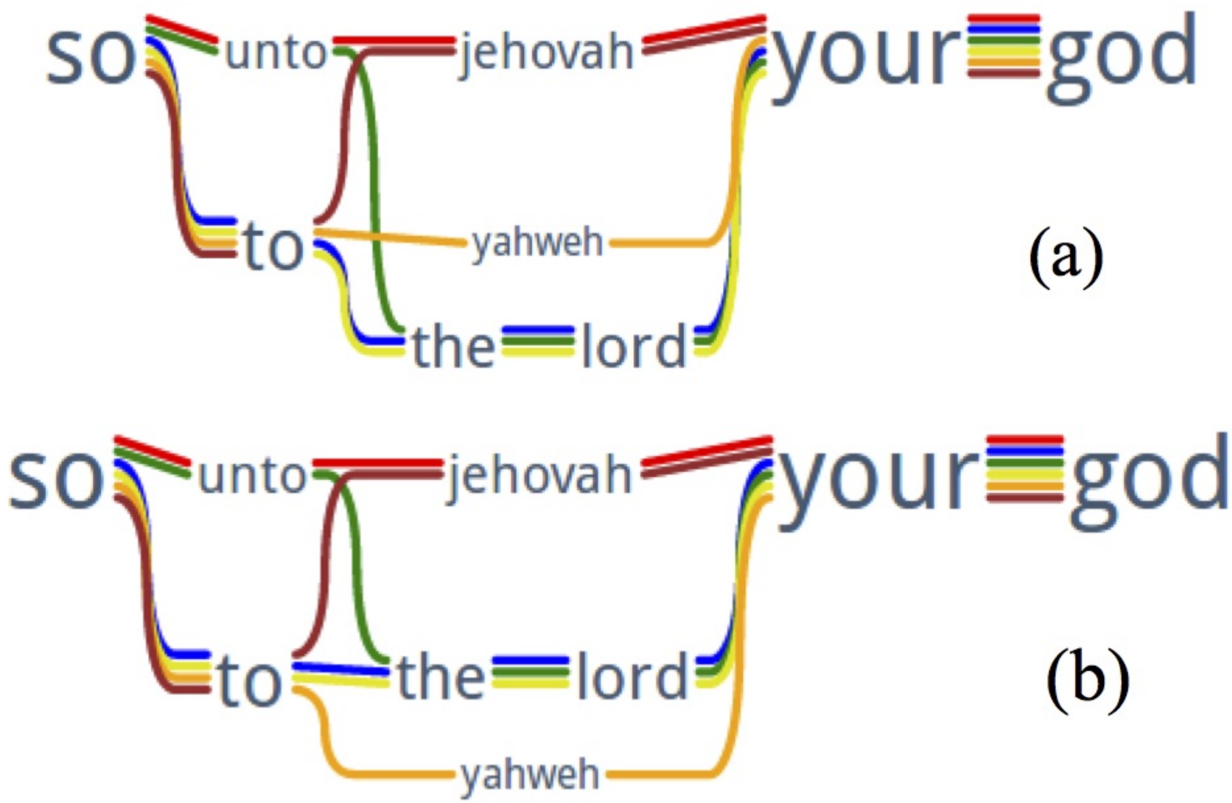
Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

Application: Storylines



Source: ABC news, Australia

Application: Text-Variant graphs

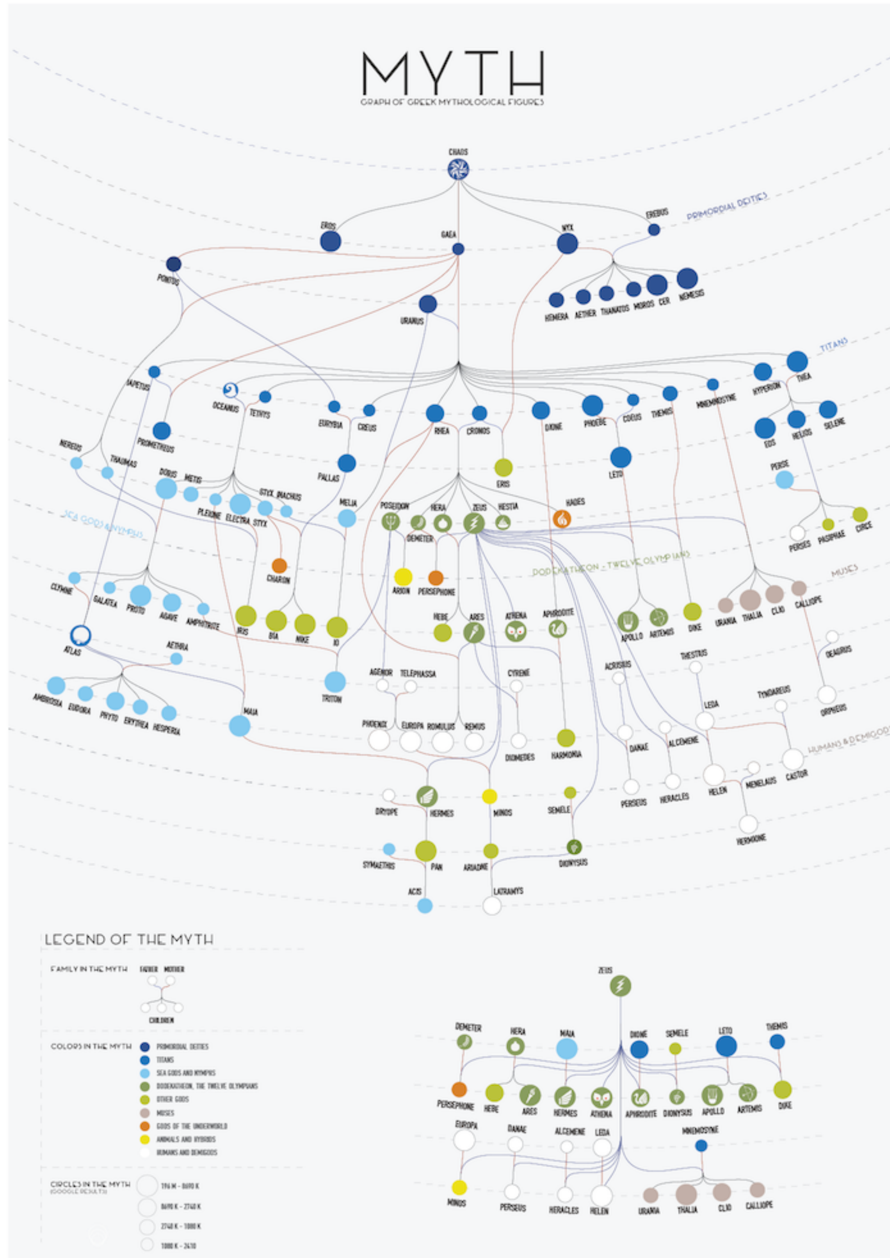


(a)

(b)

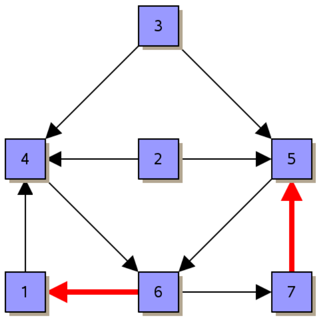
Source: Improving the Layout for Text Variant Graphs Jänicke et al.

Application: Mythological Creatures and Gods



Source: Visualization that won the Graph Drawing contest 2016. Klawitter& Mchedlidze

Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



given

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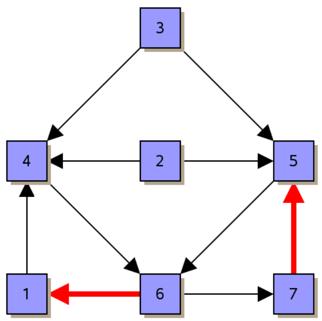

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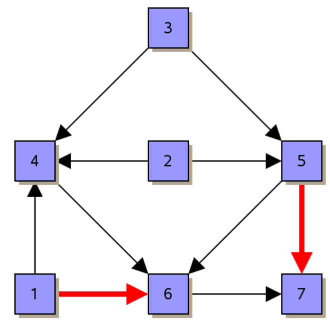

resolve cycles

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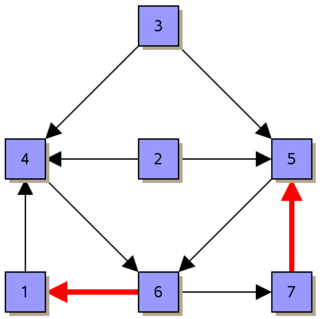

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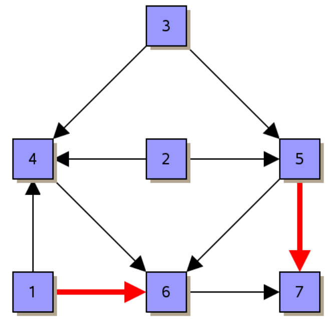

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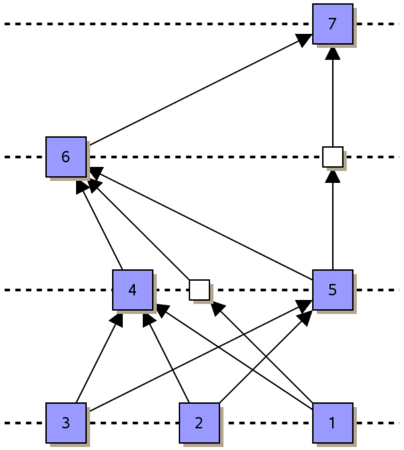

layer
assignemen

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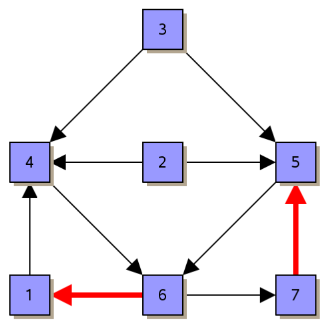

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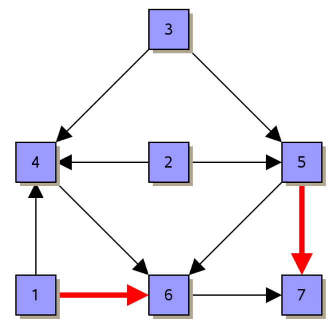

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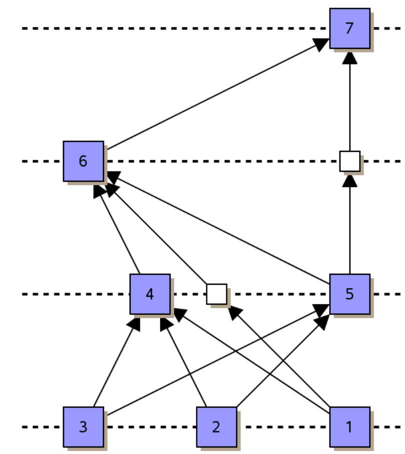

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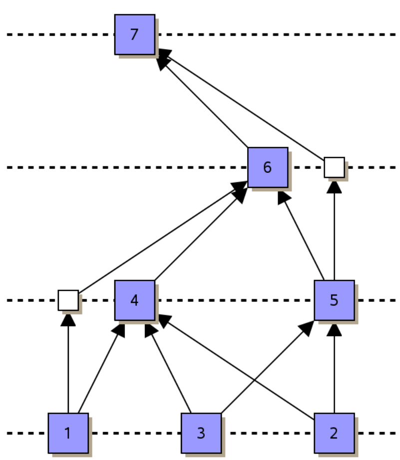
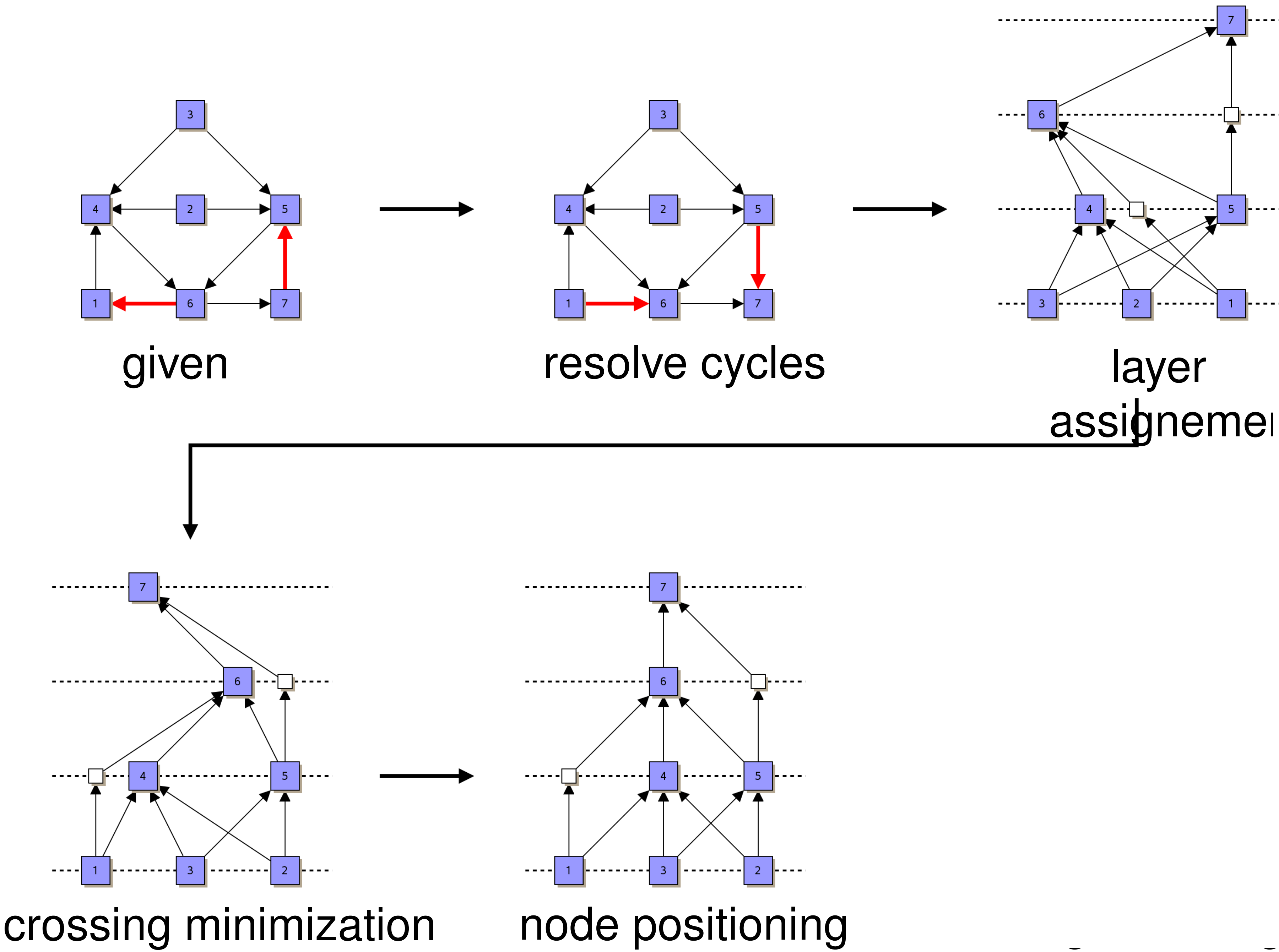


crossing minimizatio

Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



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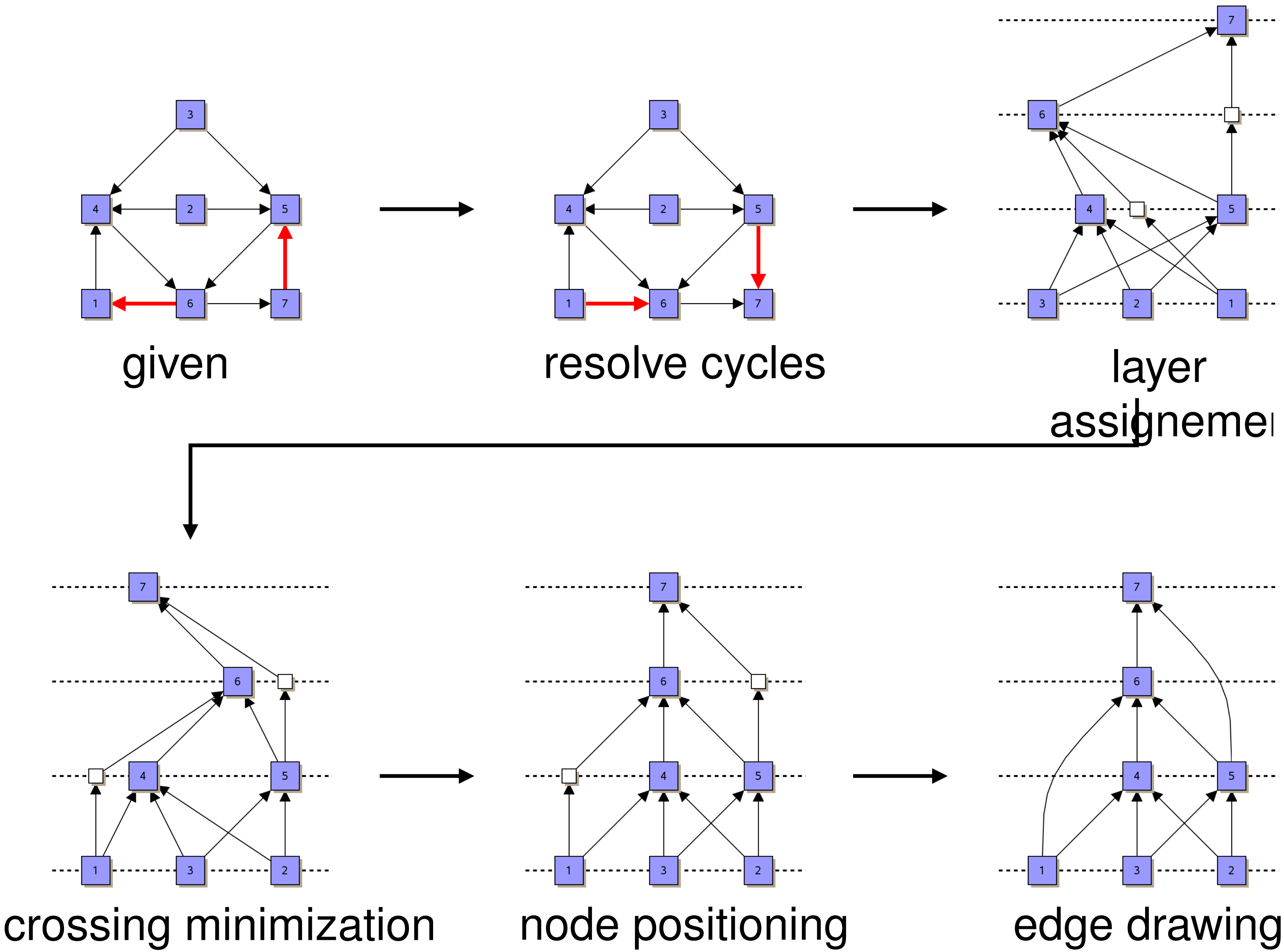
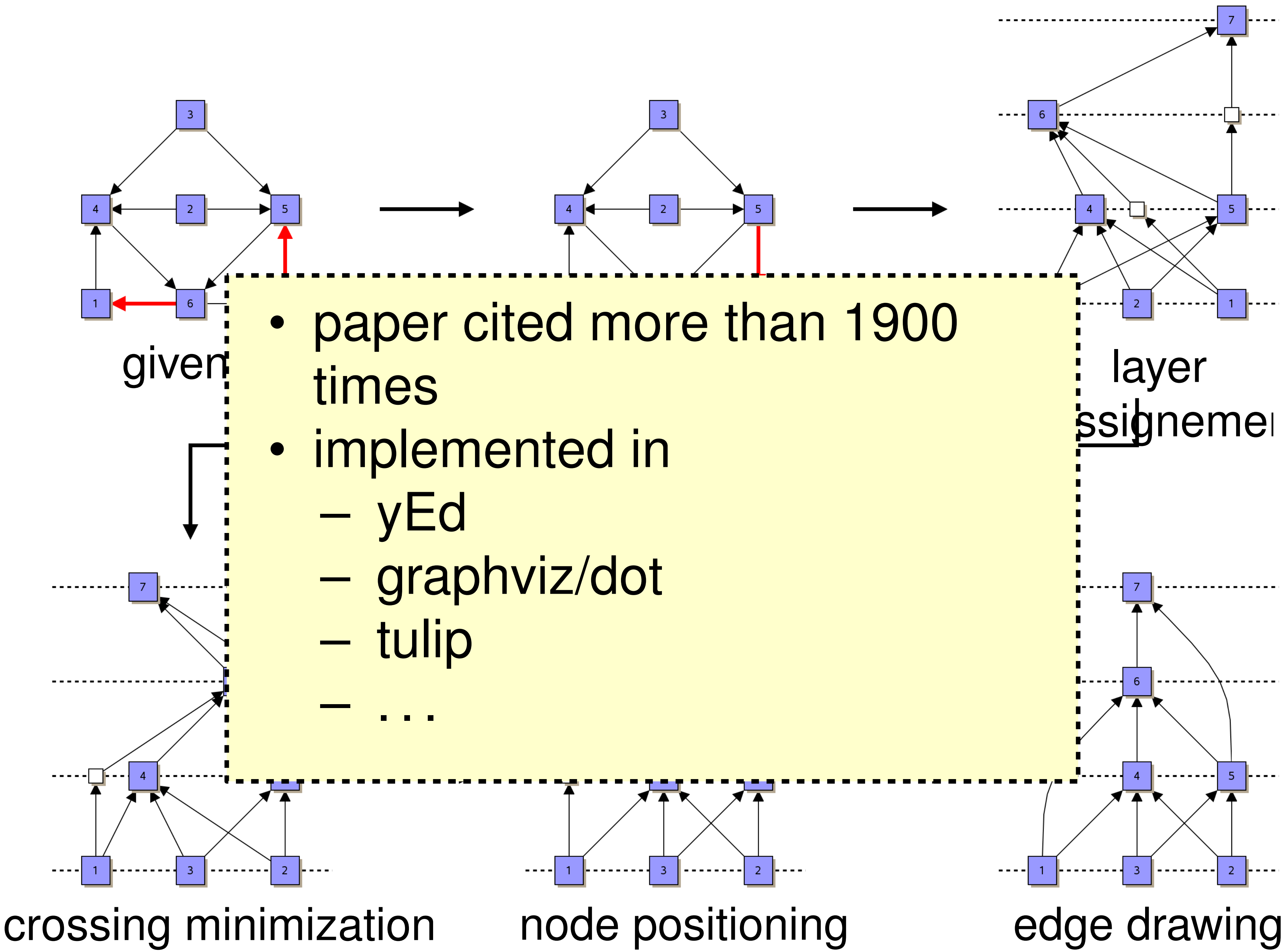


Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)

- paper cited more than 1900 times
- implemented in
  – yEd
  – graphviz/dot
  – tulip
  – . . .



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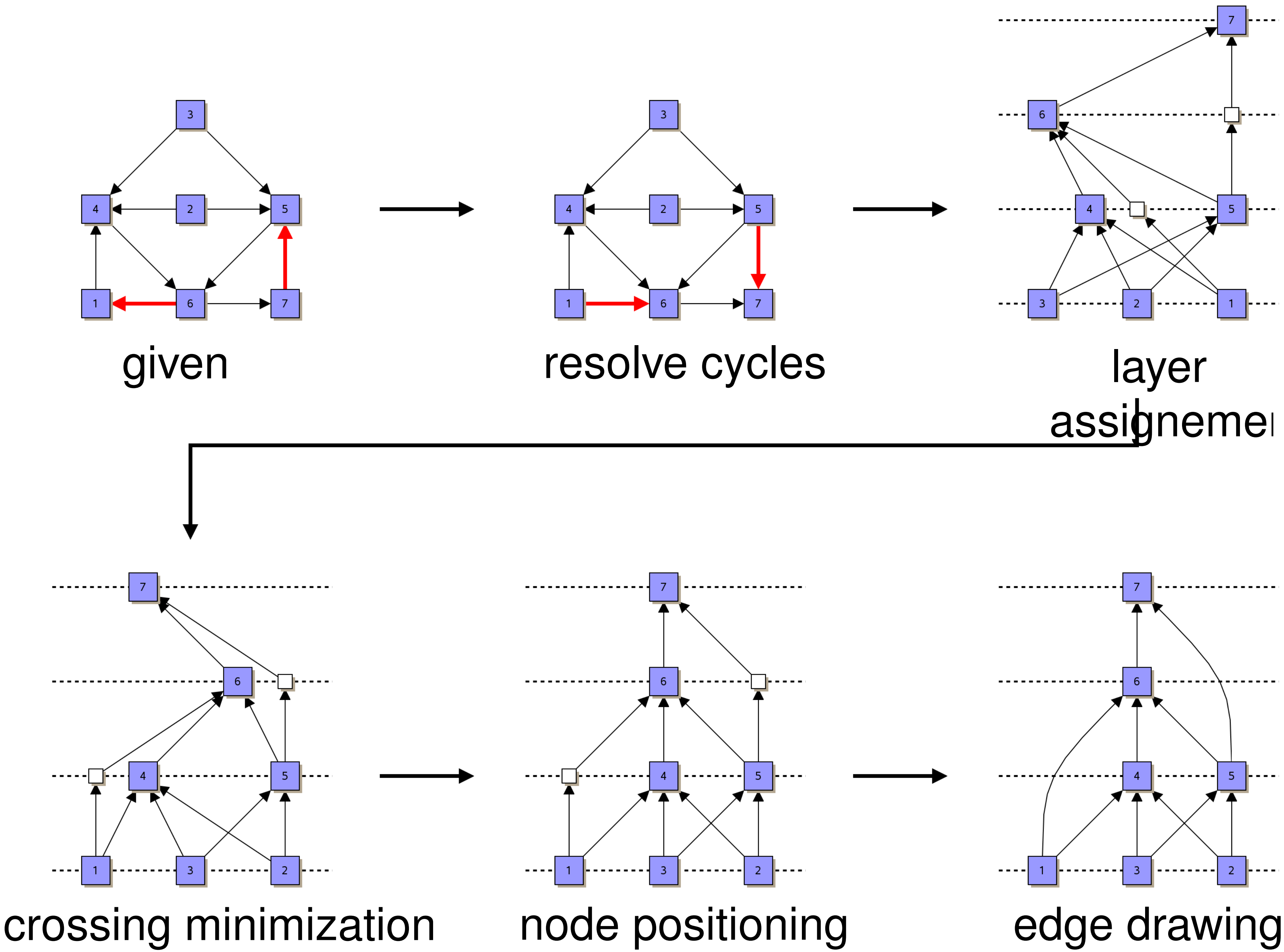

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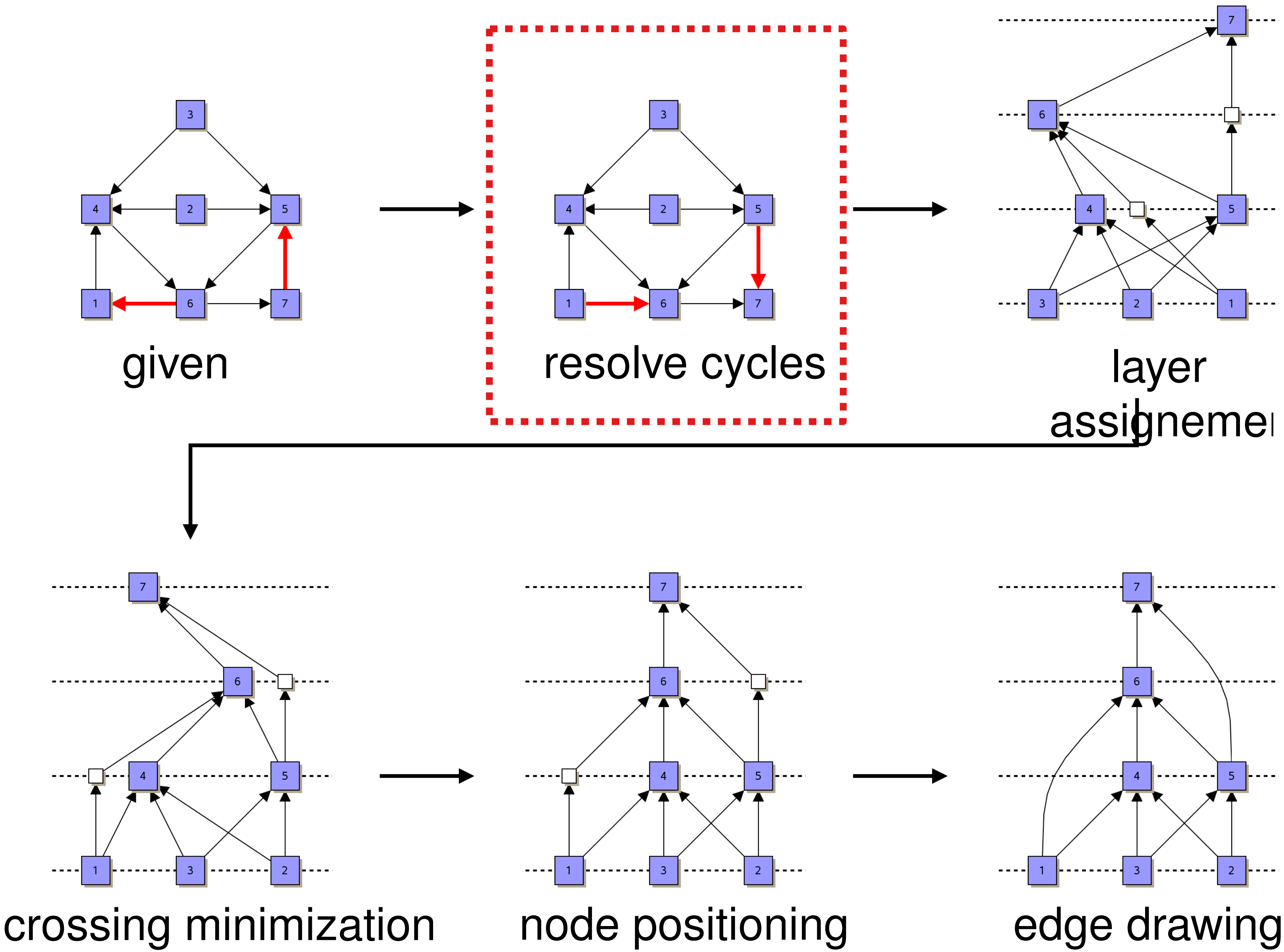

Removing Cycles

Idea:
- find maximum acyclic subgraph
- inverse the directions of the other edges

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Given: directed graph $D = (V, A)$

Find: acyclic subgraph $D' = (V, A')$ with maximum $|A'|$

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MINIMUM FEEDBACK ARC SET (FAS)

Given: directed graph $D = (V, A)$

Find: $A_f \subset A$, with $D_f = (V, A \setminus A_f)$ acyclic with minimum $|A_f|$

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Question: Is **FS** also a **FAS**? and the opposite?

Removing Cycles

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All three problems are NP-hard!

Trivial Heuristic

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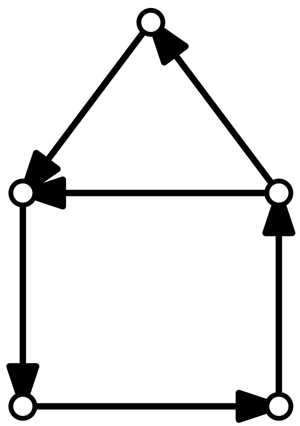

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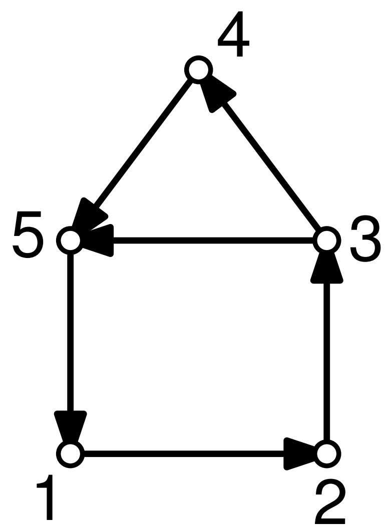

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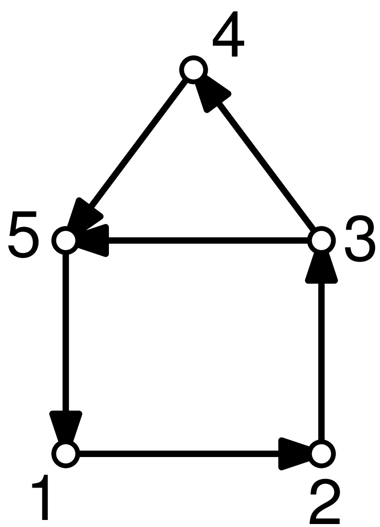

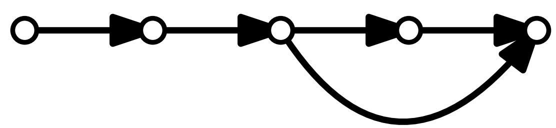

place vertices in order

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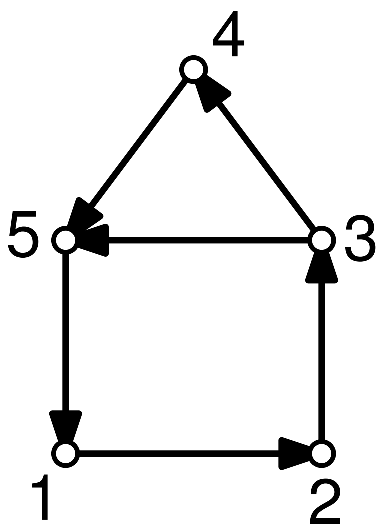

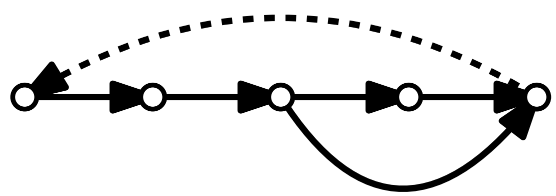

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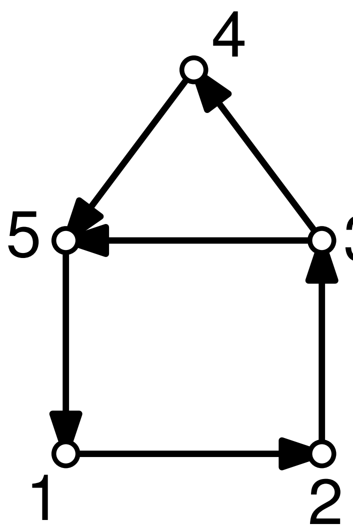

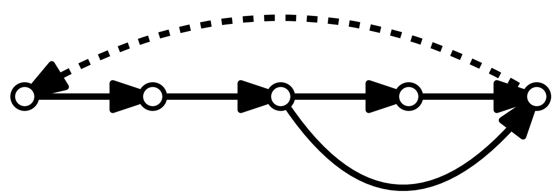

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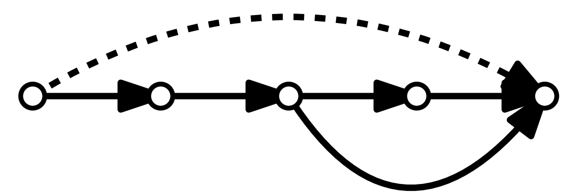

reverse all the backward edges

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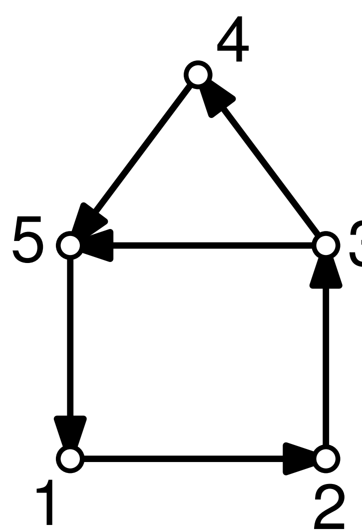

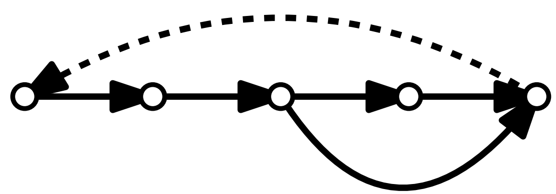

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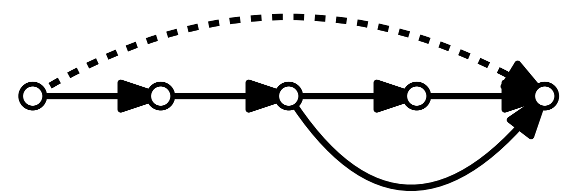


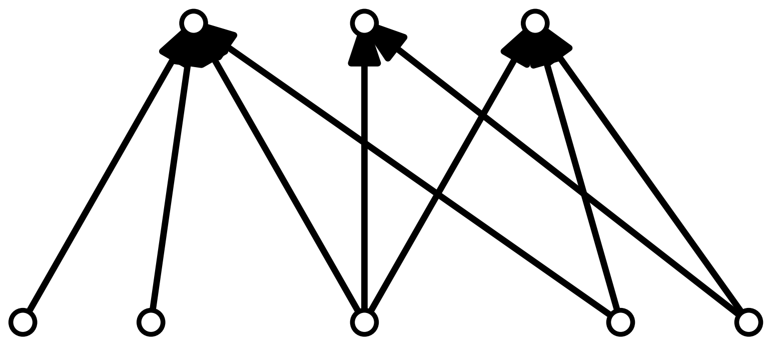
reverse all the backward edges

Benefit: Very simple to implement

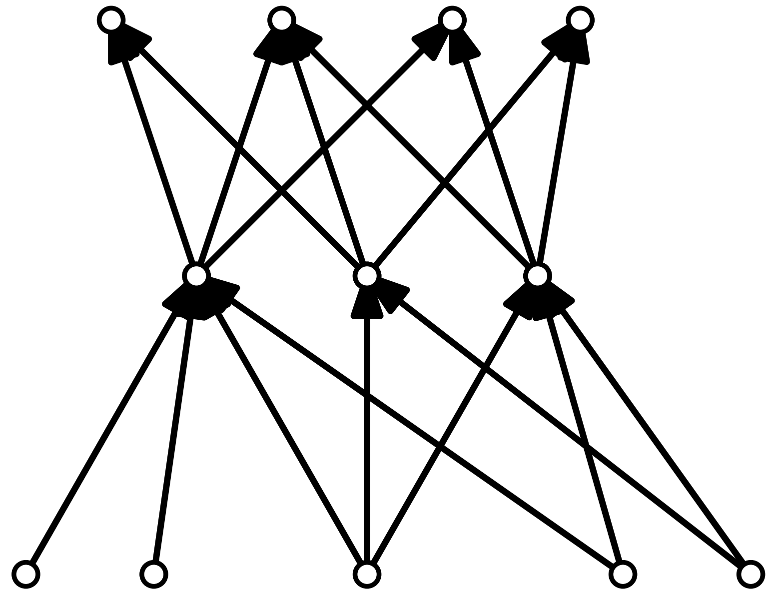
Drawback: No guarantee on the number of reversed edges

Heuristic with guarantees (Eades, Lin, Smyth 1993)

source – no incomming edge
sink – no outgoing edge



no cycles if all vertices are only sources and sinks



no cycles if graph can be eliminated by always removing either sources or sinks

$$
\begin{aligned}
N^{\rightarrow}(v) &:= \{(v,u) : (v,u) \in A\} \\
N^{\leftarrow}(v) &:= \{(u,v) : (u,v) \in A\} \\
N(v) &:= N^{\rightarrow}(v) \cup N^{\leftarrow}(v)
\end{aligned}
$$

Heuristic with guarantees (Eades, Lin, Smyth 1993)

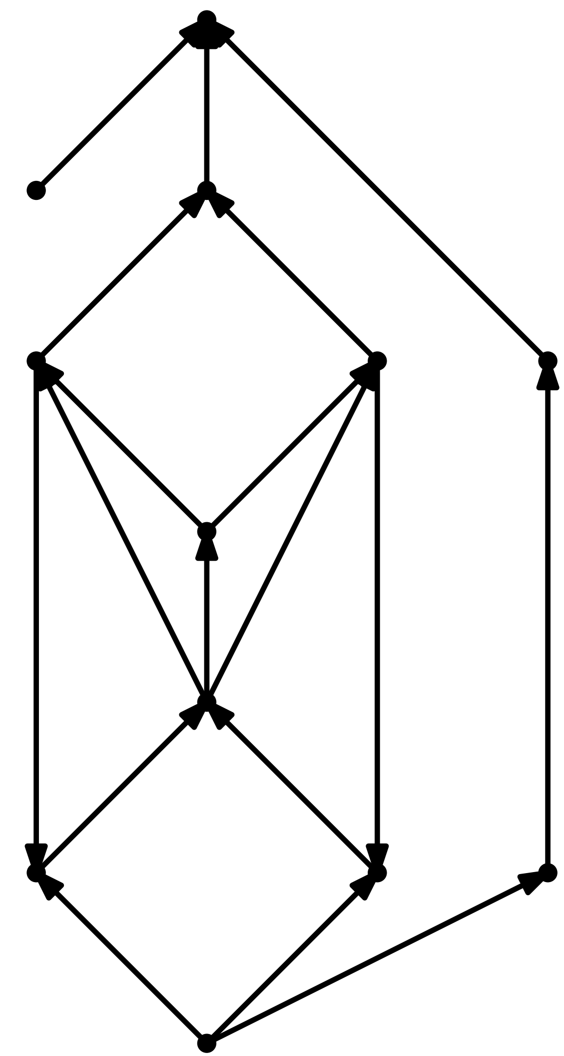
$A' := \emptyset$;

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $A' \leftarrow A' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



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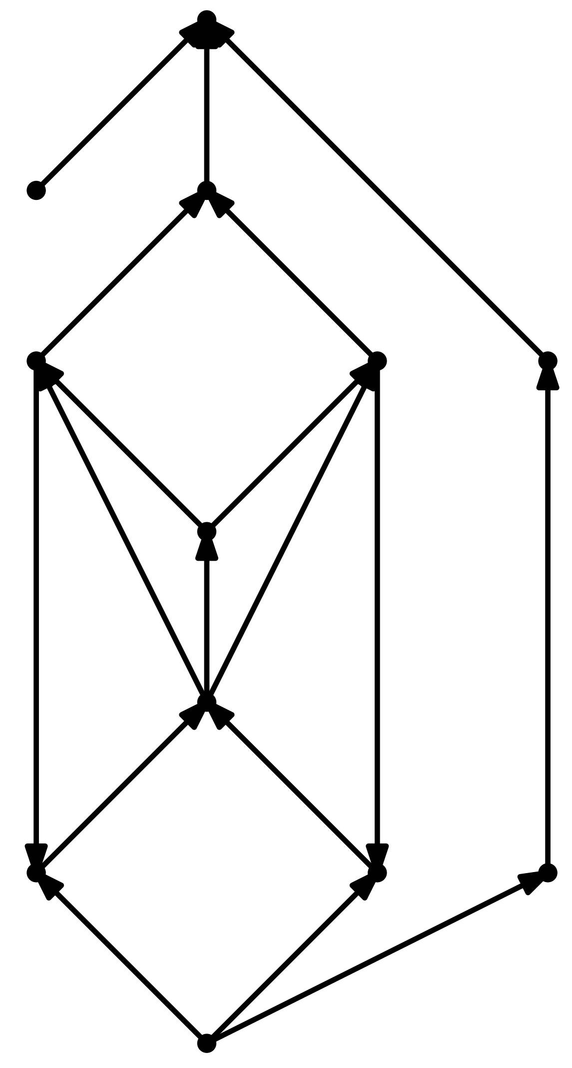

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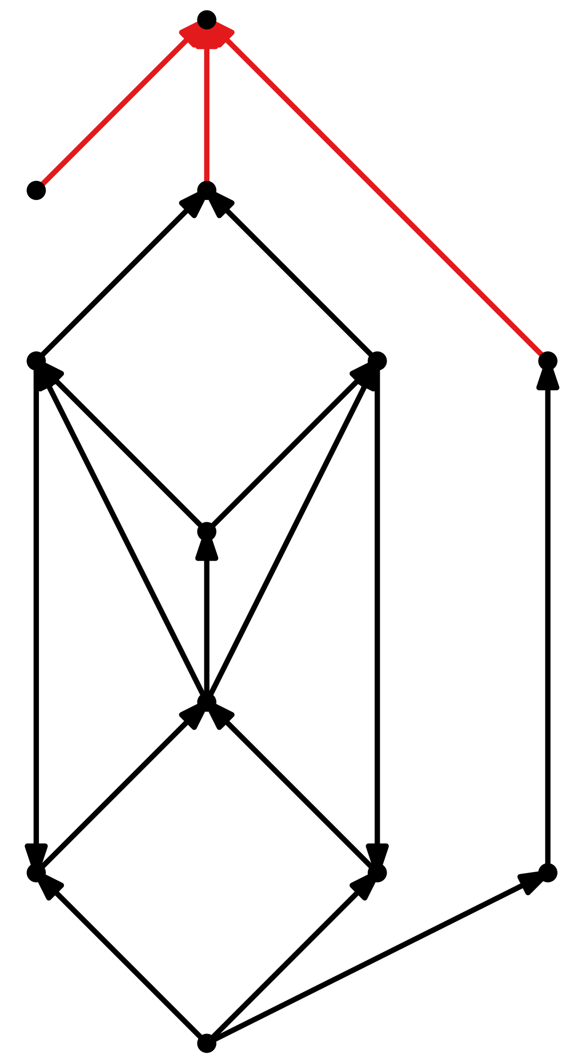
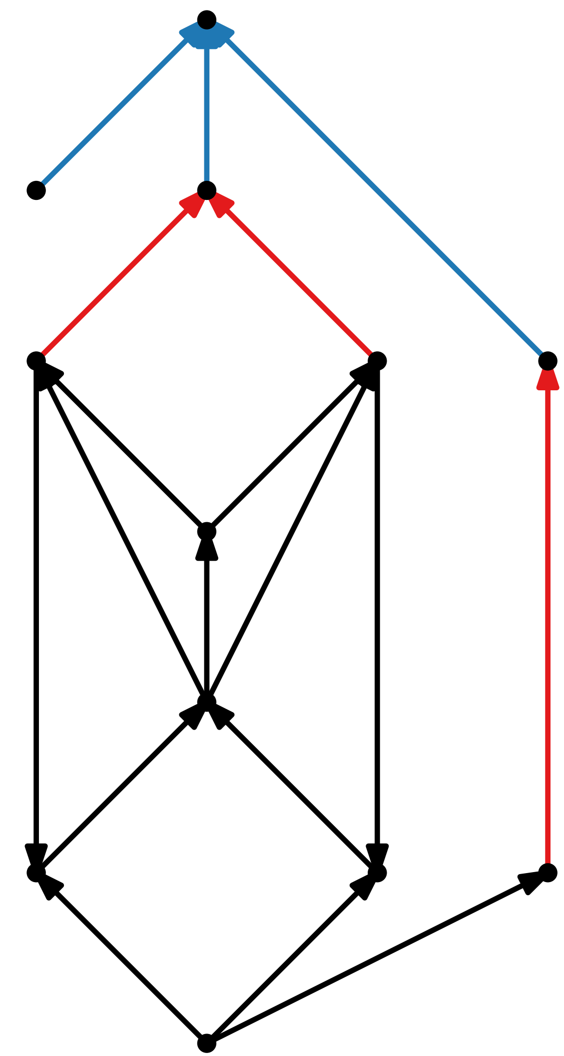


Heuristic with guarantees (Eades, Lin, Smyth 1993)

```
A' := ∅;
while V ≠ ∅ do
    while in V exists a sink v do
        A' ← A' ∪ N←(v)
        remove v and N←(v)
```



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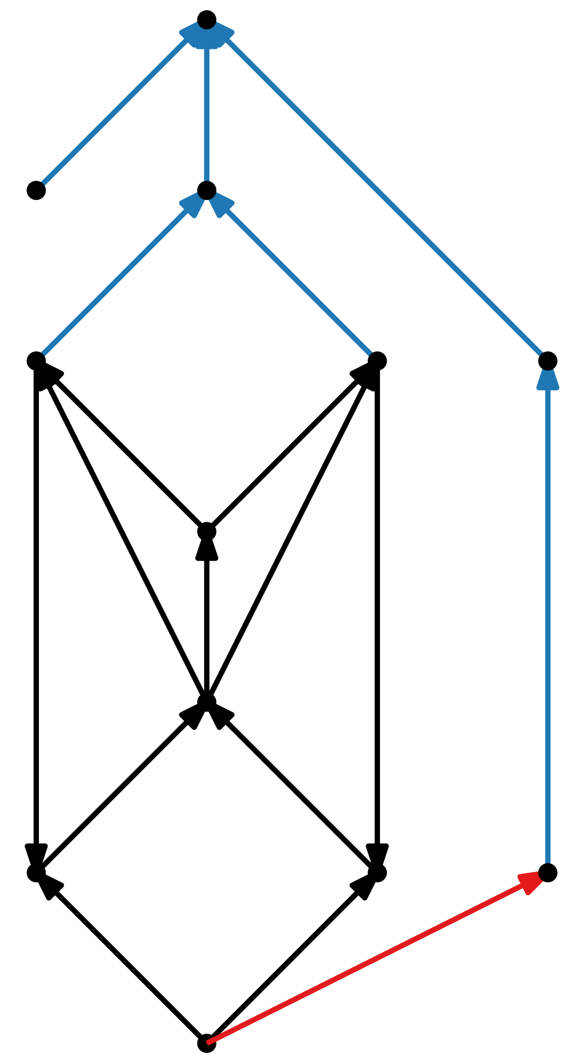

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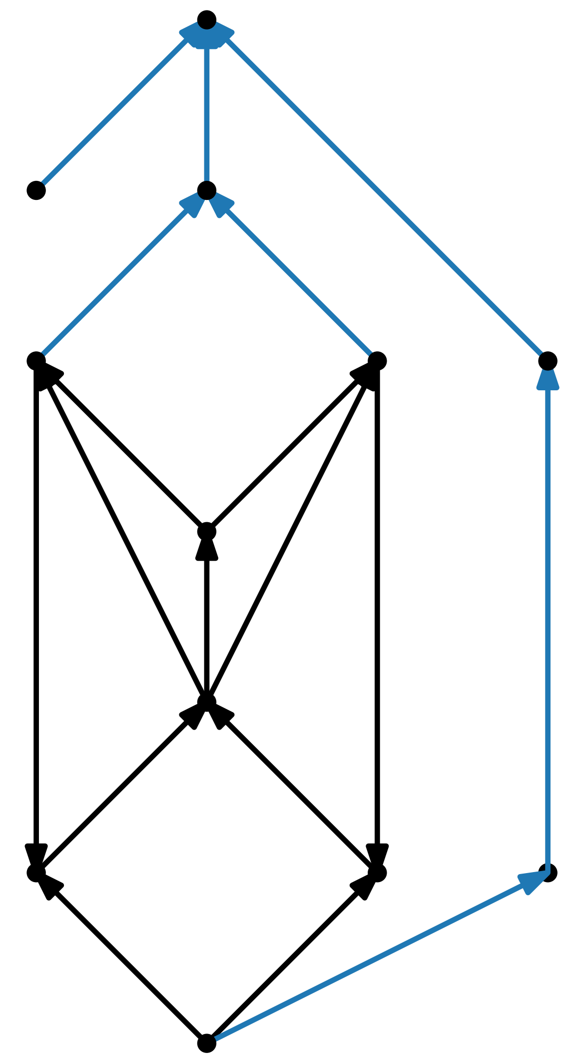
  Remove all isolated node from V



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  while in V exists a source v **do**

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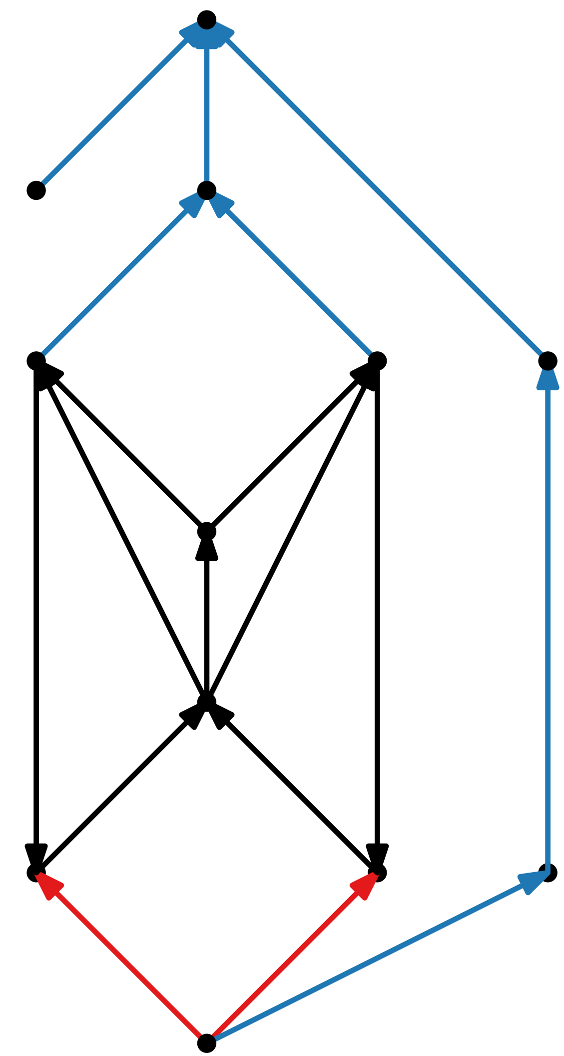

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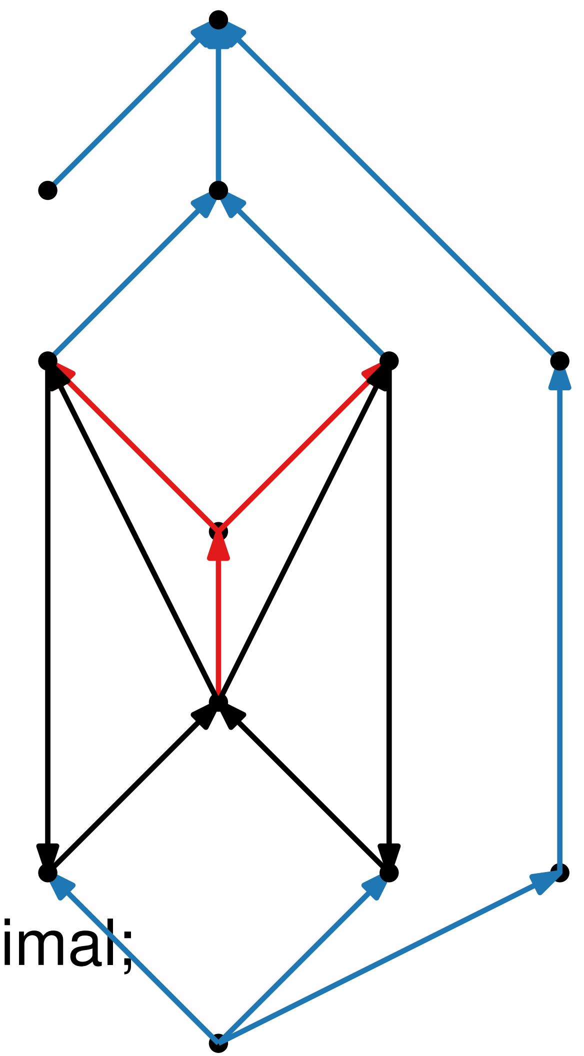
    remove v and $N^{\rightarrow}(v)$

  if $V \neq \emptyset$ **then**

    let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;

    $A' \leftarrow A' \cup N^{\rightarrow}(v)$

    remove v and $N(v)$



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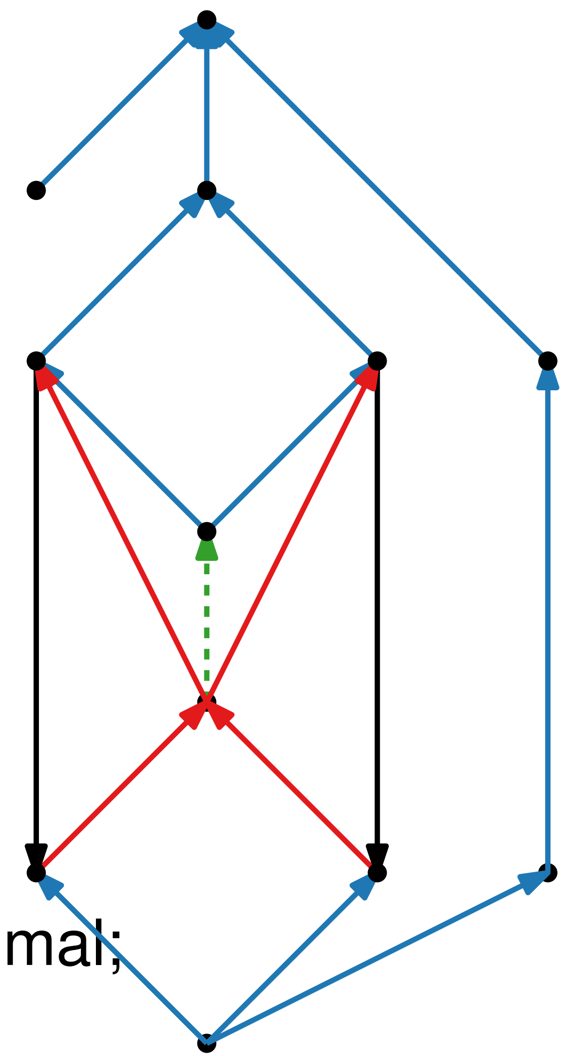

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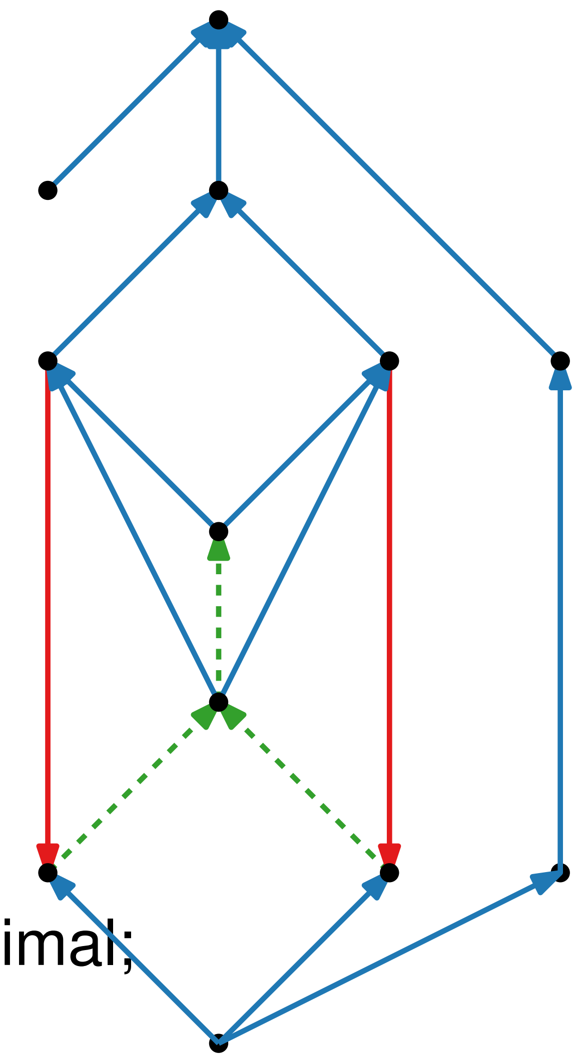

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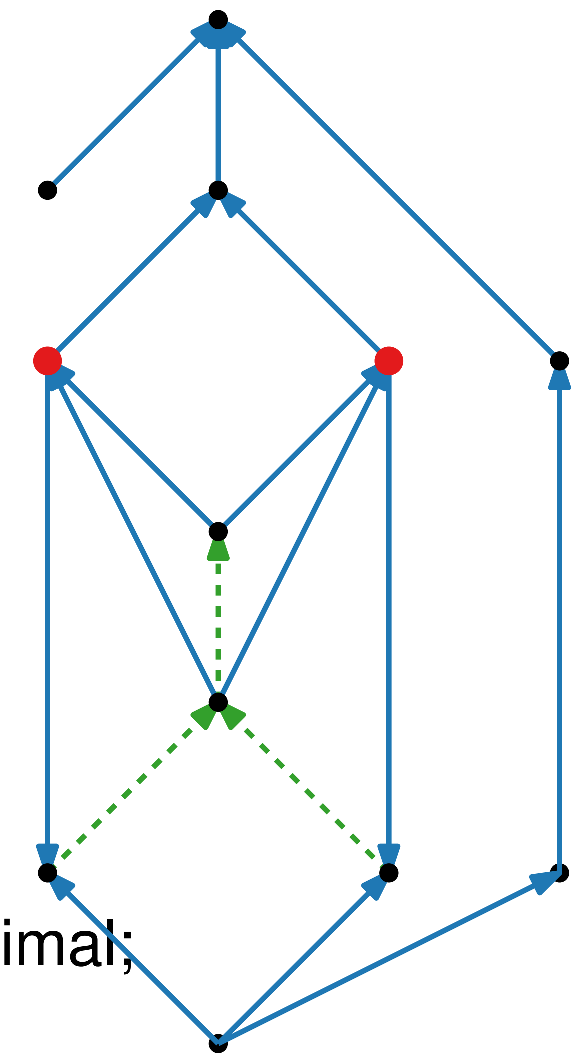

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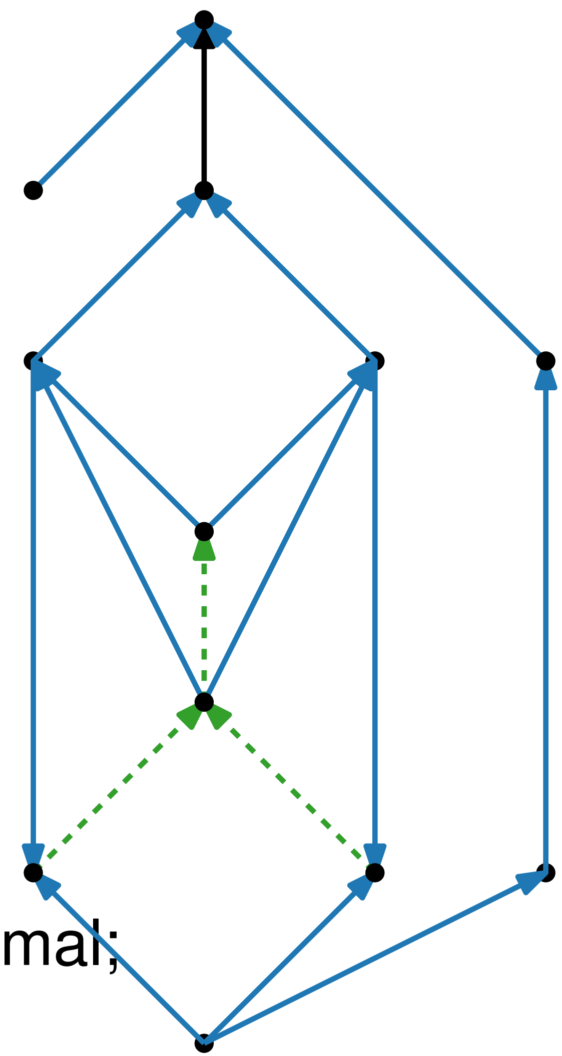

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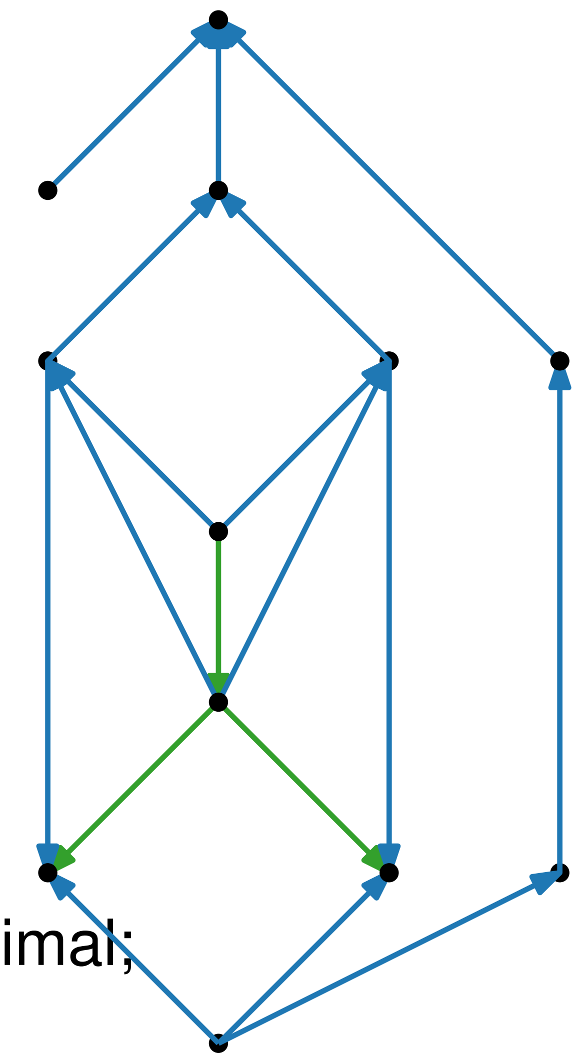

Guarantees and related work

Theorem: Let $D = (V, A)$ be a connected, directed graph without 2-cycles. Heuristic of Eades at al. computes a set of edges A' with $|A'| \geq |A|/2 + |V|/6$.
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- optimal solution integer linear programming, using branch-and-cut technique (Grötschel et al. 1985)

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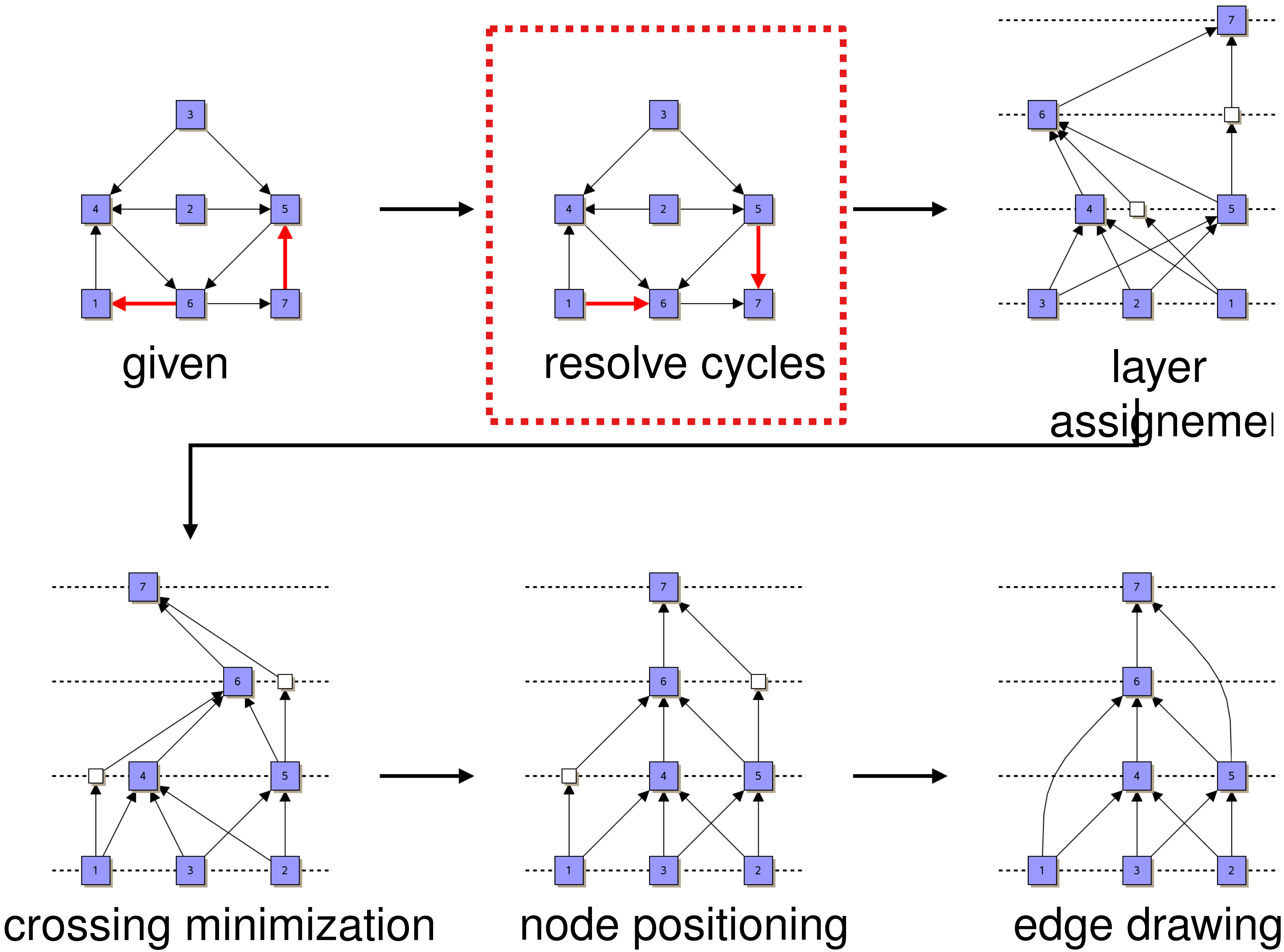

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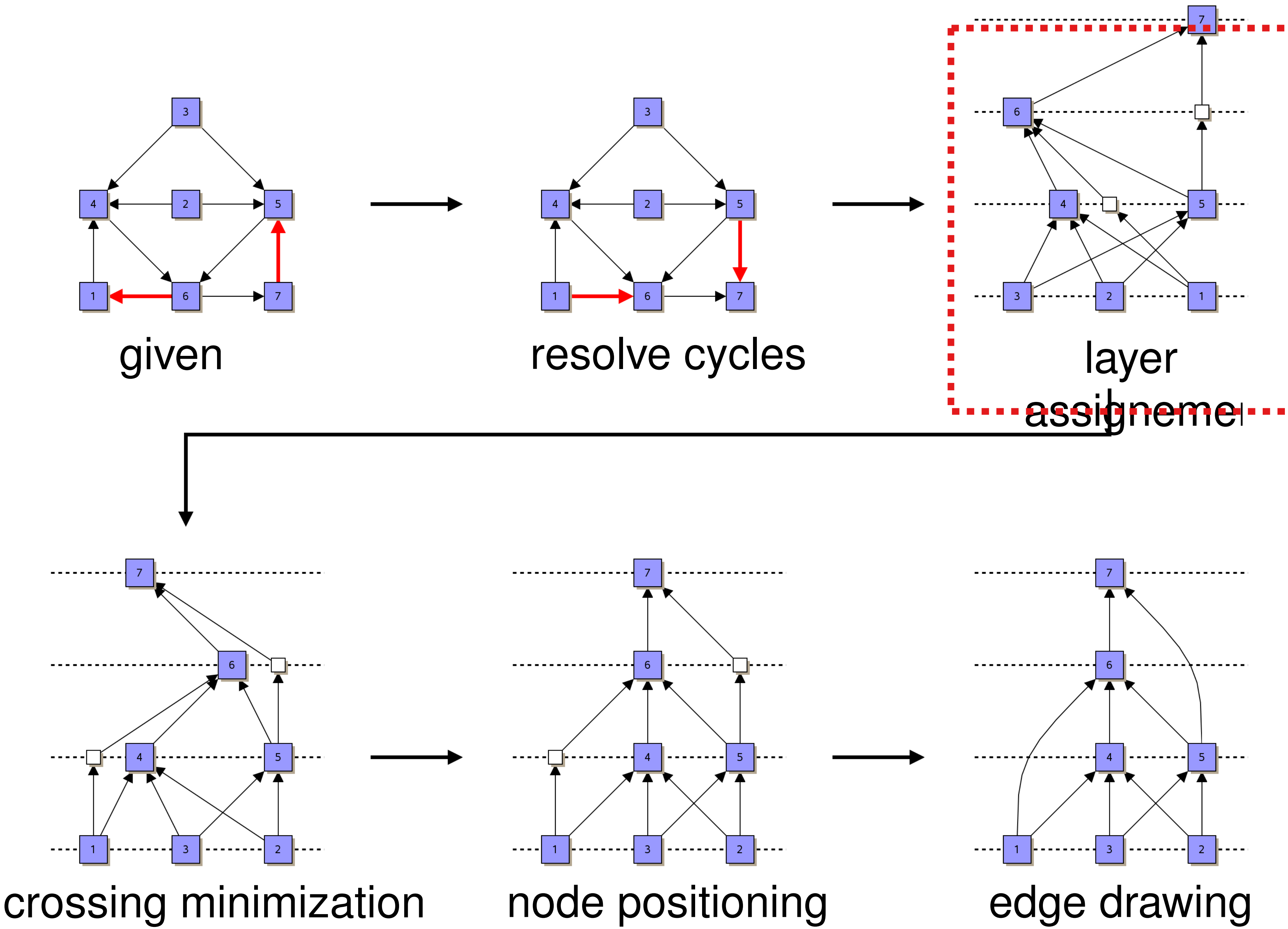

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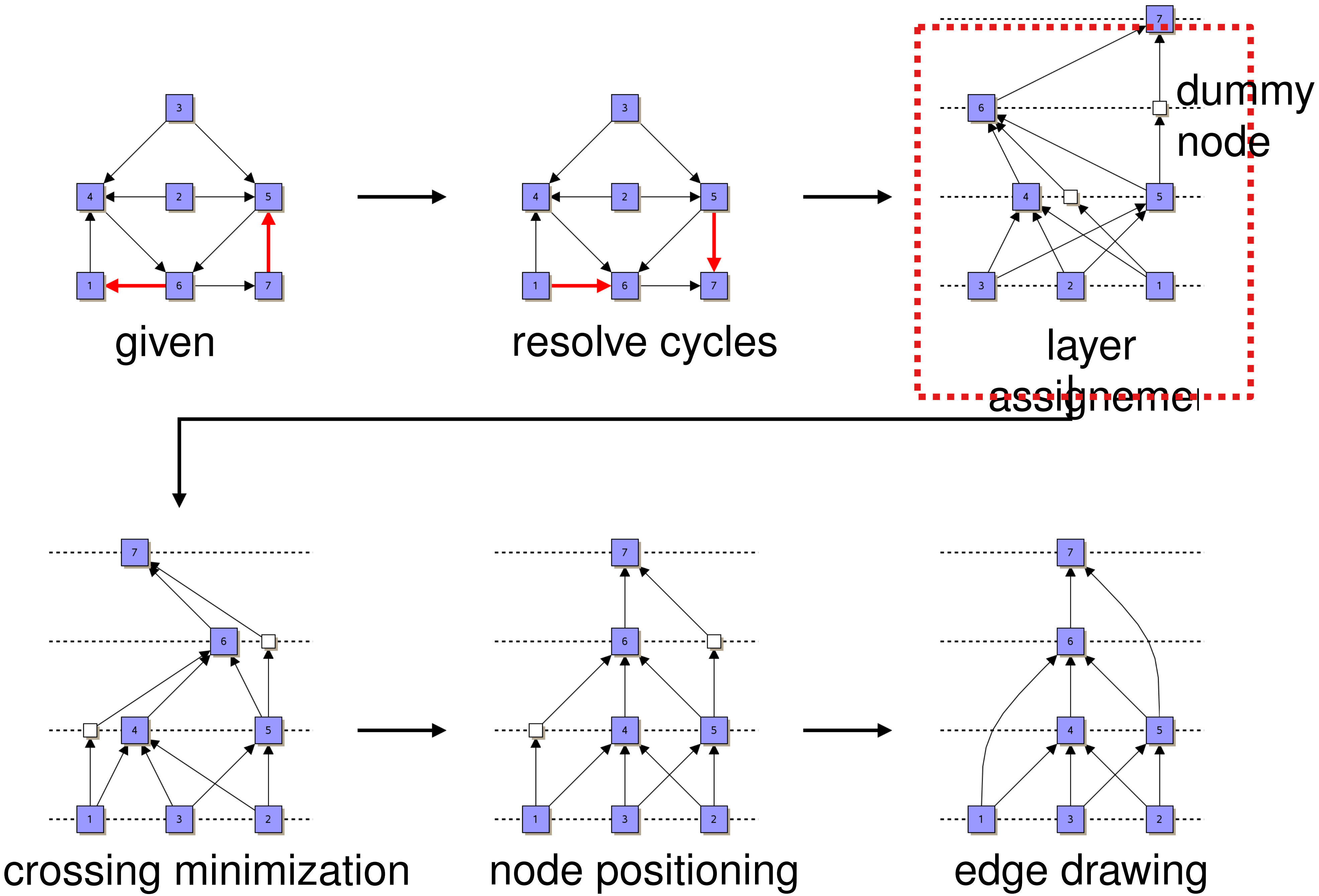

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Given.: directed acyclic graph (DAG) $D = (V, A)$

Find: Partition the vertex set V into disjoint subsets (**layers**) $L_1, \ldots, L_h$ s.t. $(u, v) \in A,\ u \in L_i,\ v \in L_j \Rightarrow i < j$

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Think for a minute and then share

What could we optimize when doing the layer assignment?

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Idea: assign each node *v* to the layer L_i, where *i* is the length of the longest simple path from a source to *v*

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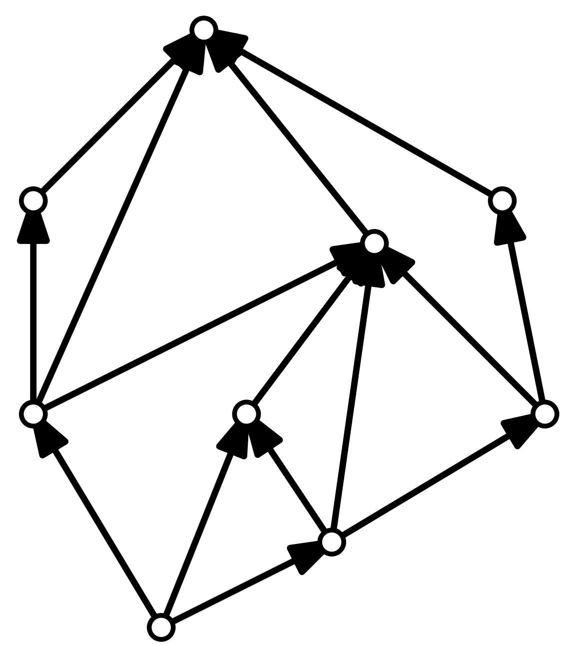

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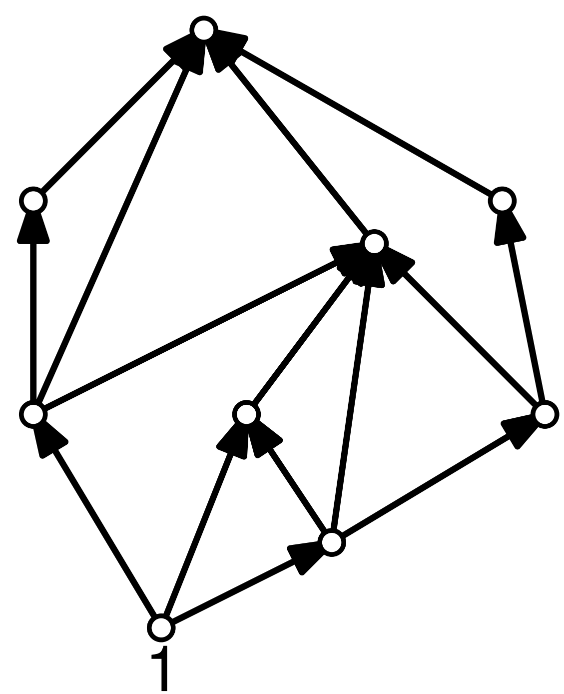

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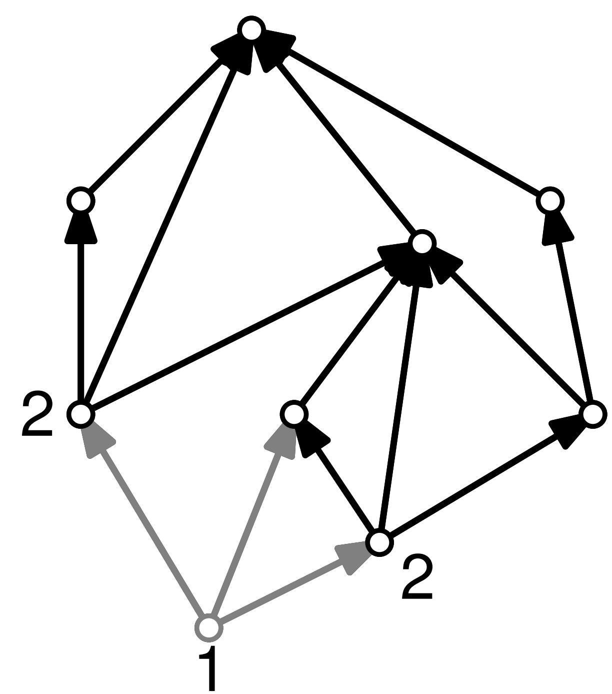

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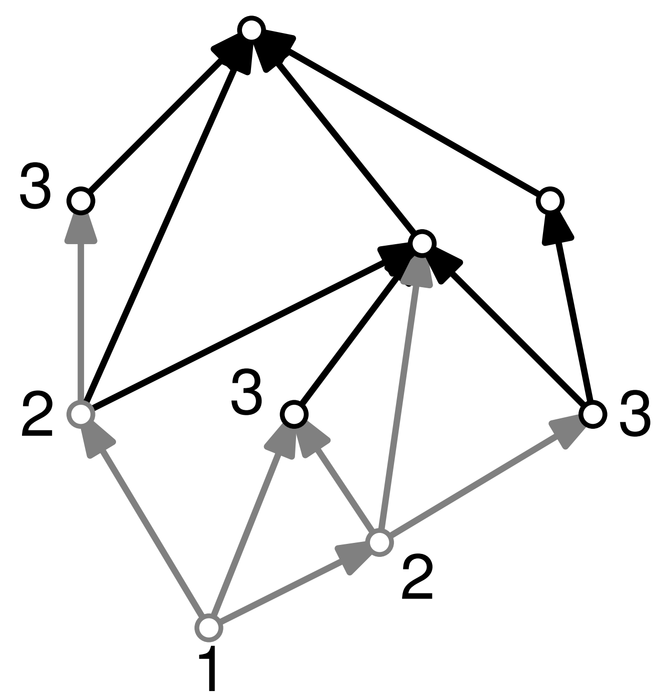

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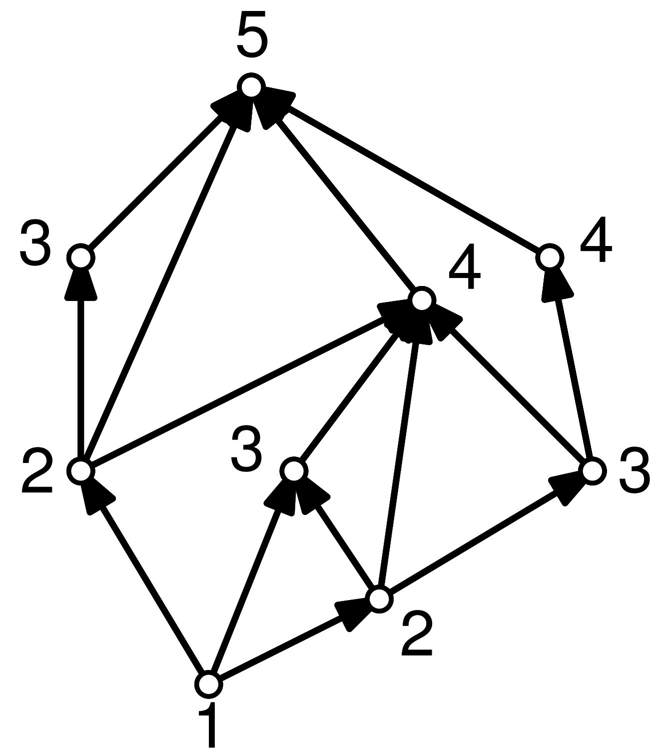

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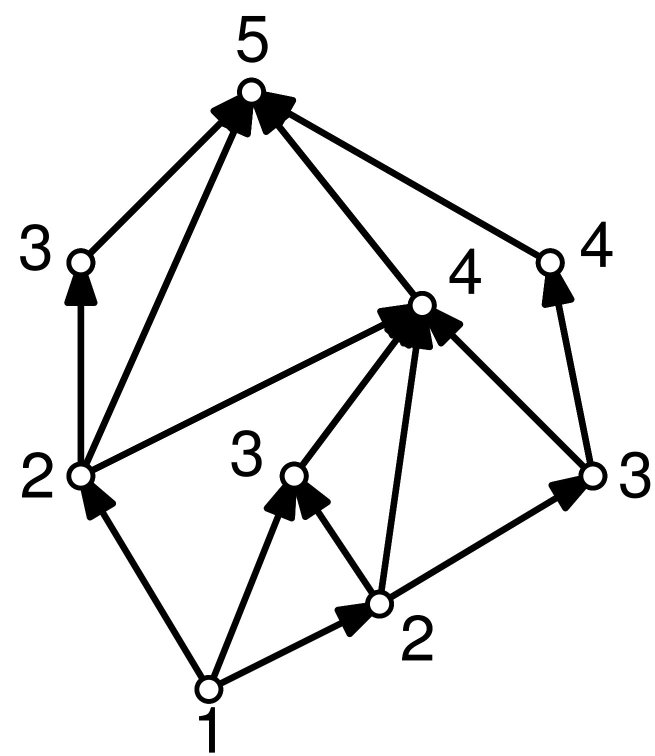

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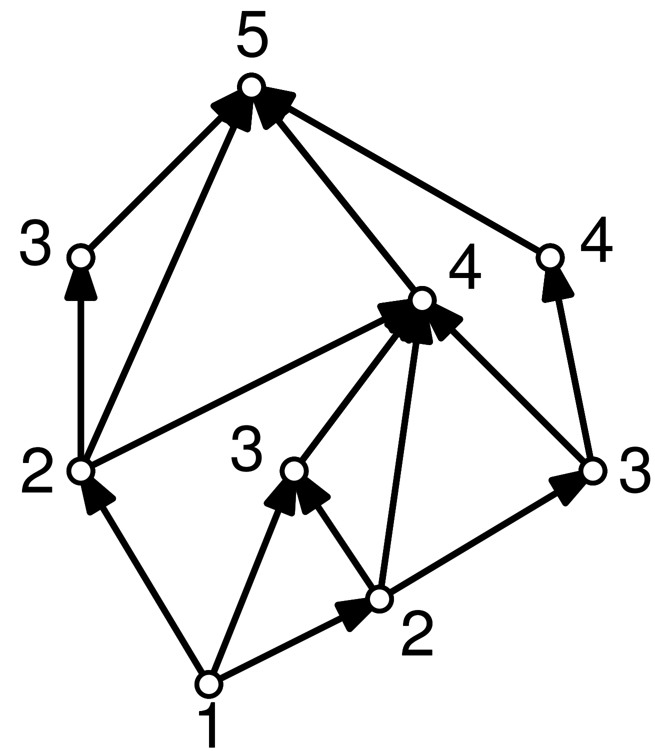

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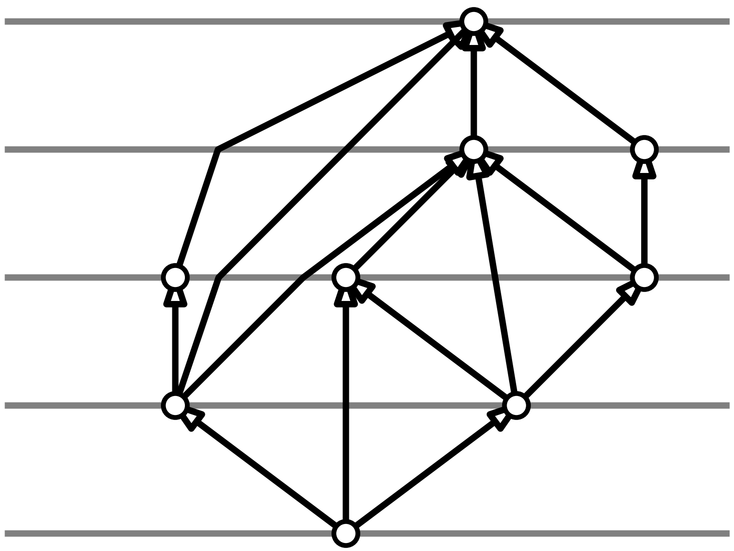

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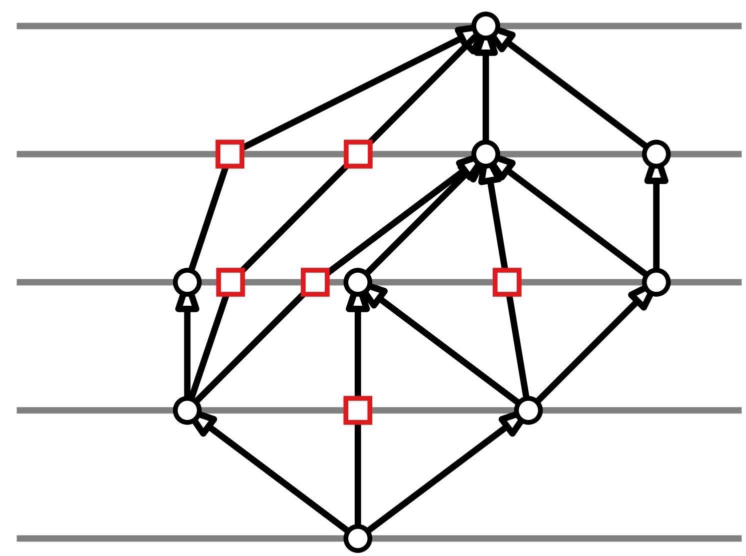

edges that span more than two layers get subdivided by dummy vertices

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Other optimization criteria:

Total edge length: $\sum_{(u,v)\in A}(y(v) - y(u))$ – with integer linear program (polynomial time) [Gansner et al 93]

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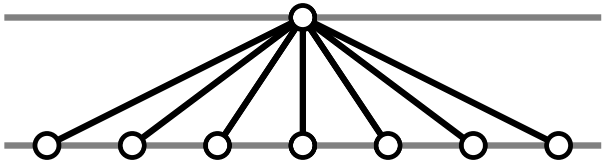

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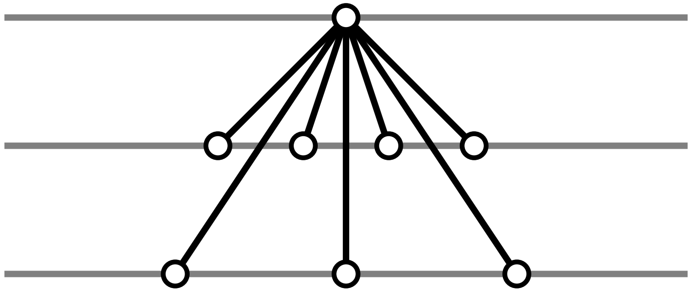

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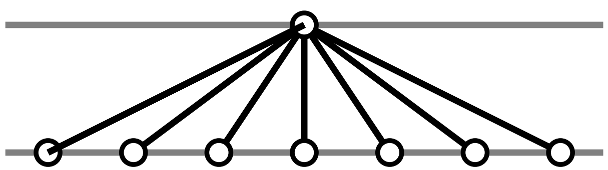

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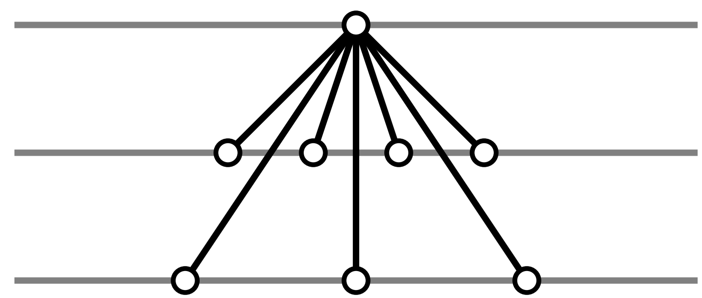

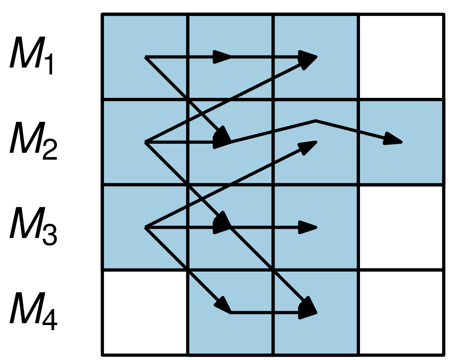

Reduction to scheduling problem – NP-hard but there is a $2 - \frac{1}{B}$-approximation algorithm

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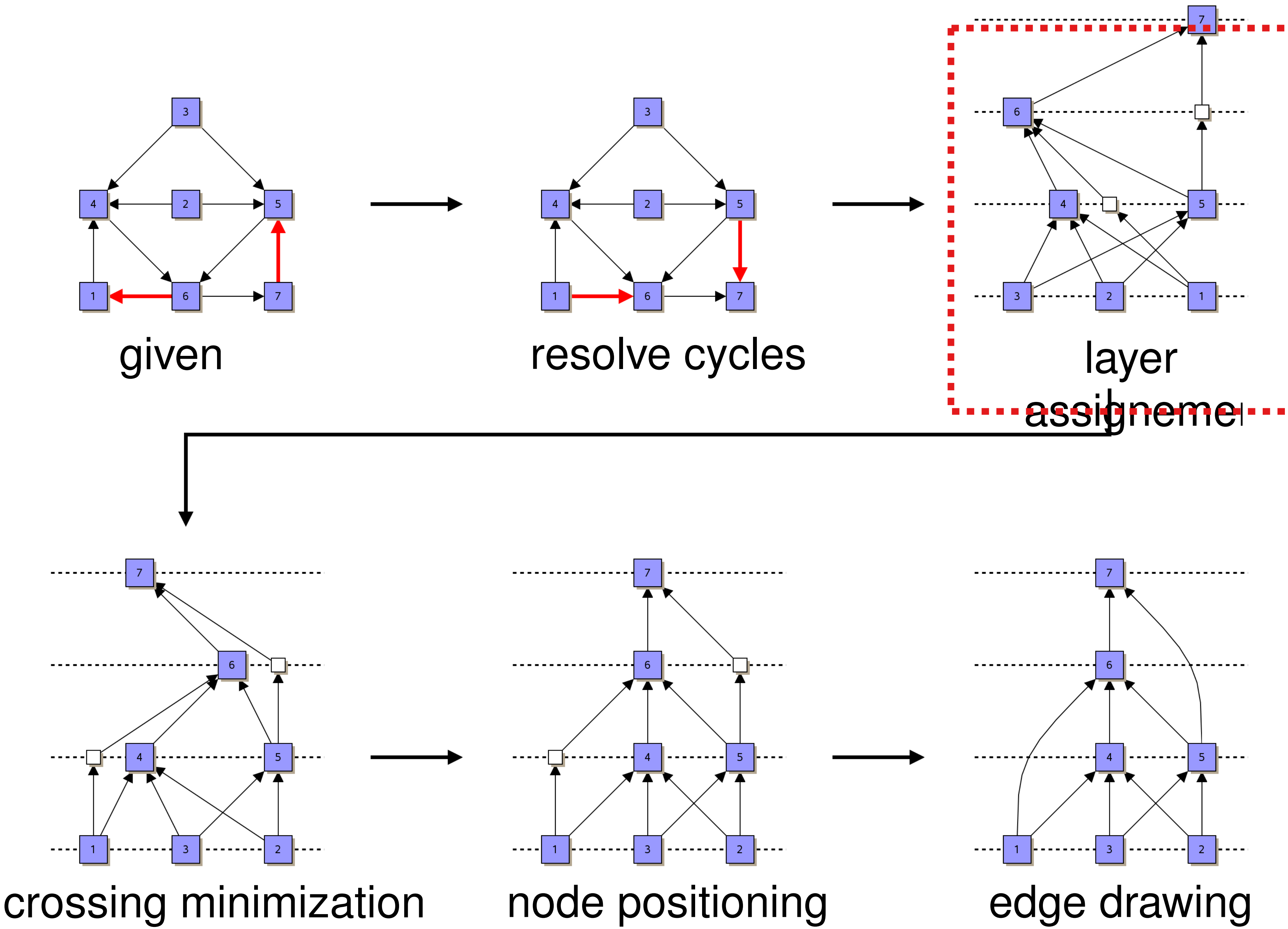

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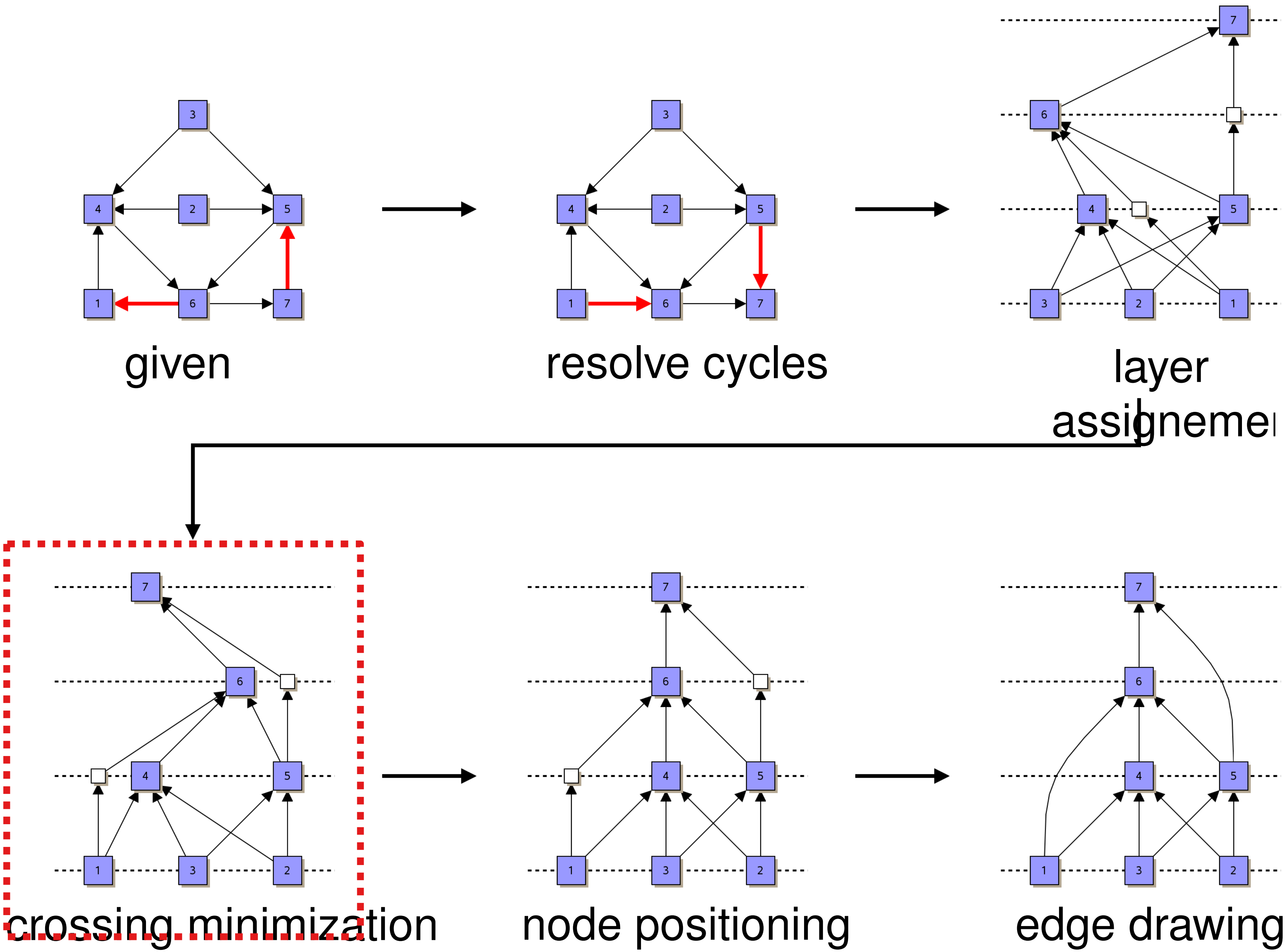

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Find: Order of the nodes on each layer, so that the number of crossing is minimized

- Number of crossings only depends on the order and not on exact coordinates
- Problem is NP-hard even for two layers (BIPARTITE CROSSING NUMBER [Garey, Johnson '83])
- No approach over several layers simultaneously
- Usually iterative optimization based on approaches for two adjacent layers (*one-sided crossing minimization*)

One-sided Crossing Minimization (OSCM)

Given: 2-Layered-Graph $G = (L_1, L_2, E)$ and ordering of the nodes O_1 of L_1

Find: Node ordering O_2 of L_2, such that the number of crossing among E is minumum

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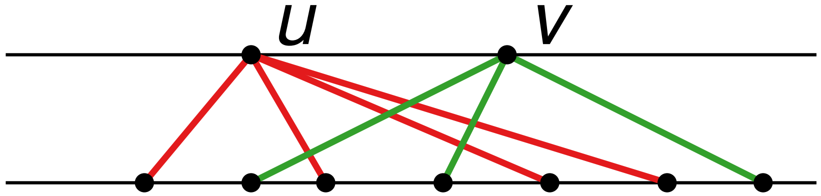

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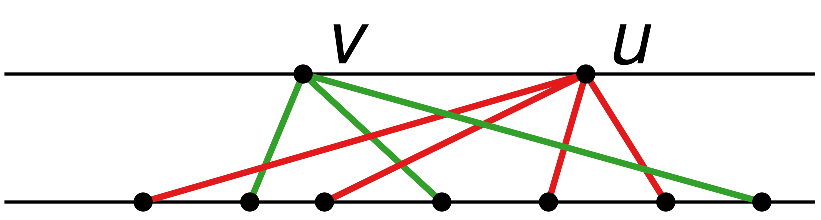

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Def: Crossing number of G with orders x_1 and x_2 for L_1 and L_2 is denoted by $\mathrm{cr}(G, O_1, O_2)$;
for fixed O_1 then $\mathrm{opt}(G, O_1) = \min_{O_2} \mathrm{cr}(G, O_1, O_2)$

- It holds that $\mathrm{cr}(G, O_1, O_2) = \sum_{O_2(u) < O_2(v)} c_{uv}$ - gives a way to compute $\mathrm{cr}(G, O_1, O_2)$ in $O(m^2)$, where m-number of edges

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- $\mathrm{cr}(G, O_1, O_2)$ can be computed in $O(m + C)$ time, where C-number of crossings [Six and Tollis 2006] $\rightarrow$ reduce to counting crossings in a circular drawing

Further Properties

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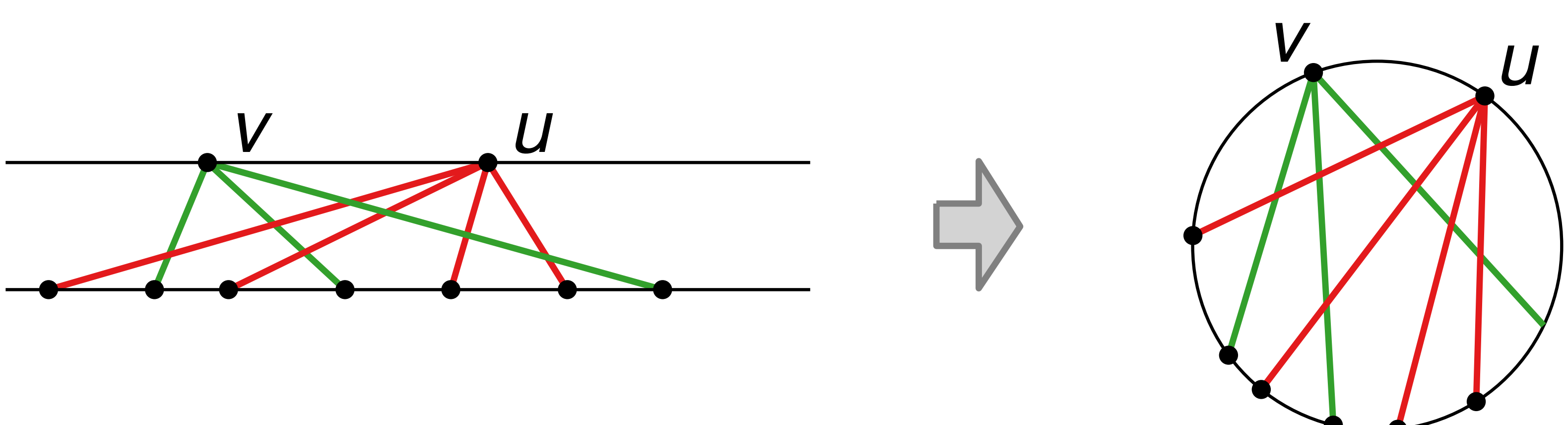

Iterative Crossing Minimization

Let $G = (V, E)$ be a DAG with layers $L_1, \dots, L_h$.

(1) compute a random ordering O_1 for layer L_1

(2) for $i = 1, \dots, h-1$ consider layers L_i and L_{i+1} and minimize $\mathrm{cr}(G, O_i, O_{i+1})$ with fixed O_i ($\rightarrow$ **OSCM**)

(3) for $i = h-1, \dots, 1$ consider layers L_{i+1} and L_i and minimize $\mathrm{cr}(G, O_i, O_{i+1})$ with fixed O_{i+1} ($\rightarrow$ **OSCM**)

(4) repeat (2) and (3) until no further improvement happens

(5) repeat steps (1)–(4) with another O_1

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Theorem 1: The One-Sided Crossing Minimization (OSCM) problem is NP-hard [Eades, Wormald 1994].

Algorithms for OSCM

Heuristics:

- Barycenter [Sugiyama et al, 81]
- Median [Eades and Wormald, 94]

Exact:

- ILP Model [Juenger and Mutzel, 97]

...and many more...

Barycenter Heuristic (Sugiyama, Tagawa, Toda 1981)

Idea: few crossing when nodes are close to their neighbours

- set

$$o_2(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} o_1(v)$$

- in case of equality introduce tiny gap

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Properties:

- trivial implementation
- quick (exactly?)
- usually very good results
- finds optimum if $\mathrm{opt}(G, o_1) = 0$
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Median-Heuristic (Eades, Wormald 1994)

Idea: use the median of the coordinates of neighbours

- for a node $v \in L_2$ with neighbours $v_1, \ldots, v_k$ set $o_2(v) = \text{med}(v) = o_1(v_{\lceil k/2 \rceil})$ and $o_2(v) = 0$ if $N(v) = \emptyset$
- if $o_2(u) = o_2(v)$ and u, v have different degree parity, place the node with odd degree to the left
- if $o_2(u) = o_2(v)$ and u, v have the same degree parity, place an arbitrary of them to the left
- Runs in time $O(|E|)$

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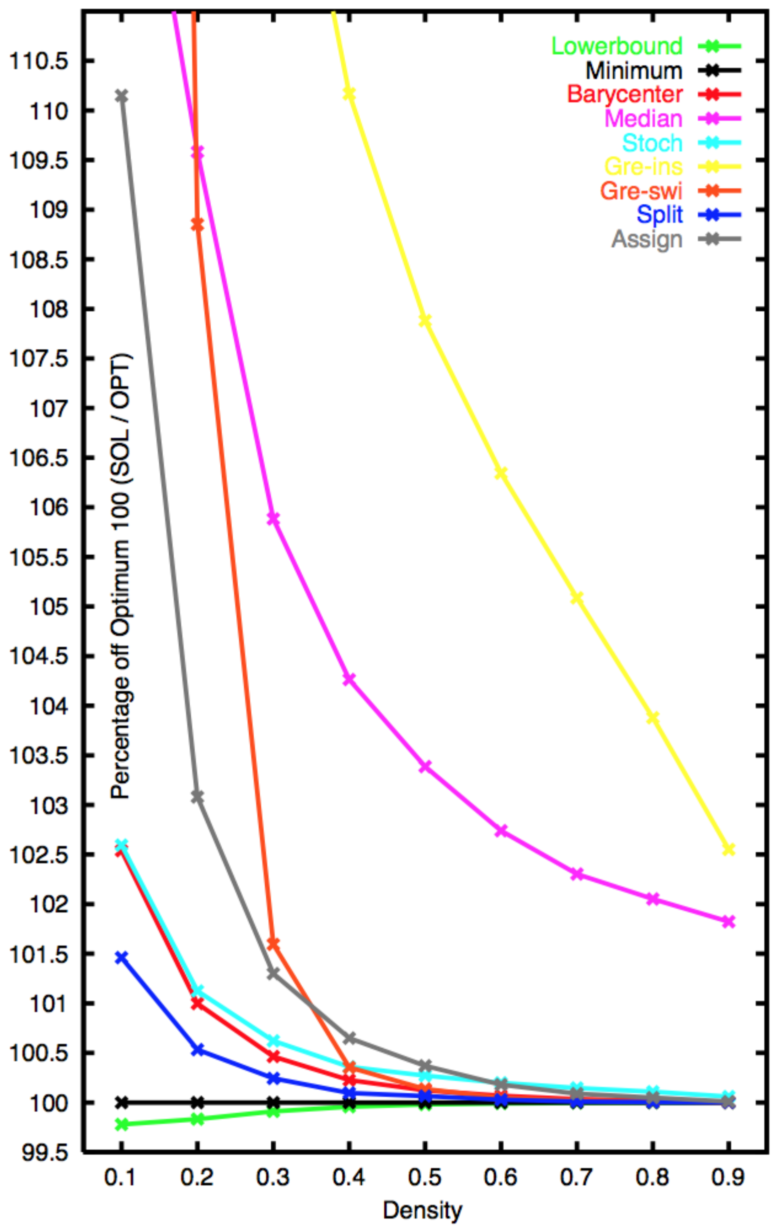
- if $o_2(u) = o_2(v)$ and u, v have the same degree parity, place an arbitrary of them to the left
- Runs in time $O(|E|)$

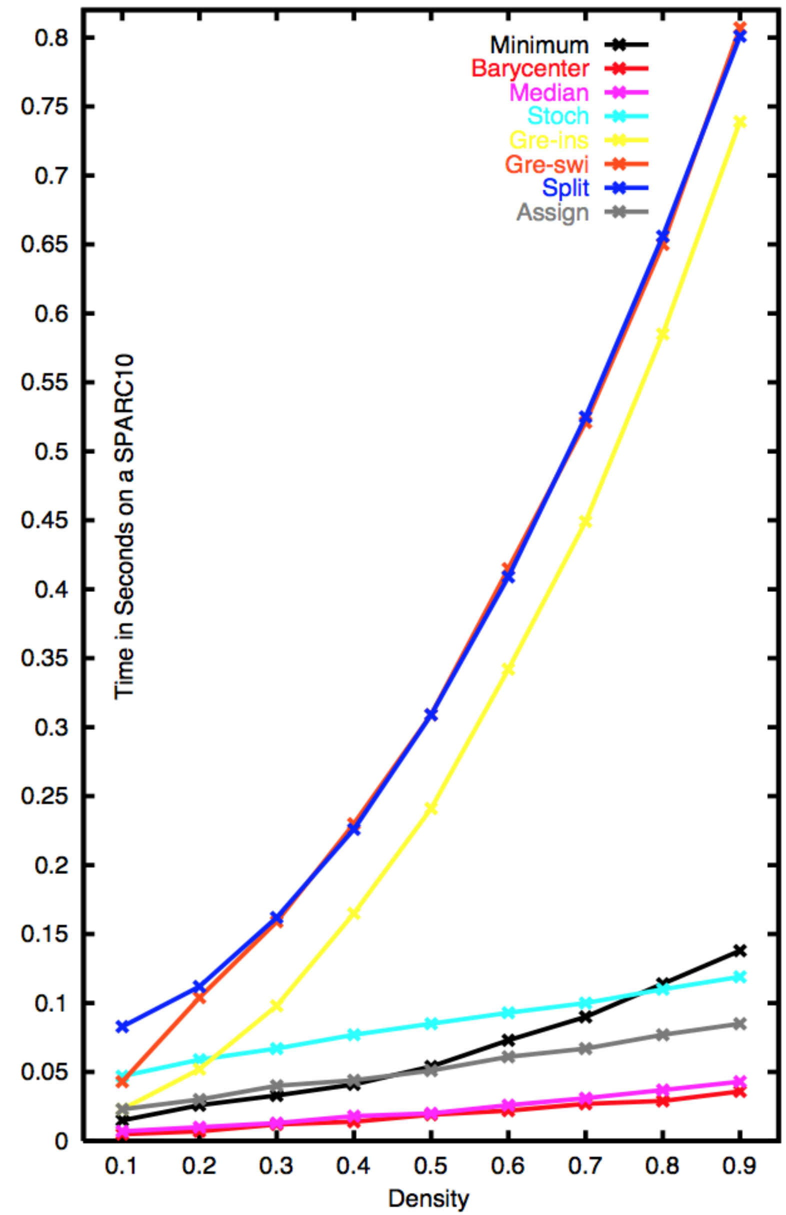
Properties:

- trivial implementation
- fast
- mostly good performance
- finds optimum when $\text{opt}(G, o_1) = 0$
- **Factor-3 Approximation**: $\text{med}(G, o_1) \leq 3\,\text{opt}(G, o_1)$

Experimental Evaluation (Jünger, Mutzel 1997)

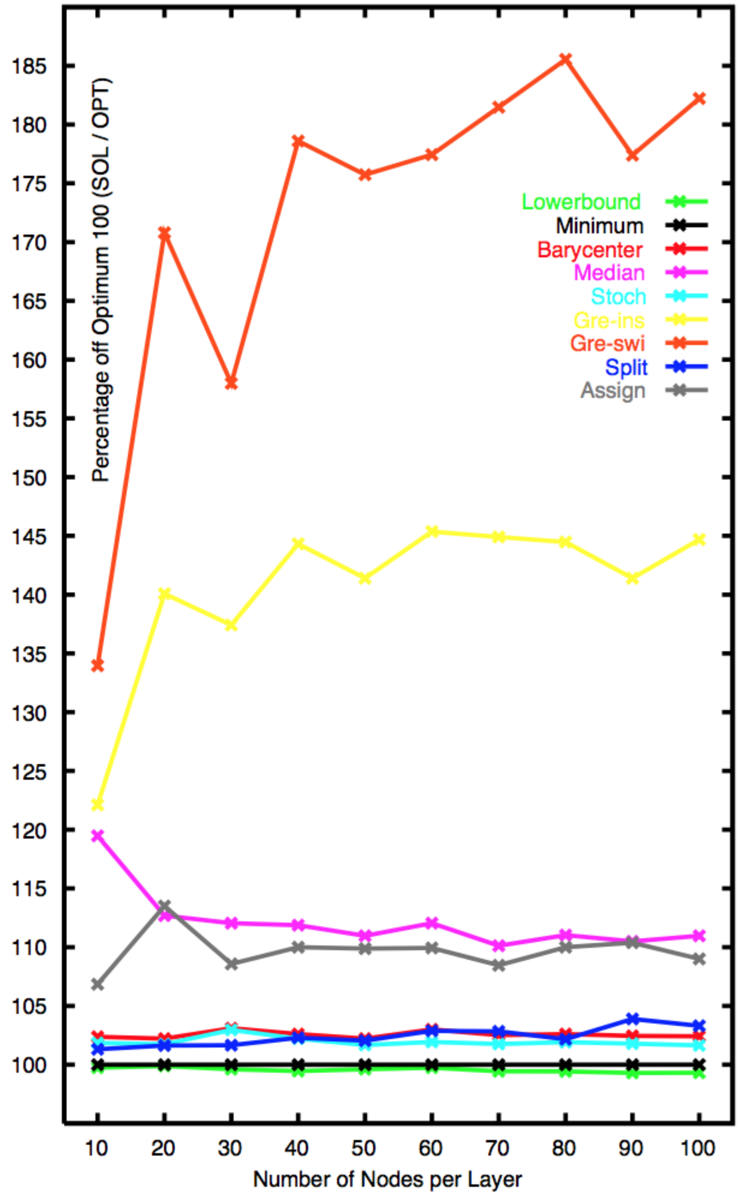


Results for 100 instances on 20 + 20 nodes with increasing density

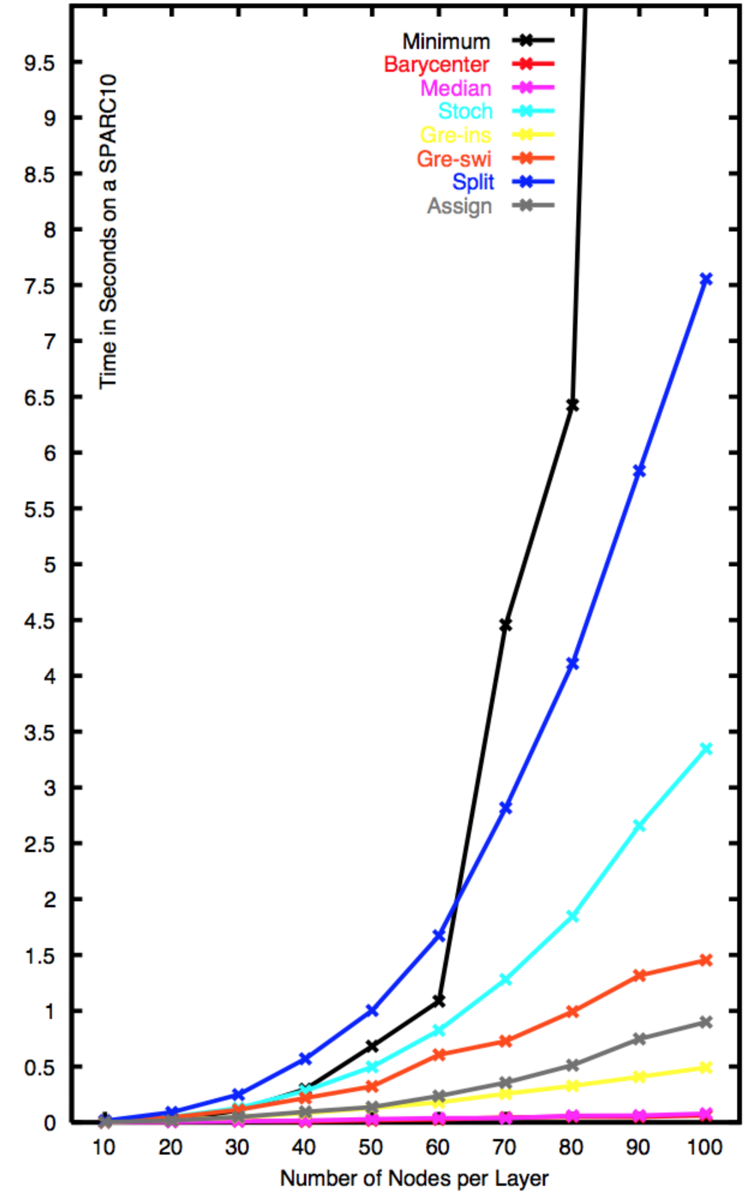


Time for 100 instances on 20 + 20 nodes with increasing density

Experimental Evaluation (Jünger, Mutzel 1997)

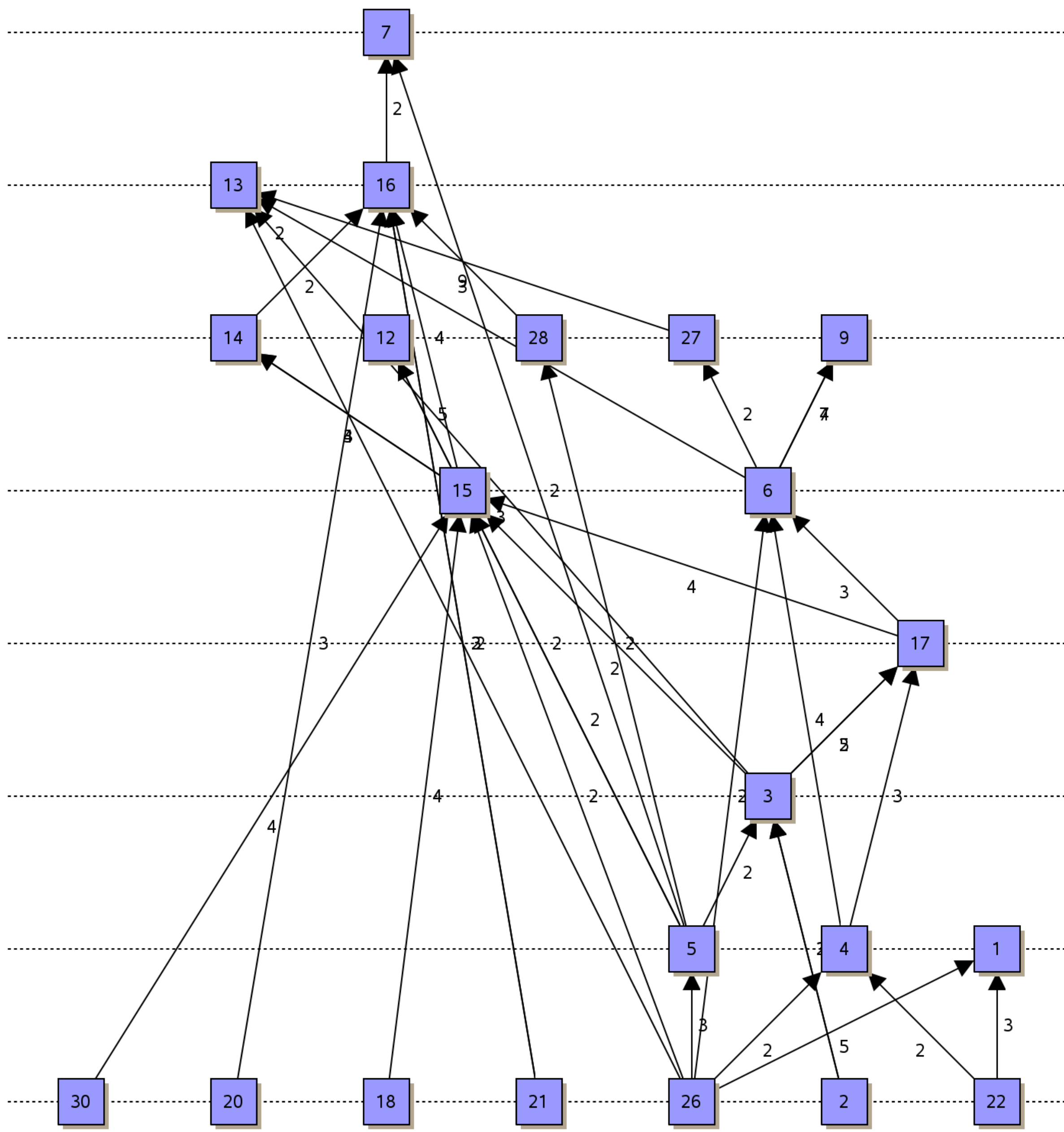


Results for 10 instances of sparse graphs with increasing size



Time for 10 instances of sparse graphs with increasing size

Example with Barycenter



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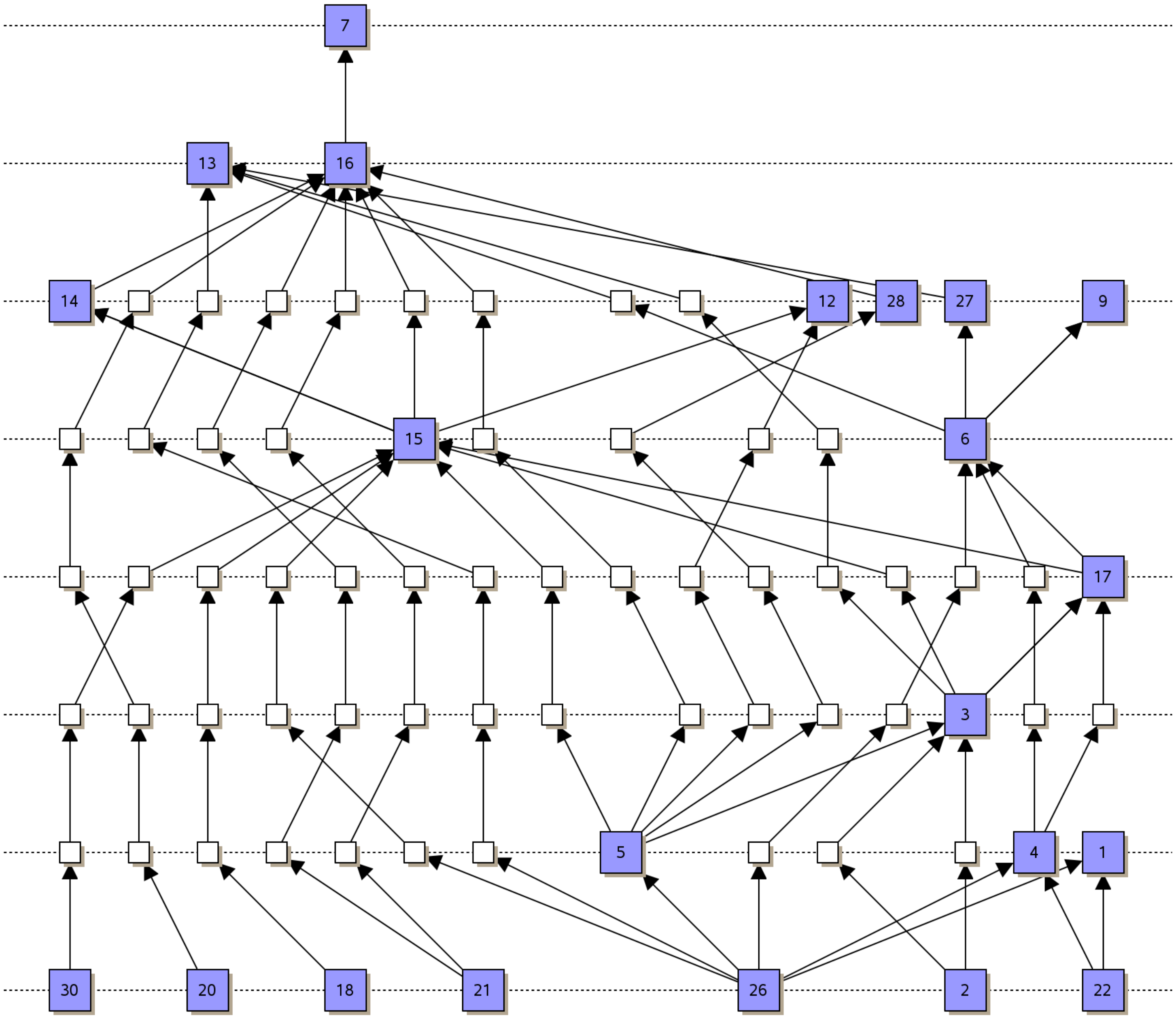

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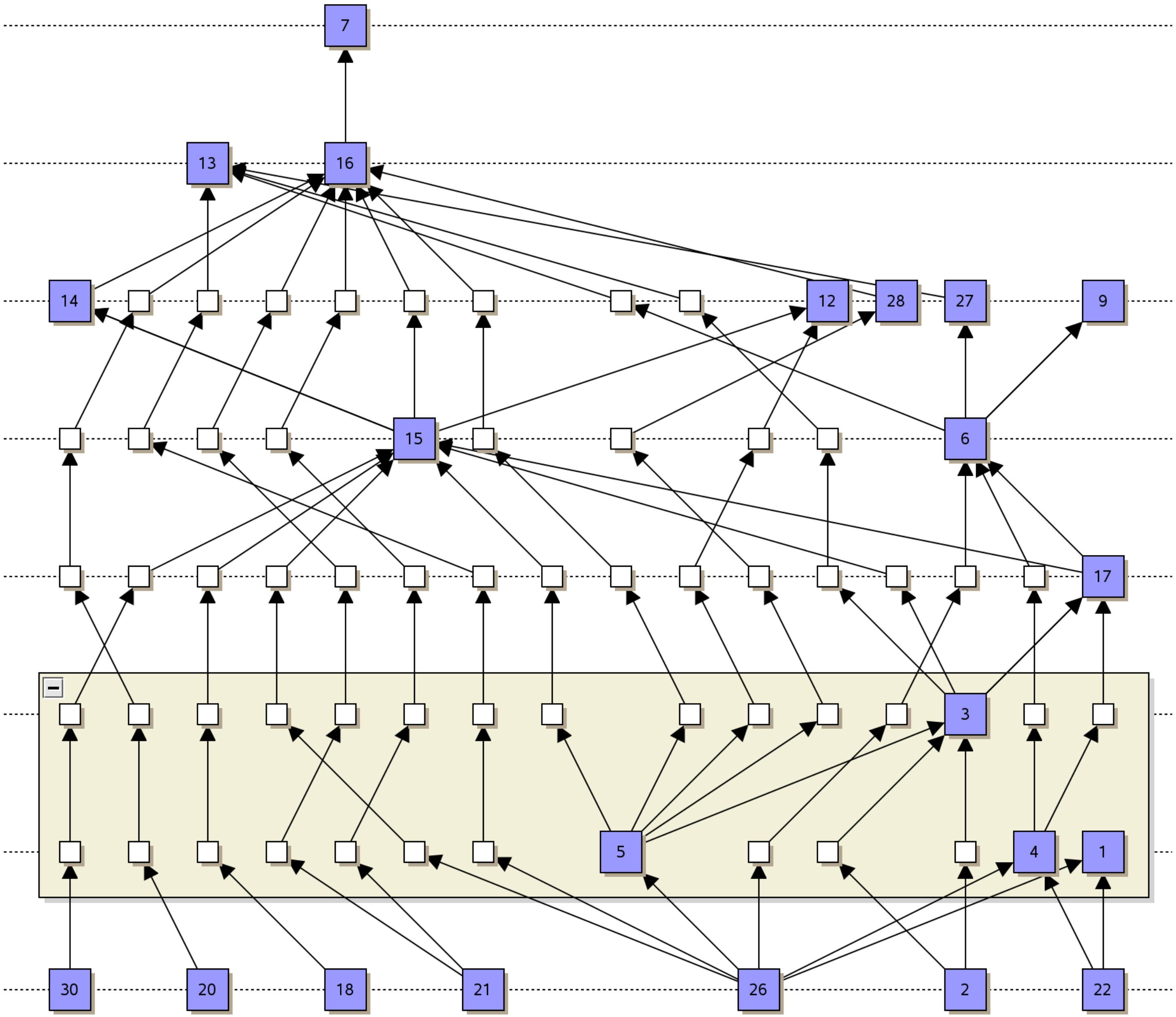

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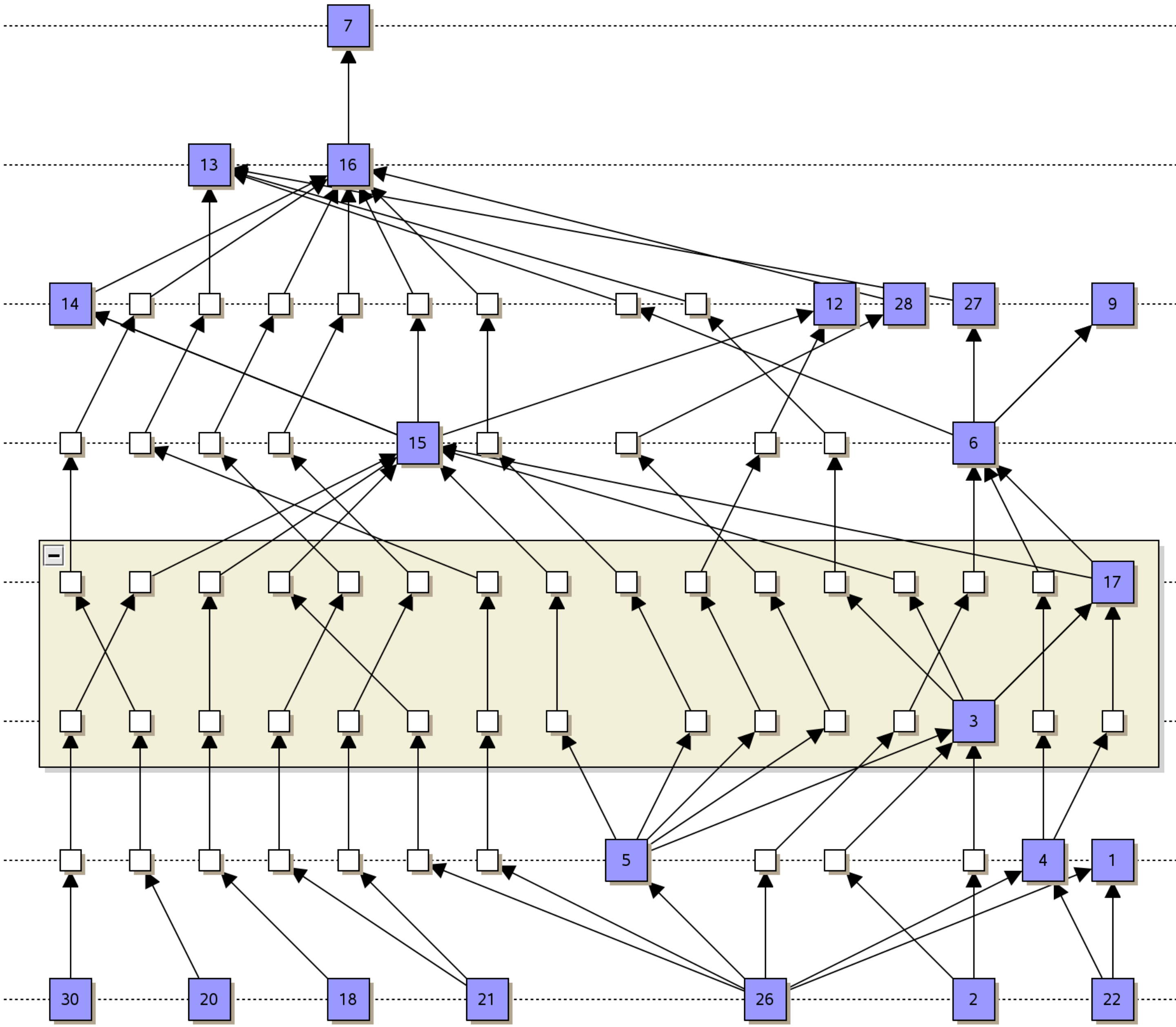

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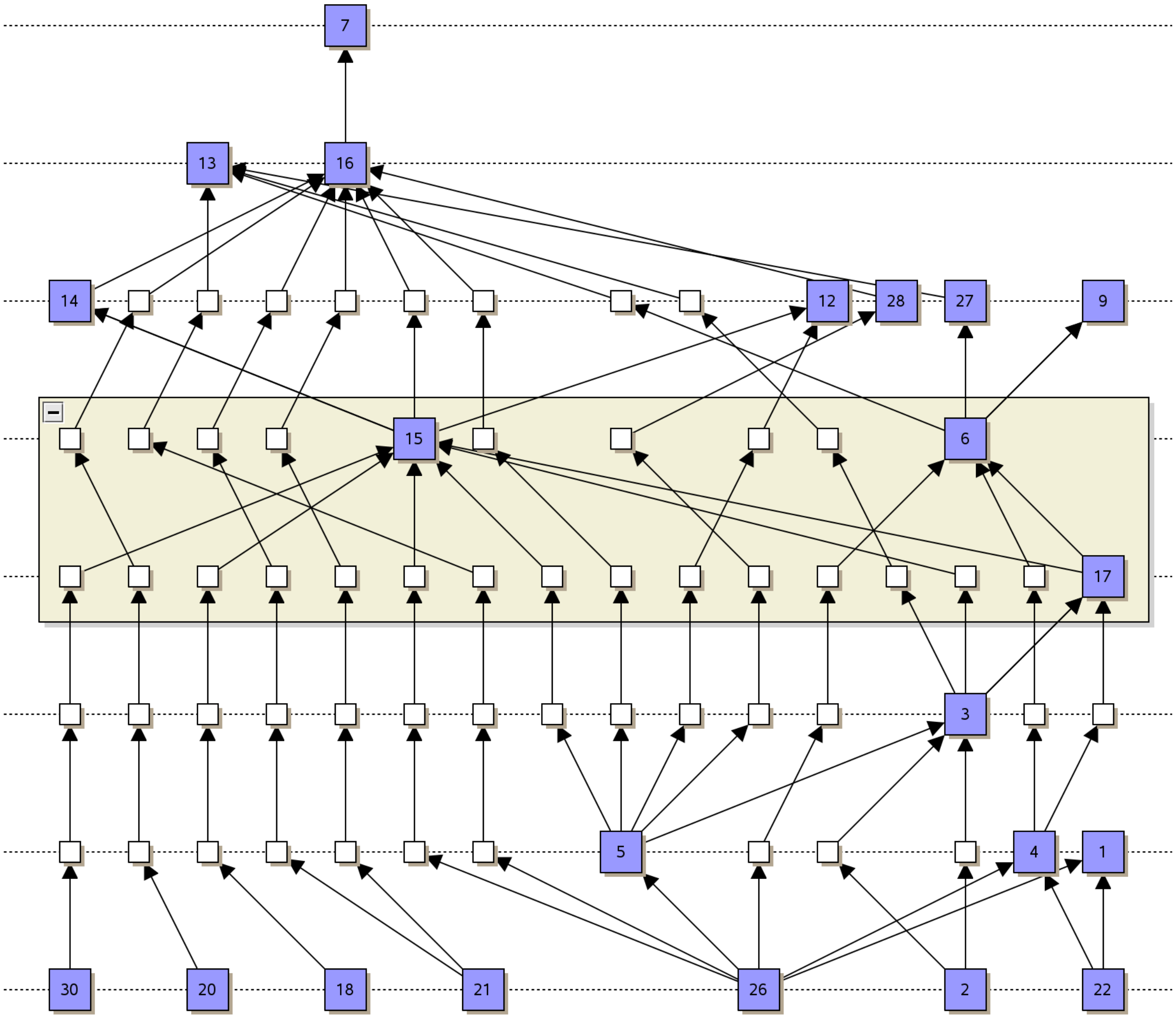

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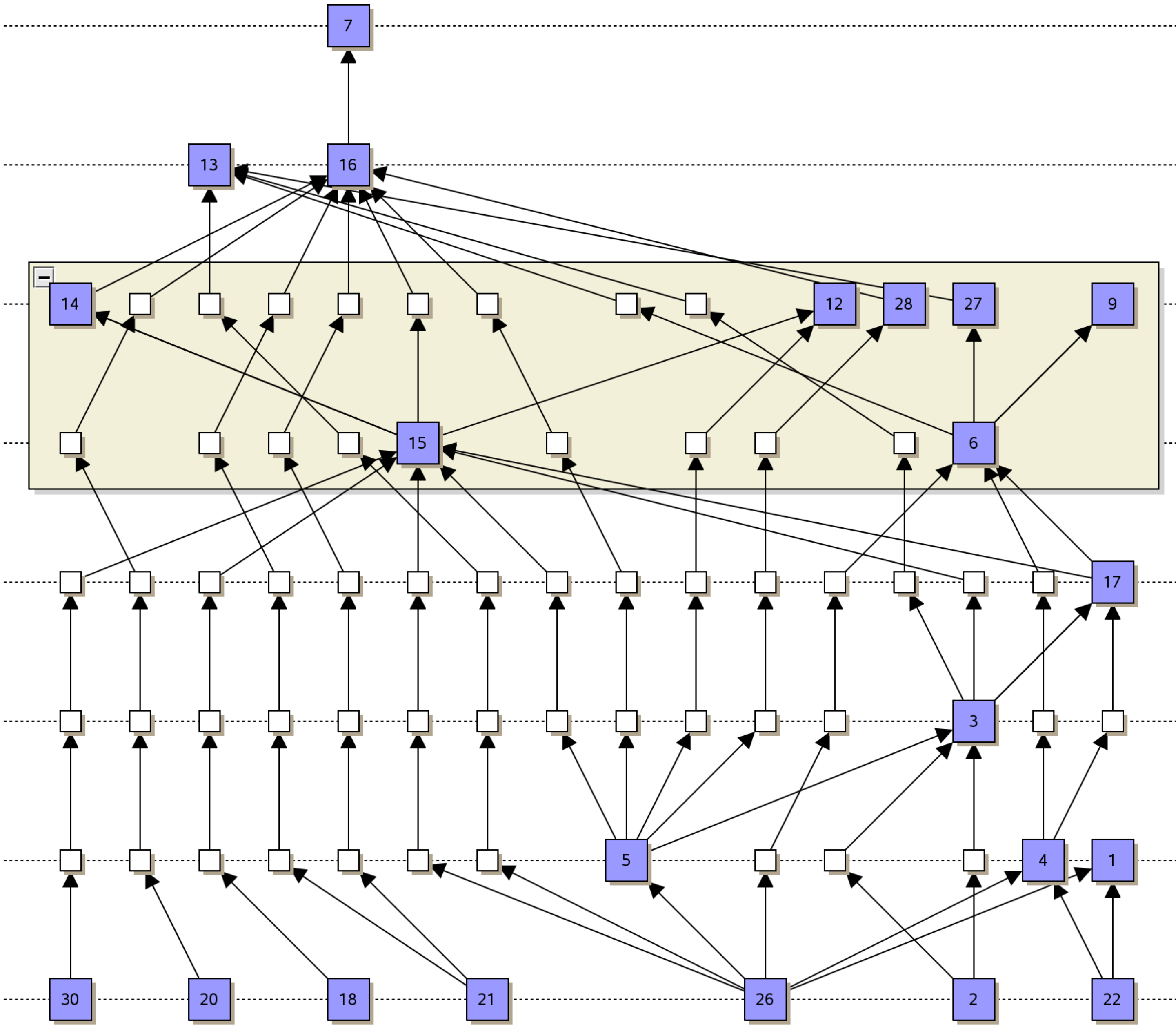

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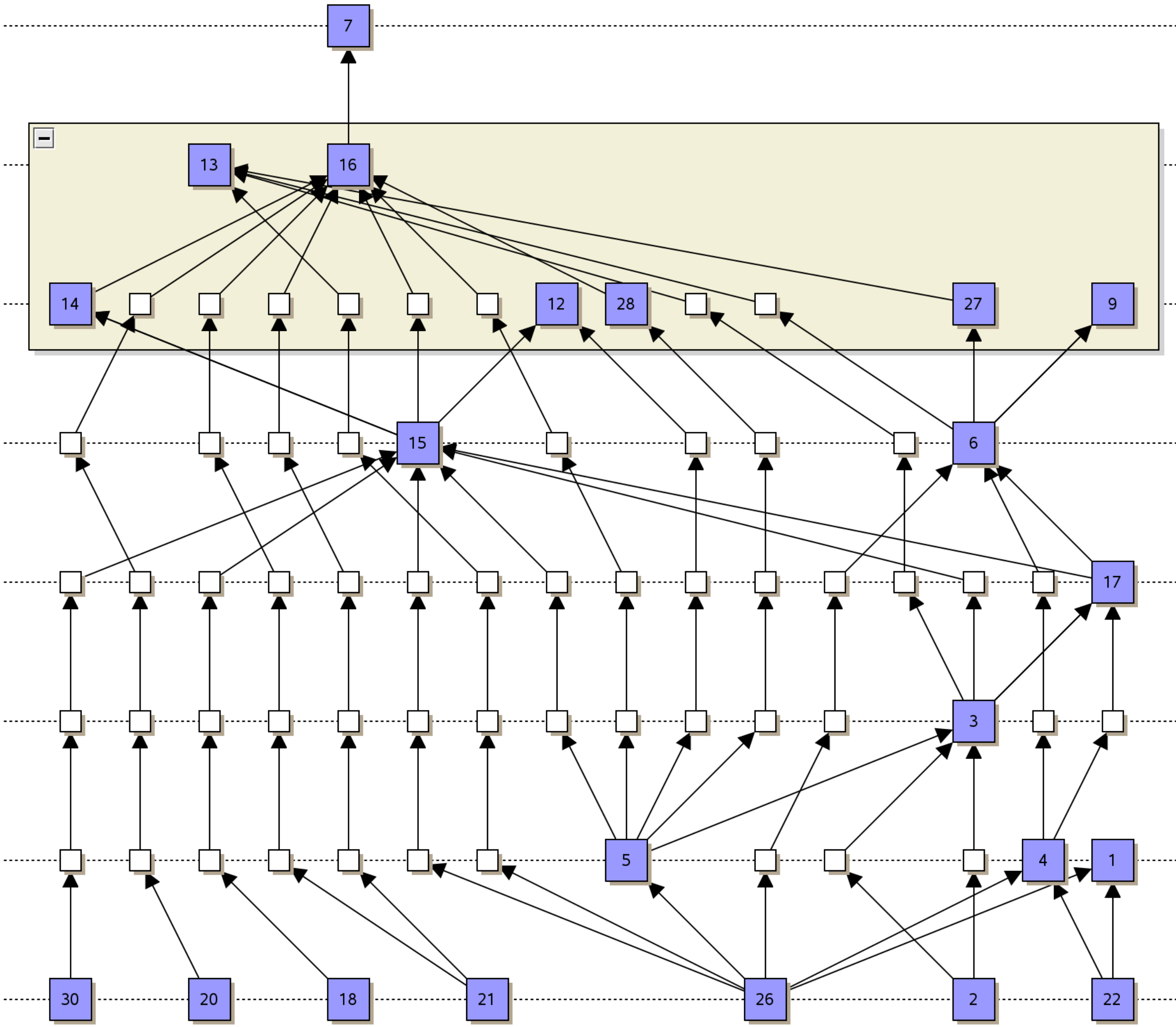

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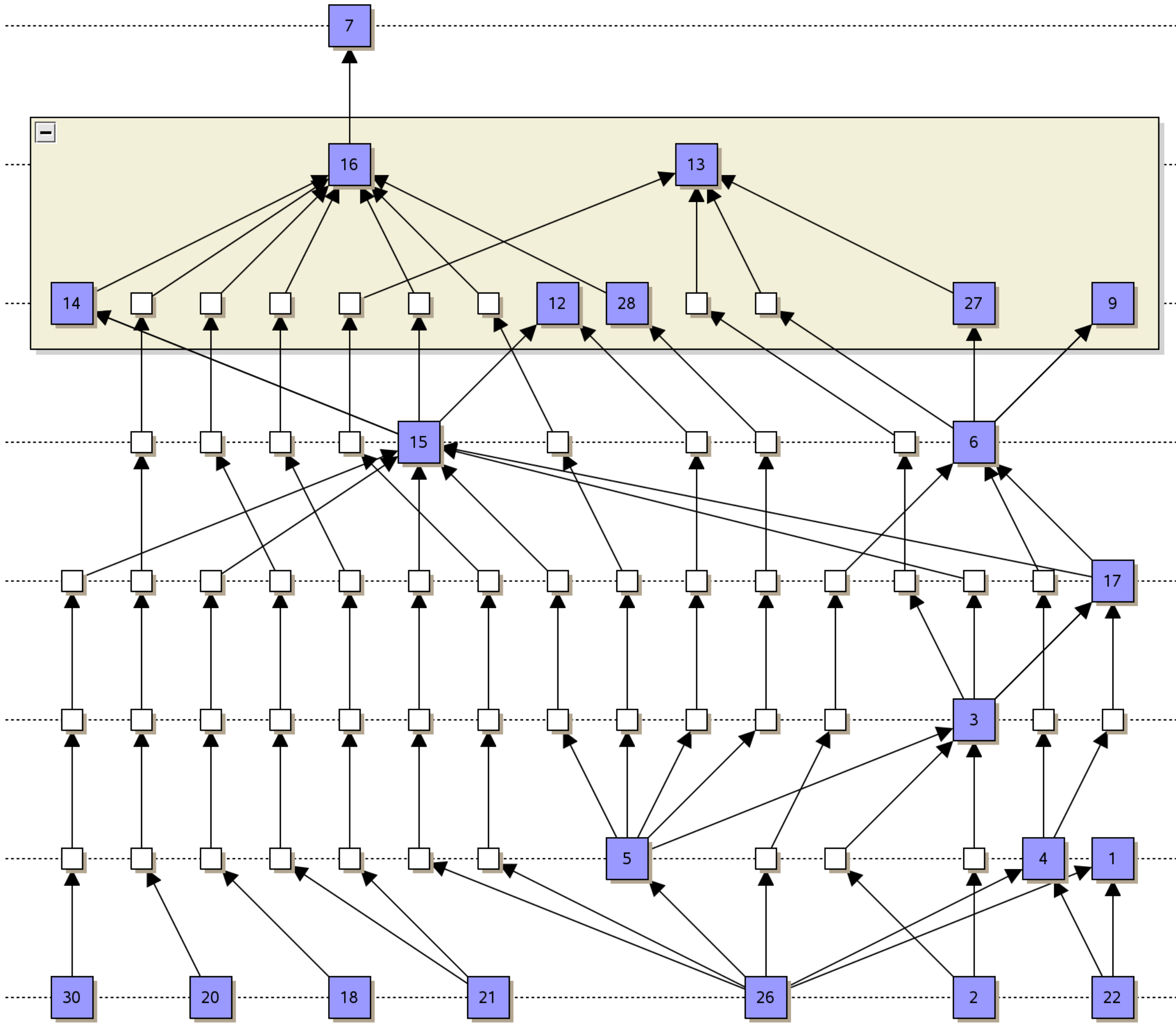

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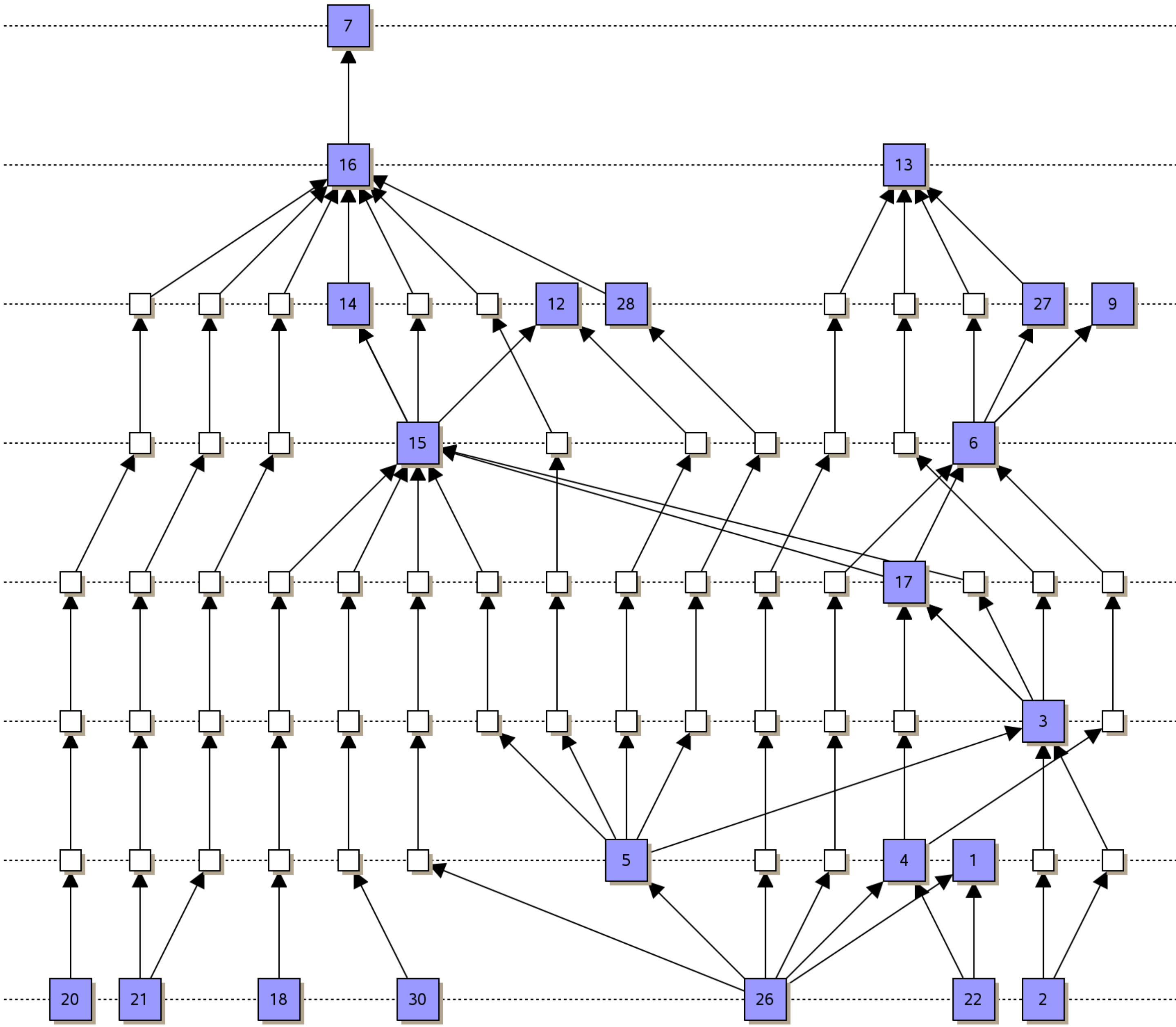
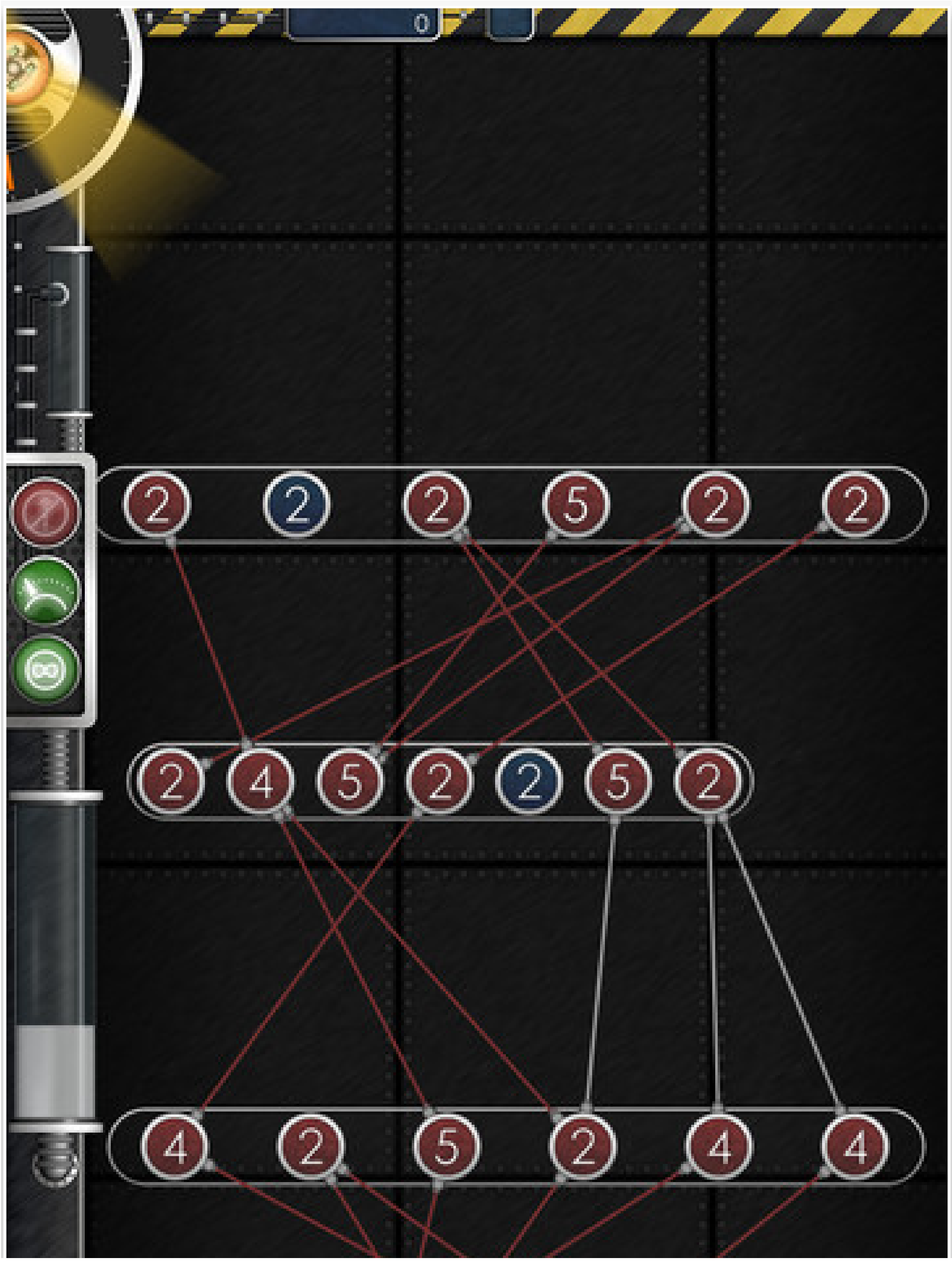


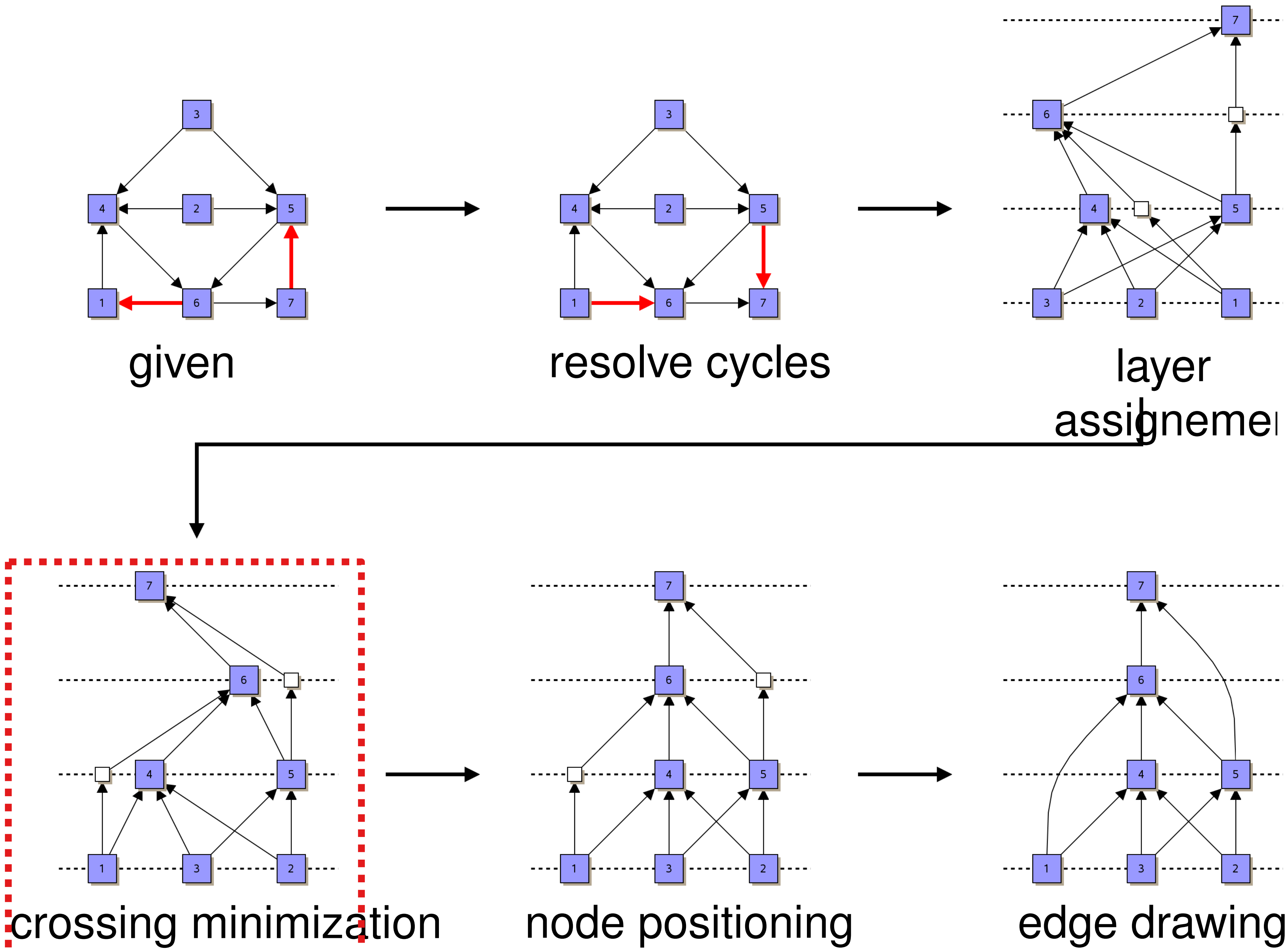
CrossingX



There was even an iPad game **CrossingX** for the OSCM Problem!

Winner of Graph Drawing Game Contest 2012

Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



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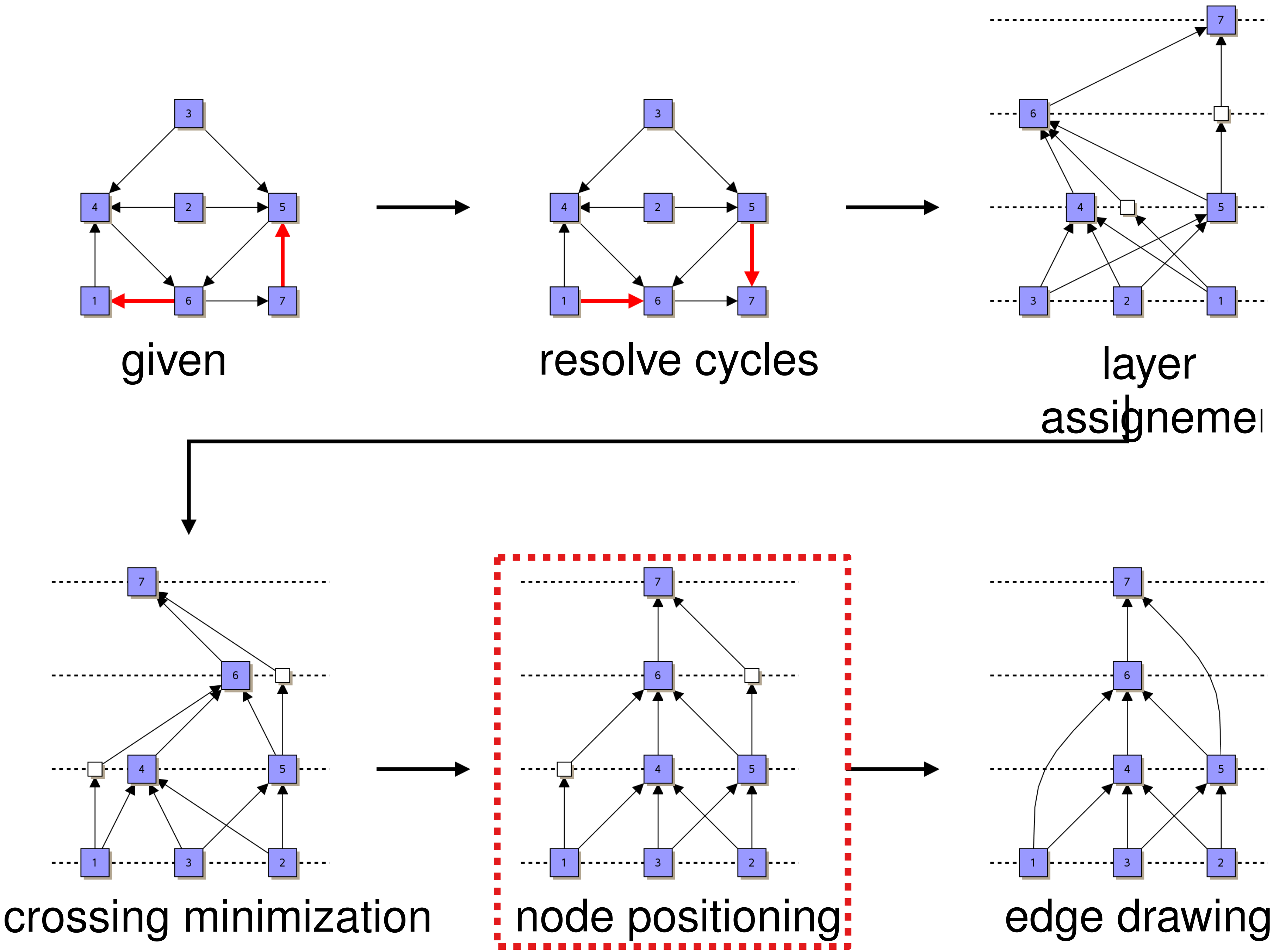
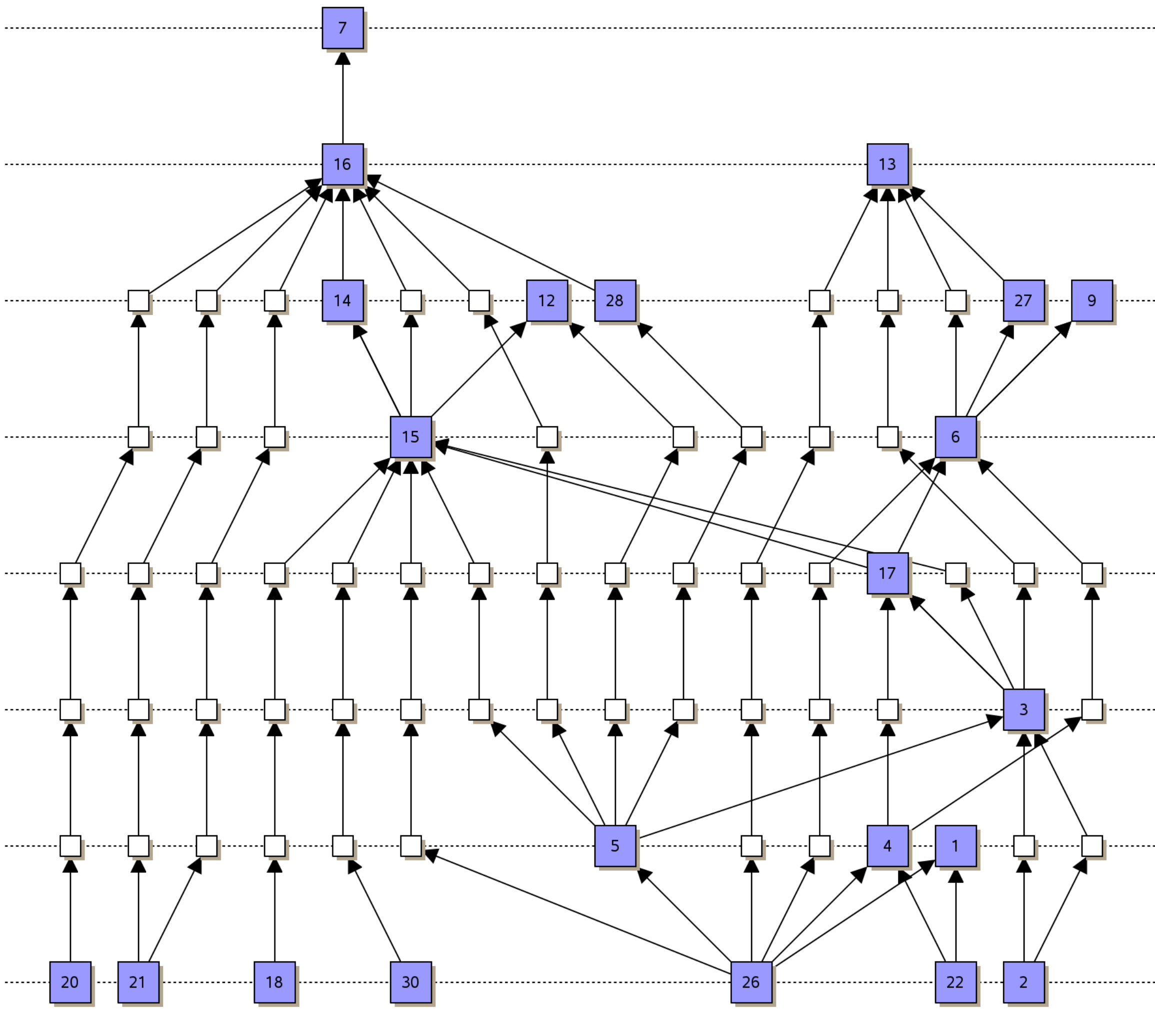


Coordinate Assignement Goals



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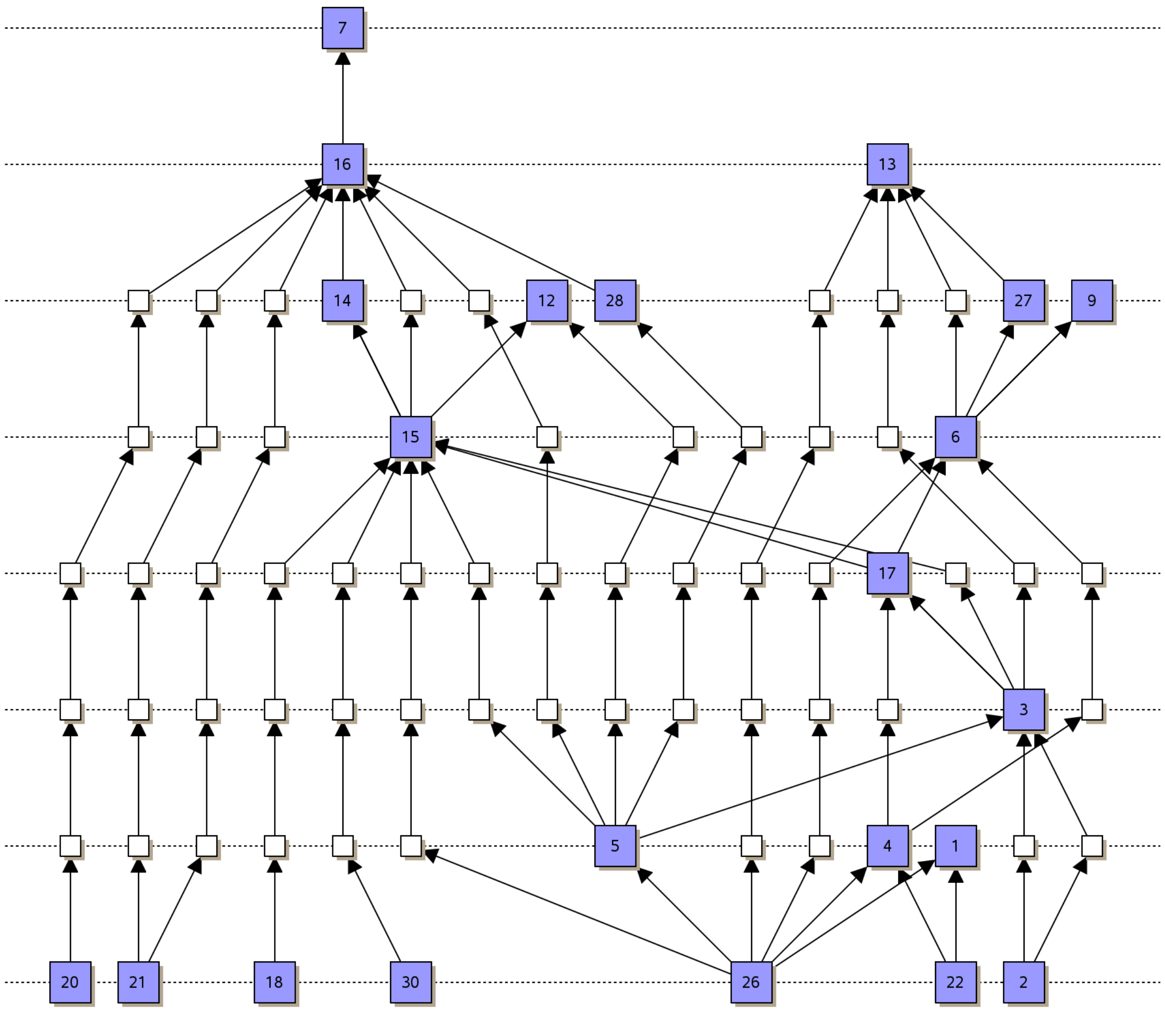
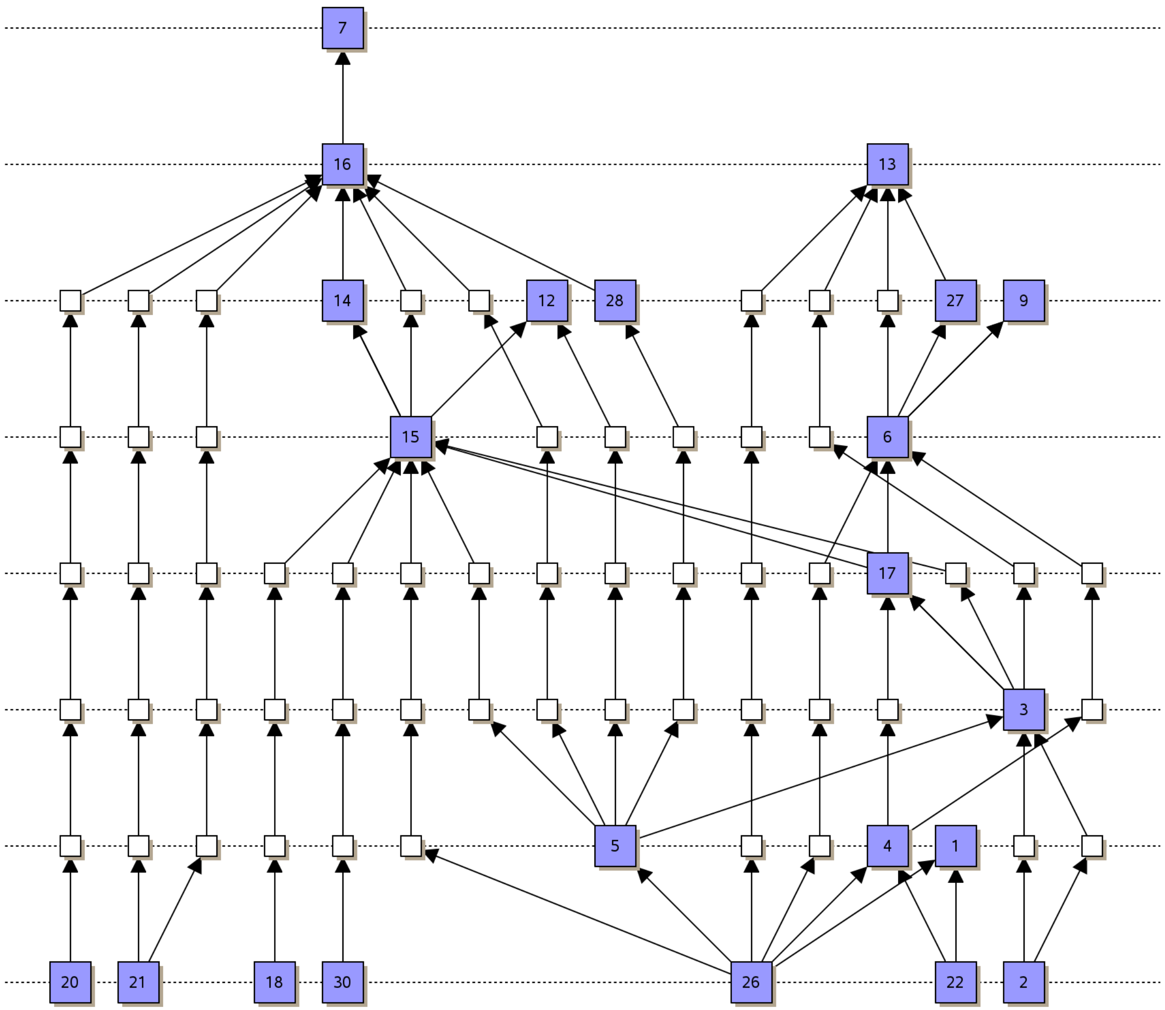


Think for a minute and then share

What could we optimize when doing the coordinate assignment?

Coordinate Assignement Goals



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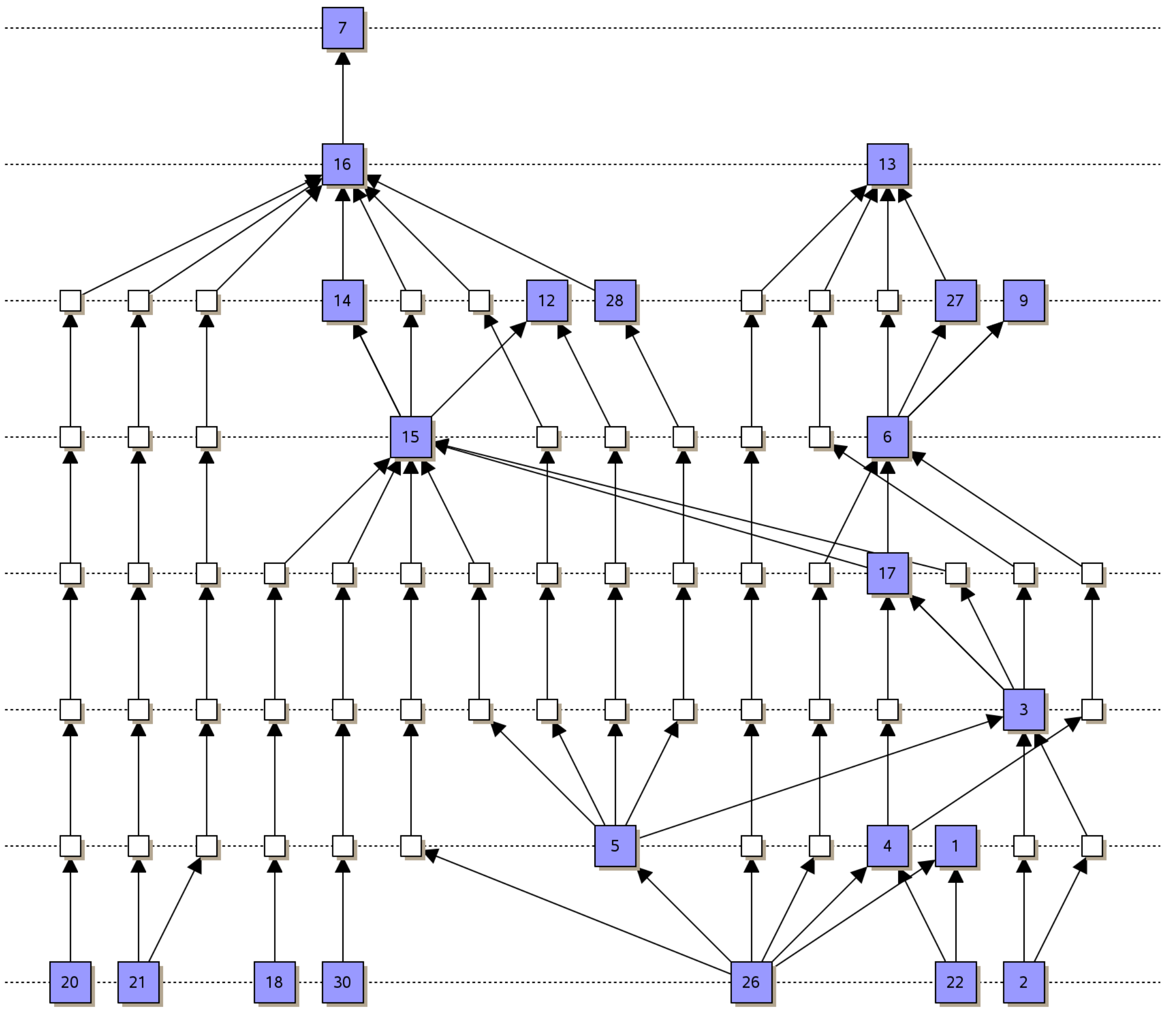

- Edges as straight as possible
- Edges as vertical as possible

Steightening Edges

Goal: minimize deviation from a straight-line for the edges with dummy-nodes

Idea: use quadratic Program

- let $p_{uv} = (u, d_1, \dots, d_k, v)$ path with k dummy nodes betwen u and v
- let $a_i = x(u) + \frac{i}{k+1}(x(v) - x(u))$ the x-coordinate of d_i when (u, v) is straight
- $g(p_{uv}) = \sum_{i=1}^{k}(x(d_i) - a_i)^2$
- minimize $\sum_{uv \in E} g(p_{uv})$
- constraints: $x(w) - x(z) \geq \delta$ for consecutive nodes on the same layer, w on the right of z (δ distance parameter)

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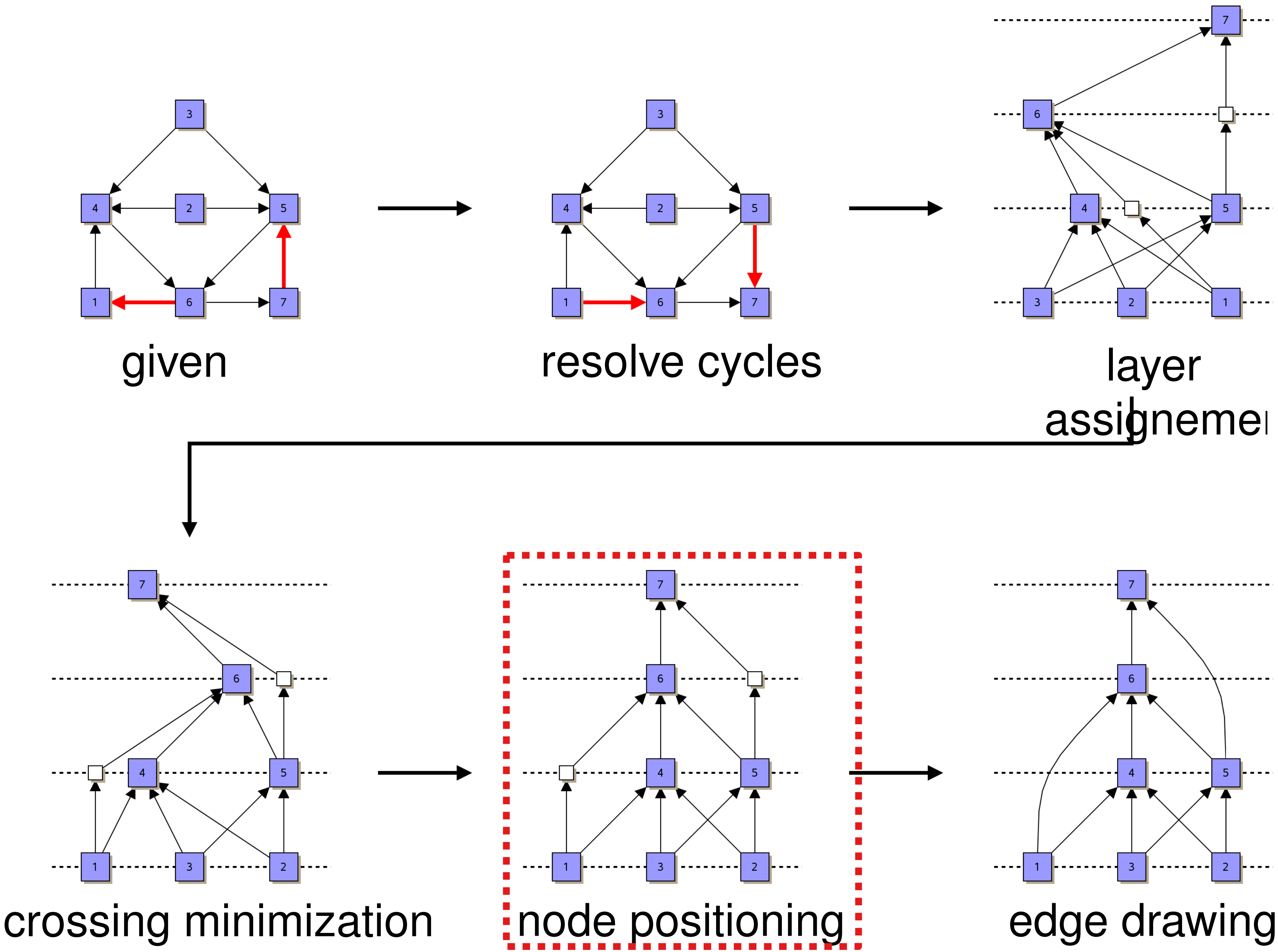
Properties:

- quadratic program is time-expensive
- straight edges increase width
- width can be exponential
- optimization function can be adapted to optimize ”verticality”

Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



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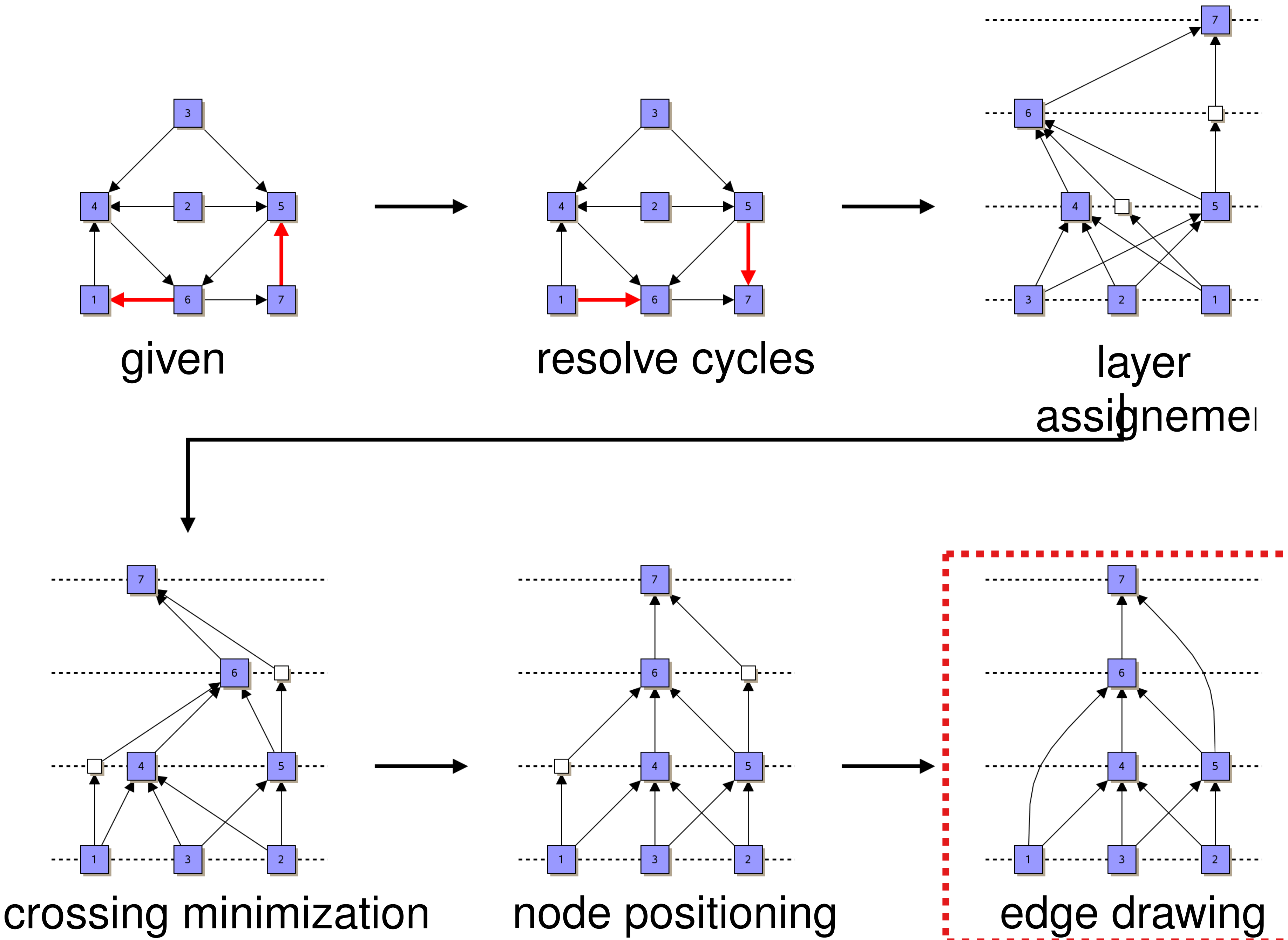
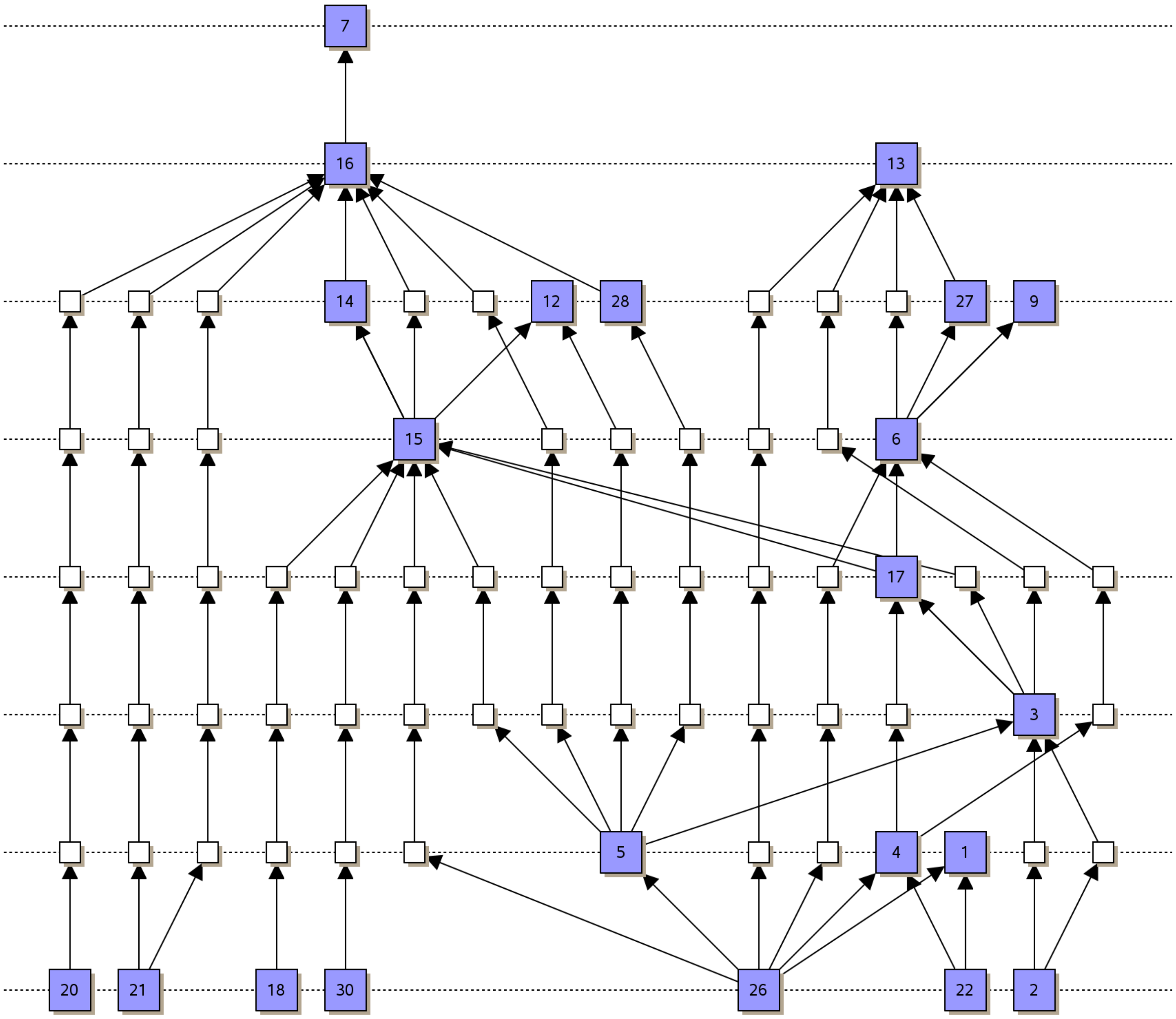


Edge Drawing



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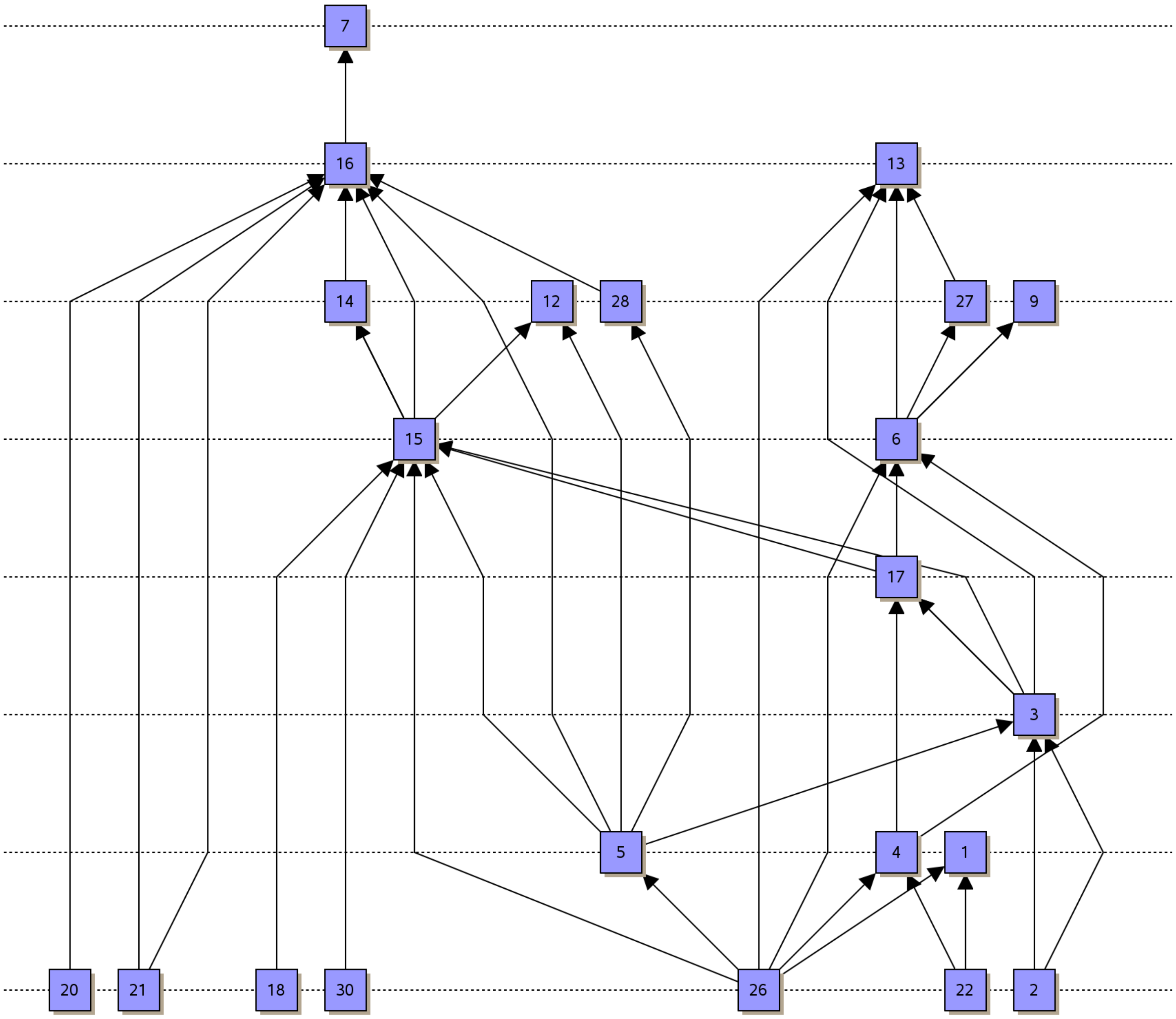

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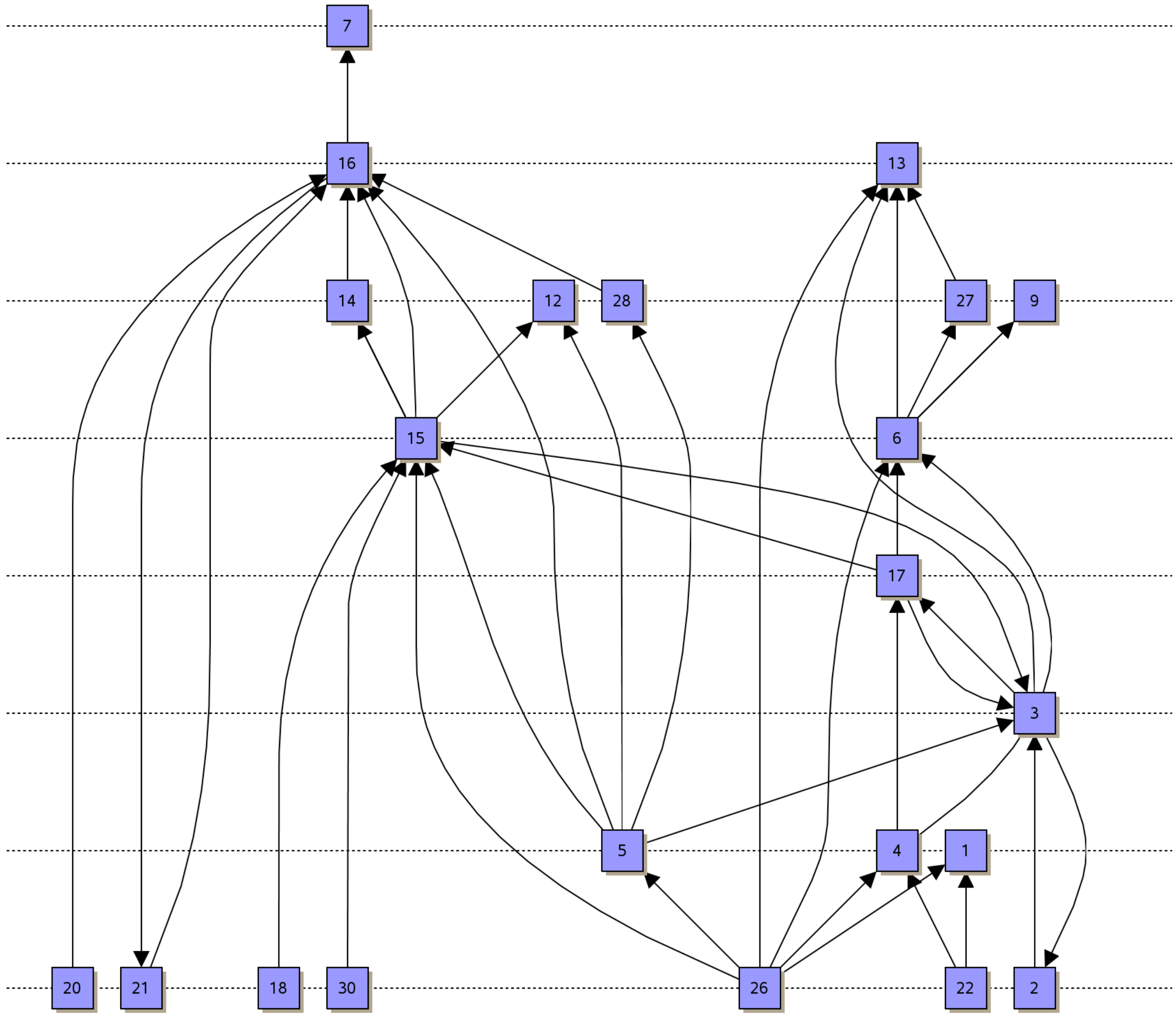
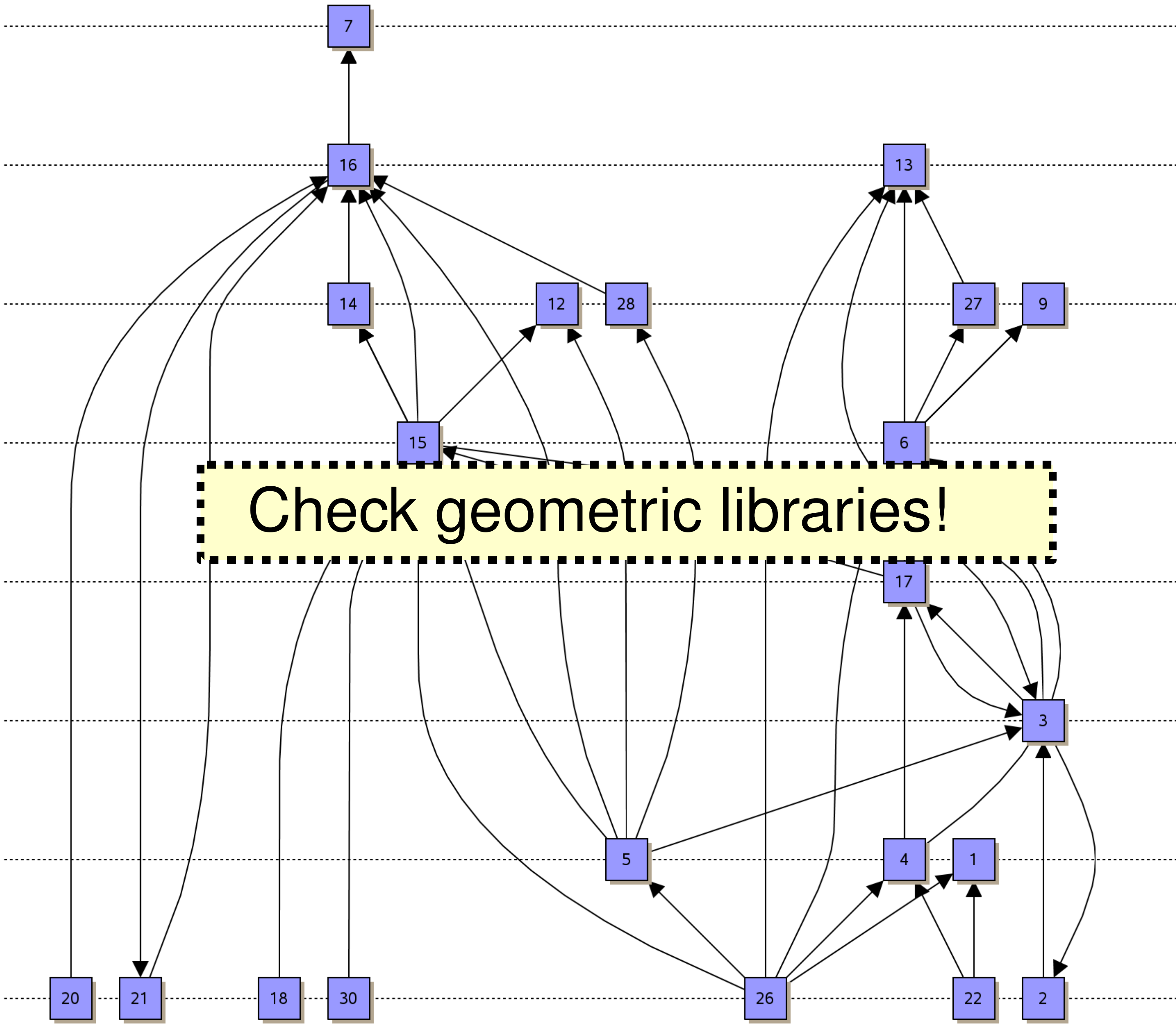


Possibility: Substitute polylines by Bézier curves

Edge Drawing



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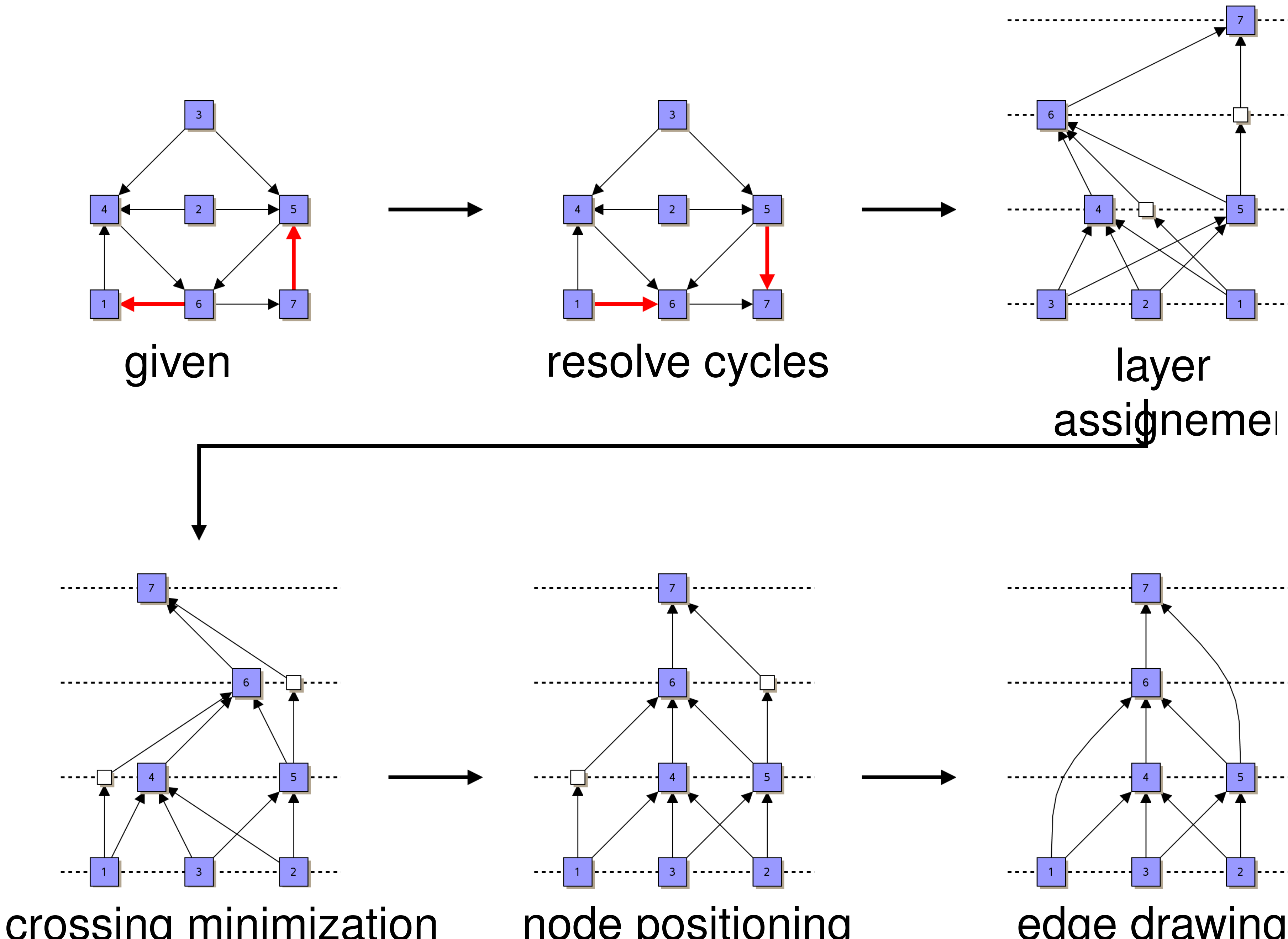
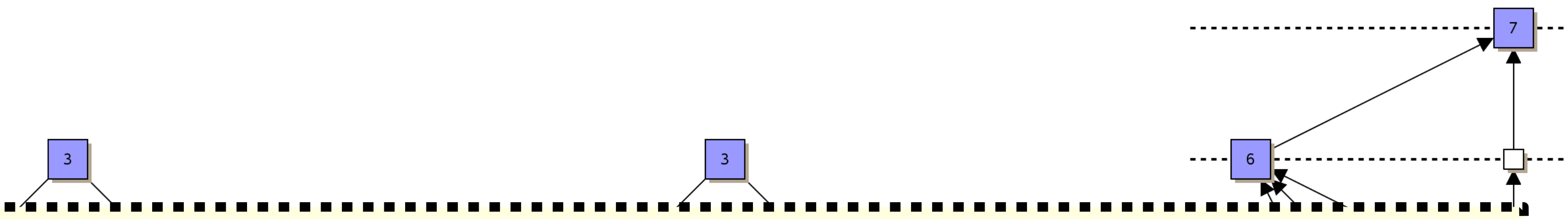
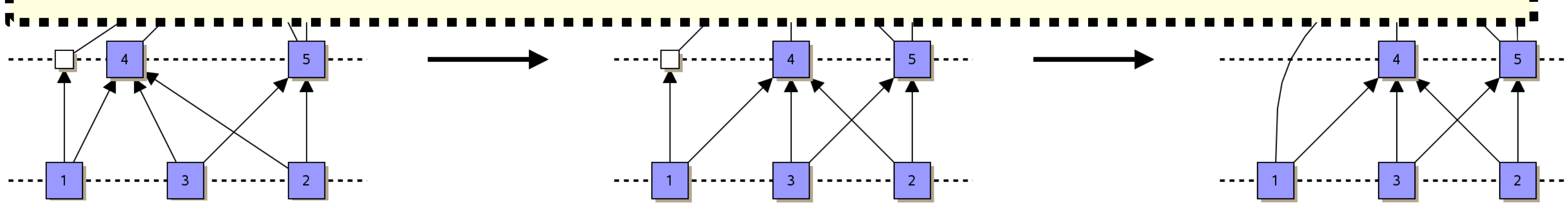
Summary


given
resolve cycles
layer assignement
crossing minimization
node positioning
edge drawing

Summary



- flexible **framework** to draw directed graphs
- sequential optimization of various criteria
- drawback: decisions taken on the previous steps influence the next steps and can not be undone
- modeling gives NP-hard problems, which can still can be solved quite well with heuristics
- many functionalities implemented in yEd and GraphVis



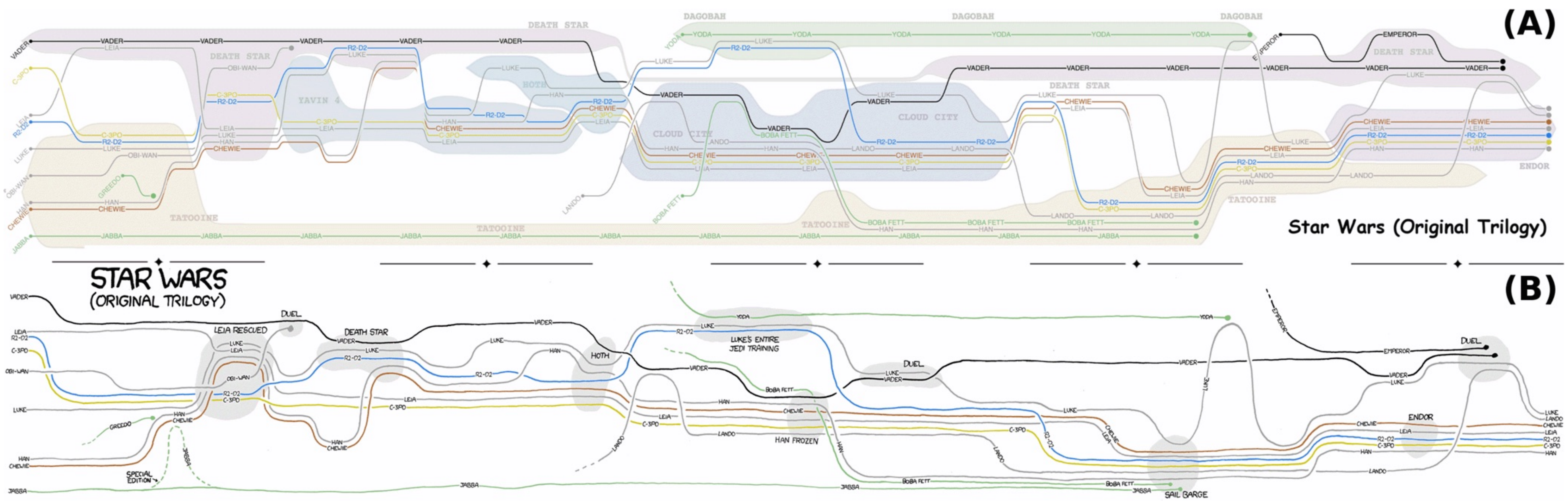
crossing minimization node positioning edge drawing

Reading

Additional Reading

Graph Drawing handbook: Chapter 13

Paper ”A framework and algorithms for circular drawings of graphs” J. Six, I. Tollis – fast computation of the number of crossings



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Next

Methods for visualization of multilayer/clustered networks