Lectures on Riemannian Geometry, Part II: Complex Manifolds

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Abstract

This is a set of introductory lecture notes on the geometry of complex manifolds. It is the second part of the course on Riemannian Geometry given at the MRI Masterclass in Mathematics, Utrecht, 2008. The first part was given by Prof. E. van den Ban, and his lectures notes can be found on the web-site of this course, http://www.math.uu.nl/people/ban/riemgeom2008/riemgeom2008.html.

Topics that we discuss in these lecture notes are : almost complex structures and complex structures on a Riemannian manifold, symplectic manifolds, Kähler manifolds and Calabi-Yau manifolds, and finally we also introduce hyperkähler geometries. Many of these structures appear in the context of string theory and other areas in theoretical physics, and these lectures notes reflect a theoretical physicist point of view on geometry.

Contents

1 Almost complex manifolds

Before we introduce and define what are almost complex manifolds, we give the definition of a complex manifold. As the word already indicates, almost complex means it is "not quite" complex, so it is useful to first recall the definition of a complex manifold:

Definition: Complex manifolds are differentiable manifolds with a holomorphic atlas. They are necessarily of even dimension, say $2n$, and allow for a collection of charts (U_j, z_j) that are one to one maps of the corresponding U_j to $\mathbb{C}^n$ such that for every non-empty intersection $U_j \cap U_k$ the maps are $z_j z_k^{-1}$ are holomorphic.

The crucial difference between a real manifold of even dimension and a complex manifold is that for the latter, the transition functions which relate the coordinates in overlapping patches are holomorphic. We will study complex manifolds in the next chapter, but discuss in this chapter the intermediate case of *almost complex manifolds*. These are a class of manifolds which are even dimensional but which are not complex, yet they inherit some of the properties of complex manifolds, as we will see below.

Definition: If a real manifold M ($\dim_{\mathbb{R}} M = m$) admits a globally defined tensor J of rank (1,1) with the property

$$J^2 = -\mathbb{I} \ , \tag{1.1}$$

then M is called an almost complex manifold. Here, $\mathbb{I}$ is the identity operator and J is a tensor field of type $(1,1)$; both operators are maps from the tangent bundle TM into itself. A globally defined $(1,1)$ tensor satisfying (1.1) is called an almost complex structure.

Locally, this implies that at each given point $p \in M$, there is an endomorphism $J_p : T_pM \to T_pM$ which satisfies $(J_p)^2 = -\mathbb{I}_p$, and which depends smoothly on $p \in M$. Here $\mathbb{I}_p$ is the identity operator acting on the tangent space T_pM at the point p. We remind that a rank $(1,1)$ tensor can be defined by introducing a basis of (real) vector fields $\partial/\partial x^\mu$ in the tangent space, and a basis of dual one-forms $\mathrm{d}x^\mu$. Here $x^\mu; \mu = 1, ..., m$ can be seen as the coordinates of the point $p \in M$. We can then write, in local coordinates

$$J_p = J_\mu{}^\nu(p) \frac{\partial}{\partial x^\nu} \otimes \mathrm{d}x^\mu \ , \tag{1.2}$$

with $J_\mu{}^\nu(p)$ real (in this real basis). It acts on vector fields

$$X = X^\mu \frac{\partial}{\partial x^\mu} \ , \tag{1.3}$$

according to

$$J(X) = (X^\mu J_\mu{}^\nu) \frac{\partial}{\partial x^\nu} \ , \tag{1.4}$$

and so

$$J^2(X) = X^\rho J_\rho{}^\nu J_\nu{}^\mu \frac{\partial}{\partial x^\mu} \ . \tag{1.5}$$

In local coordinates, the condition for an almost complex structure then translates into a matrix equation

$$J_\mu{}^\rho(p)\, J_\rho{}^\nu(p) = -\delta_\mu{}^\nu \ , \tag{1.6}$$

at any point p.

Globally, having an almost complex structure means that one can define the J_p in any patch and glue them together without encountering obstructions or singularities. There are examples where such obstructions appear; the most notable is the four-sphere S^4. It is known not to allow for an almost complex structure (see e.g. Steenrod, 1951), hence S^4 is not an almost complex manifold. It turns out that S^4 has other geometrical properties that make S^4 a quaternion-Kähler manifold. More on this in chapter 7.

Theorem: Almost complex manifolds have even dimension.

Proof: Denote the (real) dimension of M by m. J_p acts on the tangent space, and if we choose a real basis of vector fields, the $J_\mu{}^\nu(p)$ are real. It then follows that (we drop writing the base point p)

$$[\mathrm{Det}(J)]^2 = \mathrm{Det}(J^2) = \mathrm{Det}(-\mathbb{I}) = (-1)^m \ . \tag{1.7}$$

But since $[\mathrm{Det}(J)]$ is real, $[\mathrm{Det}(J)]^2$ is positive, hence m must be even and we write $m = 2n$. QED.

Complexification of the tangent space: We can complexify the tangent space by introducing linear combinations of vector fields with complex coefficients. They are of the type

$$Z = \frac{1}{2}(X + iY) \ , \tag{1.8}$$

where $X, Y \in TM$. Similarly, the complex conjugate of this vector field is denoted by

$$\bar{Z} \equiv \frac{1}{2}(X - iY) \ . \tag{1.9}$$

These vector fields specify the complexified tangent space $T_pM^{\mathbb{C}}$, and J_p acts on $T_pM^{\mathbb{C}}$ as a complex linear map, still satisfying $(J_p)^2 = -\mathbb{I}$ in each point $p \in M$. The eigenvalues of J_p can only be $\pm i$. On $TM^{\mathbb{C}}$, one can define the projection operators

$$P^\pm = \frac{1}{2}(\mathbb{I} \mp iJ) \ , \tag{1.10}$$

satisfying

$$(P^\pm)^2 = P^\pm \ , \qquad P^+ + P^- = \mathbb{I} \ , \qquad P^+P^- = 0 \ . \tag{1.11}$$

These projection operators project vector fields into the eigenspaces of J with eigenvalues $\pm i$:

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^- \ , \tag{1.12}$$

with

$$T_pM^{\pm} = \{Z \in T_pM^{\mathbb{C}} | J_pZ = \pm iZ\} \ . \tag{1.13}$$

Indeed, consider an arbitrary element $W \in T_pM^{\mathbb{C}}$, and define

$$Z \equiv P^+(W) = \frac{1}{2}(W - iJ(W)) \ , \qquad \bar{Z} \equiv P^-(W) = \frac{1}{2}(W + iJ(W)) \ . \tag{1.14}$$

It is then straightforward to show that $J(Z) = iZ$ and $J(\bar{Z}) = -i\bar{Z}$. We call elements in T_pM^+ and T_pM^- holomorphic and anti-holomorphic vectors respectively.

Matrix representation of an almost complex structure at a given point:

Since $(J_p)^2 = -\mathbb{I}_p$, J_p has eigenvalues $\pm i$. Since M is even dimensional, $m = 2n$, and following (1.14), there are equal number of $+i$ and $-i$ eigenvalues. Locally, in a given point p, this implies that one can choose a basis of $2n$ real vector fields in the tangent space T_pM such that the almost complex structure takes the form

$$J_p = \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{pmatrix} \ , \tag{1.15}$$

or equivalently, in a basis of complex vector fields on $T_pM^{\mathbb{C}}$

$$J_p = \begin{pmatrix} i\mathbb{I}_{n\times n} & 0 \\ 0 & -i\mathbb{I}_{n\times n} \end{pmatrix} \ . \tag{1.16}$$

One can choose a basis of complex vector fields in $T_pM^{\mathbb{C}}$ consisting of $\partial/\partial z^a; a = 1, ..., n$, and their complex conjugates. Their duals are denoted by $\mathrm{d}z^a$, and complex conjugates, such that the almost complex structure can be written as

$$J_p = i\frac{\partial}{\partial z^a} \otimes \mathrm{d}z^a - i\frac{\partial}{\partial \bar{z}^a} \otimes \mathrm{d}\bar{z}^a \ . \tag{1.17}$$

We repeat that this property only holds pointwise, that is, with respect to a given point $p \in M$. Locally in a patch, an almost complex structure can *not* be written as in (1.17) as one cannot keep the matrix elements of J_p constant after varying the base point p.

Exercise 1.1: Consider a manifold of dimension 2. Show that, in a real basis on T_pM, the most general form for the matrix representation of J is given by

$$J = \begin{pmatrix} a & b \\ -\frac{1+a^2}{b} & -a \end{pmatrix} \ , \tag{1.18}$$

where a and b are real and $b \neq 0$. Furthermore, find a change of basis such that J takes the canonical form (1.15).

Solution: Consider a change of basis using the invertible matrix

$$\Lambda = \frac{1}{b}\begin{pmatrix} b & 0 \\ -a & 1 \end{pmatrix} \ . \tag{1.19}$$

Exercise 1.2: Repeat the same as in exercise 1.1, but now for arbitrary even dimension $m = 2n$.

Theorem: Complex manifolds are almost complex.

Proof: We should prove that on complex manifolds, one can construct a globally defined almost complex structure. First, complex manifolds allow for a holomorphic atlas, this means there exists local complex coordinates z^a in a neighborhood U of any given point $p \in M$. We can now define the tensor

$$J = i\frac{\partial}{\partial z^a} \otimes \mathrm{d}z^a - i\frac{\partial}{\partial \bar{z}^a} \otimes \mathrm{d}\bar{z}^a \ . \tag{1.20}$$

This object is well defined in the patch U, this in contrast to almost complex manifold, where (1.22) is only defined at a given point (the reason being that on an almost complex manifold one can not introduce complex coordinates that vary holomorphic on U). Second, to have it globally defined, we need to show that it keeps its form on the overlap of two patches $(U, z) \cap (V, w)$. Since for complex manifolds, the transition functions are holomorphic, i.e. the coordinate transformations $z^a(w)$ are holomorphic, it then follows that

$$\frac{\partial}{\partial z^a} \otimes \mathrm{d}z^a = \frac{\partial z^a}{\partial w^c}\frac{\partial w^b}{\partial z^a}\frac{\partial}{\partial w^b} \otimes \mathrm{d}w^c = \frac{\partial}{\partial w^a} \otimes \mathrm{d}w^a \ . \tag{1.21}$$

So

$$J = i\frac{\partial}{\partial w^a} \otimes \mathrm{d}w^a - i\frac{\partial}{\partial \bar{w}^a} \otimes \mathrm{d}\bar{w}^a \ , \tag{1.22}$$

which proves the theorem. QED. Notice that the converse is not necessarily true, since the transition functions need not to be holomorphic. So not all almost complex manifolds are complex. In the next chapter, we formulate the condition for an almost complex manifold to be complex.

Theorem: An almost complex manifold is orientable. (Without proof)

Remark: Not all even dimensional spaces are almost complex. As already mentioned before, the four-sphere S^4 is not almost complex. In fact, the only spheres of even dimension that are almost complex are S^2 and S^6, all other are not even almost complex. The two-sphere turns out to be even a complex manifold (see next chapter).

(p,q) **forms on almost complex manifolds**: The projectors $P^{\pm}$ can be defined to act on forms as follows. In real coordinates, the components of the projectors are just matrices $P_\mu^{\pm\nu}$. They act on a real one-form $\theta \equiv \theta_\mu \mathrm{d}x^\mu$ as

$$P^+\theta \equiv P_\mu^{+\nu}\theta_\nu\,\mathrm{d}x^\mu\ , \qquad P^-\theta \equiv P_\mu^{-\nu}\theta_\nu\,\mathrm{d}x^\mu\ , \tag{1.23}$$

and we further introduce the notation

$$\theta^{(1,0)} \equiv P^+\theta\ , \qquad \theta^{(0,1)} \equiv P^-\theta\ . \tag{1.24}$$

They satisfy $\theta = \theta^{(1,0)} + \theta^{(0,1)}$, and we call them $(1,0)$ and $(0,1)$ forms respectively.

Exercise 1.3: Let (M,J) be an almost complex manifold. Thus, $J^2 = -\mathbb{I}$ on the tangent space T_pM at any point $p \in M$.

- Show that the eigenvalues of J on T_pM are $\pm i$.
- Show that these eigenvalues $+i$ and $-i$ have equal multiplicities.
- Let Z be any holomorphic vector field on M, i.e. $JZ = iZ$, and let θ be a one-form on M. Show that $\theta(Z) = 0$ on any holomorphic vector field Z if and only if θ is of type $(0,1)$, i.e. $P^+\theta = 0$ and $P^-\theta = \theta$ with $P^{\pm} = \frac{1}{2}(\mathbb{I} \mp iJ)$.

Solution:

- The eigenvalue equation $JX = \lambda X$, for X in the complexified tangent space and $\lambda \in \mathbb{C}$, implies $J^2X = \lambda^2 X = -X$. Hence $\lambda^2 = -1$ and so $\lambda = \pm i$.
- This follows from the fact that the eigenvectors of J come in pairs: if Z is a vector field with eigenvalue $+i$, satisfying $JZ = iZ$, then its complex conjugate $\bar{Z}$ is an eigenvector with eigenvalue $-i$, $J\bar{Z} = -i\bar{Z}$. This can be most easily seen by decomposing the complexified vector fields in terms of real vector fields $Z = X + iY, \bar{Z} = X - iY$, and then use $JX = -Y, JY = X$.
- First we decompose $\theta = \theta^{(1,0)} + \theta^{(0,1)}$. Then we compute, with $Z = Z^\mu \partial_\mu$ in local coordinates,

$$\theta(Z) = P^+\theta(Z) + P^-\theta(Z) = P_\mu^{+\nu}\theta_\nu Z^\mu + P_\mu^{-\nu}\theta_\nu Z^\mu\ . \tag{1.25}$$

For any holomorphic vector field Z we have that $P^+Z = Z$ and $P^-Z = 0$, and so $\theta(Z) = \theta^{(1,0)}(Z)$. If now $\theta(Z) = 0$, $\forall Z$, then it follows that $\theta^{(1,0)} = 0$, hence θ is of type $(0,1)$. Conversely, if θ is of type $(0,1)$, $\theta = \theta^{(0,1)}$ and $\theta^{(1,0)} = 0$, then the above calculation shows that $\theta(Z) = 0$ on any holomorphic vector field Z.

Similarly, one can define higher (p,q) forms, for example, when $p+q=2$,

$$\omega^{(2,0)}_{\mu\nu} \equiv P^{+\rho}_{\mu} P^{+\sigma}_{\nu} \omega_{\rho\sigma} \ , \quad \omega^{(1,1)}_{\mu\nu} \equiv \Big(P^{+\rho}_{\mu} P^{-\sigma}_{\nu} + P^{-\rho}_{\mu} P^{+\sigma}_{\nu} \Big) \omega_{\rho\sigma} \ , \quad \omega^{(0,2)}_{\mu\nu} \equiv P^{-\rho}_{\mu} P^{-\sigma}_{\nu} \omega_{\rho\sigma} \ . \tag{1.26}$$

Notice that

$$\omega = \omega^{(2,0)} + \omega^{(1,1)} + \omega^{(0,2)} \ . \tag{1.27}$$

It is an easy exercise to see that the exterior derivative of a $(1,0)$ form is not just a linear combination of a $(2,0)$ and a $(1,1)$ form, but there is also a $(0,2)$ part. This $(0,2)$ part will in fact be absent when the manifold is complex. It should be clear how to generalize this to arbitrary (p,q):

Exercise 1.4: Prove that for an arbitrary k-form on an almost complex manifold, one can define (p,q) tensors of the type defined above, with $k = p+q$ such that

$$\omega = \sum_{p+q=k} \omega^{(p,q)} \ . \tag{1.28}$$

Moreover, show that one can decompose

$$\mathrm{d}\big(\omega^{(p,q)}\big) = (\lambda_1)^{(p-1,q+2)} + (\lambda_2)^{(p,q+1)} + (\lambda_3)^{(p+1,q)} + (\lambda_4)^{(p+2,q-1)} \ , \tag{1.29}$$

for some $p+q+1$-forms $\lambda_1, ...\lambda_4$.

Comment 1.1: In anticipation of the next chapter, for complex manifolds the first and the last terms on the right hand side in (1.29) will be absent, $\lambda_1 = \lambda_4 = 0$. Regardless of this, even for almost complex manifolds one can define operators ∂ and $\bar{\partial}$ by

$$\partial\omega^{(p,q)} = (\lambda_3)^{(p+1,q)} \ , \qquad \bar{\partial}\omega^{(p,q)} = (\lambda_2)^{(p,q+1)} \ , \tag{1.30}$$

but only for complex manifolds will we have that $\mathrm{d} = \partial + \bar{\partial}$.

2 Complex manifolds

In this chapter, we study complex manifolds in some more detail, and discuss some examples. Furthermore, we state the conditions for an almost complex manifold to be complex.

We start by repeating the definition of complex manifolds:

Definition: Complex manifolds are differentiable manifolds with a holomorphic atlas. They are necessarily of even dimension, say $2n$, and allow for a collection of charts (U_j, z_j) that are one to one maps of the corresponding U_j to $\mathbb{C}^n$ such that for every non-empty intersection $U_j \cap U_k$ the maps are $z_j z_k^{-1}$ are holomorphic.

Example 1: The (unit) two-sphere S^2, which is the subset of $\mathbb{R}^3$, defined by

$$x^2 + y^2 + z^2 = 1 \ , \tag{2.1}$$

is a complex manifold. One can use stereographic projection from the North Pole to the real plane $\mathbb{R}^2$ with coordinates X, Y given by

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right) \ . \tag{2.2}$$

This can be done for any point except the North Pole itself (corresponding to $z = 1$). To include the North Pole, we introduce a second chart, in which we stereographically project from the South Pole:

$$(U, V) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right) \ , \tag{2.3}$$

which holds for any point on S^2 except for the South Pole (at $z = -1$). In both patches, we can now define complex coordinates

$$Z = X + iY \ , \quad \bar{Z} = X - iY \ , \quad W = U - iV \ , \quad \bar{W} = U + iV \ , \tag{2.4}$$

and show that on the overlap of the two patches, the transition function is holomorphic. Indeed, on the overlap we compute that

$$W = \frac{1}{Z} \ . \tag{2.5}$$

This expression relates the coordinate W to Z in a holomorphic way. Hence the two-sphere is a complex manifold which can be identified with $\mathbb{C} \cup \{\infty\}$.

Theorem: Any orientable two-dimensional Riemannian manifold is a complex manifold.

<u>Proof</u>: By definition, Riemannian manifolds admit a positive definite metric. In two dimensions, one can always choose coordinates x, y in a neighborhood of any point such that the metric tensor is of the form

$$\mathrm{d}s^2 = \lambda^2(x, y)\big(\mathrm{d}x^2 + \mathrm{d}y^2\big) \ , \tag{2.6}$$

for a given real function λ. In complex coordinates $z = x + iy$, this reads

$$\mathrm{d}s^2 = \lambda^2 (z, \bar{z})\, \mathrm{d}z \mathrm{d}\bar{z} \ . \tag{2.7}$$

In another patch, with complex coordinate $w = u + iv$, one can similarly write

$$\mathrm{d}s^2 = \mu^2 (w, \bar{w})\, \mathrm{d}w \mathrm{d}\bar{w} \ , \tag{2.8}$$

for some function μ. If the two patches overlap, one can change coordinates from w to z,

$$\mathrm{d}w = \frac{\partial w}{\partial z}\mathrm{d}z + \frac{\partial w}{\partial \bar{z}}\mathrm{d}\bar{z} \ , \tag{2.9}$$

and complex conjugate. Since the metric is a tensor, hence globally defined, one can equate (2.7) to (2.8) on the overlap. Using (2.9), this leads to the requirement

$$\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial z} = 0 \ . \tag{2.10}$$

Hence the transition function is either holomorphic or anti-holomorphic. If it were anti-holomorphic, $\partial w / \partial z = 0$, and the Cauchy-Riemann equations then imply

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \ , \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \ , \tag{2.11}$$

and hence the Jacobian takes the form

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & -\frac{\partial u}{\partial x} \end{pmatrix} \ . \tag{2.12}$$

But this means that $\det J < 0$ and so the manifold would not be orientable (by definition). This contradicts the assumption of the theorem, so this means that the transition function is holomorphic ($\partial w / \partial \bar{z} = 0$), and the manifold is complex. QED.

Example 2: The complex projective space $\mathbb{C}P^n$ is a complex manifold of dimension $2n$.

This can be seen as follows: $\mathbb{C}P^n$ is constructed by first considering the space $\mathbb{C}^{n+1}/0$, with coordinates $z^1, ..., z^{n+1}$, where not all the $z^a; a = 1, ..., n+1$ are simultaneously zero. On this space, we quotient by an equivalence relation, by identifying

$$(z^1, ..., z^{n+1}) \approx \lambda(z^1, ..., z^{n+1}) \ , \tag{2.13}$$

for any non-zero complex λ. Any point in $\mathbb{C}^{n+1}/0$ defines a line from the origin through the point $(z^1, ..., z^{n+1})$, and the equivalence relation relates each two points on the line by rescaling. Hence the complex projective space is the set of lines through the origin in $\mathbb{C}^{n+1}$. We can construct an atlas by choosing coordinates defined on the charts

$$U_j = \{z^a; a = 1, ..., n+1 | z^j \neq 0\} \ , \qquad \zeta^a_{[j]} \equiv \frac{z^a}{z^j} \ , \tag{2.14}$$

for fixed j. These $n+1$ charts cover the entire space since the origin was left out in the complex plane. The coordinates $\zeta^a_{[j]}$ are well defined on U_j since $z^j \neq 0$. Furthermore, they are invariant under the equivalence relation (2.13). There are only n independent coordinates since $\zeta^j_{[j]} = 1$, so $\mathbb{C}P^n$ has complex dimension n. On the overlap of two patches $(U_j, \zeta^a_{[j]}) \cap (U_k, \zeta^a_{[k]})$, we have

$$\zeta^a_{[j]} = \frac{z^a}{z^k}\frac{z^k}{z^j} = \frac{\zeta^a_{[k]}}{\zeta^j_{[k]}} \ , \tag{2.15}$$

hence the transition functions are holomorphic. The z^i are called *homogeneous coordinates* whereas the $\zeta^a_{[j]}, \forall j$ are called *inhomogeneous coordinates*.

Exercise 2.1: The weighted projective space $\mathcal{W}P^n_{[\omega_1,...,\omega_{n+1}]}$, of complex dimension n, can be defined similarly as the complex projective space, but with a different equivalence relation,

$$(z^1, z^2, ..., z^{n+1}) \approx (\lambda^{\omega_1} z^1, \lambda^{\omega_2} z^2, ..., \lambda^{\omega_{n+1}} z^{n+1}) \ , \tag{2.16}$$

and λ a complex number, and ω_a positive integers. Compared with $\mathbb{C}P^n$, we rescale the coordinates with a different weight. Show that the weighted projective space is complex.

Definition: Let (M, J) be an almost complex manifold. If the Lie bracket of any two holomorphic vector field is again a holomorphic vector field, then the almost complex structure is said to be *integrable*.

We remind that the Lie bracket is defined on the space of vector fields, and acts on functions according to

$$[X, Y]f \equiv X(Y(f)) - Y(X(f)) \ . \tag{2.17}$$

In local coordinates x^μ, the vector field $Z \equiv [X, Y]$ has components

$$Z^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu \ , \tag{2.18}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$ form a basis on the tangent space T_xM. We further remind, see (1.14) that holomorphic and anti-holomorphic vector fields are those vector fields satisfying $JZ = iZ$ and $J\bar{Z} = -i\bar{Z}$.

Definition of the Nijenhuis tensor: for any two vector fields X, Y, we define the Nijenhuis tensor N as

$$N(X,Y) \equiv [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \ . \tag{2.19}$$

Exercise 2.2: Show that the Nijenhuis tensor can be written, in local coordinates, as

$$N_{\mu\nu}{}^{\rho} = (\partial_\mu J_\nu{}^{\sigma})\, J_\sigma{}^{\rho} - J_\mu{}^{\sigma}(\partial_\sigma J_\nu{}^{\rho}) - (\mu \leftrightarrow \nu) \ . \tag{2.20}$$

As a consequence, on a complex manifold, where the the complex structure can be brought into its canonical and constant form (1.22), the Nijenhuis tensor vanishes.

Theorem: An almost complex structure J on a manifold M is integrable if and only if $N(X,Y) = 0$ for any two vector fields X and Y.

Proof: Clearly, the Nijenhuis tensor can be extended to the complexified tangent space $T_xM^{\mathbb{C}}$. Now, consider any two holomorphic vector fields, Z and W. If the Nijenhuis tensor vanishes for any two vector fields, so it does on holomorphic vector fields. We then find the identity

$$0 = N(Z,W) = 2\Big([Z,W] + iJ[Z,W]\Big) \ . \tag{2.21}$$

But this implies $J[Z,W] = i[Z,W]$ which means that $[Z,W]$ is holomorphic. This proves one side of the theorem.

Conversely, suppose that J is integrable and hence $[Z,W]$ is holomorphic for any two holomorphic vector fields Z and W, that is, $J[Z,W] = i[Z,W]$. Now take any two vector fields X and Y. By means of the projection operators (1.14), we can decompose

$$X = Z + \bar{Z} \ , \qquad Y = W + \bar{W} \ , \tag{2.22}$$

with Z, W holomorphic and $\bar{Z}, \bar{W}$ anti-holomorphic. It is straightforward to show that $N(Z,\bar{W}) = N(\bar{Z},W) = 0$. The remaining component of the Nijenhuis tensor are then

$$N(X,Y) = N(Z,W) + N(\bar{Z},\bar{W}) \ . \tag{2.23}$$

For holomorphic vector fields, we further compute

$$N(Z,W) = 2\Big([Z,W] + iJ[Z,W]\Big) \ , \tag{2.24}$$

but this also vanishes when J is integrable. Similarly for the complex conjugate $N(\bar{Z},\bar{W})$. Hence $N(X,Y) = 0$. This proves the other side of the theorem. QED.

As we mentioned above, for a complex manifold, the Nijenhuis tensor vanishes, and so the almost complex structure is integrable. The converse is in fact also true, as was proven by Newlander and Nirenberg in 1957:

The Newlander-Nirenberg theorem: Let (M, J) be an almost complex manifold. If J is integrable, the manifold M is complex. (Without proof).

As a result of Exercise 2.2, the opposite also holds, namely that all complex manifolds have integrable complex structures. An almost complex structure which is integrable is called a *complex structure*.

Exercise 2.3: With the definition of the Dolbeault-like operators on an almost complex manifold as in (1.30), show that the condition for ∂ to square to zero is that the Nijenhuis tensor vanishes. Show furthermore that for a complex manifold

$$\mathrm{d} = \partial + \bar{\partial} \ . \tag{2.25}$$

This proves the statements made in Comment 1.1 in the previous chapter, and allows us to define the Dolbeault cohomologies on a complex manifold with respect to the operators ∂ and $\bar{\partial}$.

3 Symplectic manifolds

In this chapter we discuss symplectic manifolds. They arise naturally as the phase space in classical Hamiltonian mechanics, as we will see. Moreover, symplectic manifolds form a subset in the space of almost complex manifolds, which is one of the main theorems that we prove in this chapter.

Definition: A symplectic manifold (M, ω) is a manifold M equipped with a non-degenerate closed two-form ω. Such a form is called a symplectic form. In local coordinates x^μ on M,

$$\omega = \omega_{\mu\nu}(x)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ , \qquad \mathrm{d}\omega = 0 \ . \tag{3.1}$$

The condition of being non-degenerate means that $\omega_{\mu\nu}$ is invertible and we denote its inverse by $\omega^{\mu\nu}$ such that

$$\omega^{\mu\nu}\omega_{\nu\rho} = \delta^\mu_\rho \ . \tag{3.2}$$

An invertible antisymmetric matrix has an even number of rows and columns, so symplectic manifolds are therefore necessarily of even real dimension. In more intrinsic terms, without reference to coordinates, the condition of non-degeneracy is equivalent to requiring that the n-th wedge product is nowhere vanishing,

$$\omega^n \equiv \omega \wedge \omega \wedge ... \wedge \omega \neq 0 \ . \tag{3.3}$$

This condition can be rephrased as the determinant of $\omega_{\mu\nu}(x)$ being everywhere nonzero.

Example 1: $\mathbb{R}^{2n}$ is a symplectic manifold. This can be seen by writing $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, with coordinates x^i and $y_i; i = 1, ..., n$ respectively. We then define the two-form

$$\omega = \mathrm{d}x^i \wedge \mathrm{d}y_i \ . \tag{3.4}$$

Clearly, this two-form is globally defined on $\mathbb{R}^{2n}$, it is closed, and non-degenerate. In fact, as a matrix, the symplectic form is given by (in the convention $\omega = \frac{1}{2}\omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$)

$$\omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix} \ . \tag{3.5}$$

Example 2: The cotangent bundle T^*M of a manifold M is symplectic. Suppose that x^i are local coordinates on the base manifold M. On top of the base coordinates x^i, there are fibre coordinates p_i. A one-form in $T^*_x M$ can then be constructed as

$$\theta \equiv p_i \, \mathrm{d}x^i \ . \tag{3.6}$$

θ is called the canonical one-form on the cotangent bundle. So T^*M as a manifold carries local coordinates (x^i, p_i) where the x are coordinates on the base and the p are coordinates in the fibre. A symplectic form can then be defined as

$$\omega \equiv \mathrm{d}\theta = \mathrm{d}p_i \wedge \mathrm{d}x^i \ . \tag{3.7}$$

Clearly, ω is closed and non-degenerate. In classical mechanics, the x^i can be thought of as the coordinates of the particle propagating on a manifold M, and p_i its momenta. The total set of coordinates (x^i, p_i) define the phase space, which is the cotangent bundle of M.

Example 3: The complex projective space $\mathbb{C}P^n$ is symplectic. We will show this in the next chapter, where we show that $\mathbb{C}P^n$ is in fact Kähler, and we prove that Kähler manifolds are symplectic.

Lagrangian submanifold: Consider a symplectic manifold (M, ω) of dimension $2n$. A submanifold L of half dimension (so, $\dim L = n$) is called *Lagrangian* when ω restricted to L is zero, $\omega_{|L} = 0$.

Darboux's Theorem: If (M, ω) is a symplectic manifold of dimension $2n$, there exist coordinates $x^i, y_i; i = 1, ..., n$ in the neighborhood of each point $p \in M$ such that the symplectic form takes the canonical form

$$\omega = \mathrm{d}x^i \wedge \mathrm{d}y_i \ . \tag{3.8}$$

(Without proof, or see e.g. [6]). Notice that this construction of ω depends on the point p, and hence, (3.8) is only true locally in the neighborhood of the point p.

Symplectomorphisms: Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A diffeomorphism $f : M_1 \to M_2$ is called a *symplectomorphism* if

$$f^* \omega_2 = \omega_1 \ , \tag{3.9}$$

where $f^*\omega_2$ is the pull-back of ω_2 under f to M_1. If $(M_1, \omega_1) = (M_2, \omega_2)$, then symplectomorphisms leave the symplectic form invariant. When applied to the phase space in classical mechanics, symplectomorphisms are also called canonical transformations, i.e. those diffeomorphisms that leave the canonical two-form (3.7) invariant. As a consequence of Darboux's theorem, any two symplectic manifolds are *locally* symplectomorphic.

Theorem: Let (M, ω) be a symplectic manifold. Then every differentiable function $H : M \to \mathbb{R}$ determines a vector field X_H which generates a symplectomorphism in the sense of

$$\mathcal{L}_{X_H} \omega = 0 \ , \tag{3.10}$$

where $\mathcal{L}_{X_H}$ is the Lie-derivative along X_H.

<u>Proof</u>: The symplectic two-form defines a map $\omega : TM \to T^*M$, with $\omega(X)$ a one-form in T^*M for any $X \in TM$. Since the symplectic form is non-degenerate, there is also an inverse map, such that there is an isomorphism between one-forms and vector fields. In local coordinates, this statement means that the components of a one-form $X_\mu \mathrm{d}x^\mu$ can be obtained from a vector field $X^\mu \partial_\mu$ (and vice-versa) according to

$$X_\mu = \omega_{\mu\nu} X^\nu \ , \qquad X^\mu = \omega^{\mu\nu} X_\nu \ . \tag{3.11}$$

Consider now a real function H on M. The one-form $\mathrm{d}H$ then defines a vector field with components

$$X^\mu_H = \omega^{\mu\nu} \partial_\nu H \ . \tag{3.12}$$

In a coordinate-free notation, this reads

$$i_{X_H}\omega = \mathrm{d}H \ , \tag{3.13}$$

where i denotes the interior product. The Lie-derivative can then be computed from

$$\mathcal{L}_{X_H}\omega \equiv i_{X_H}\mathrm{d}\omega + \mathrm{d}(i_{X_H}\omega) \ , \tag{3.14}$$

by defintion of the Lie-derivative acting on forms. The first term on the right hand side is zero because ω is closed. The second term is also zero because of (3.13), so $\mathcal{L}_{X_H}\omega = 0$. The Lie-derivative generates infinitesimal diffeomorphisms, and so along X_H it generates infinitesimal symplectomorphisms. Hence we have proven our theorem. QED.

Remark: The function H is called a Hamiltonian and X_H is called a *Hamiltonian vector field*. For more on this, see e.g. [6]. The converse of this theorem is also true, locally. That is, any generator of a symplectomorphism comes from a Hamiltonian vector field. This can be seen from (3.14). Suppose we have a symplectomorphism such that the left hand side is zero for some vector field X. The first term on the right hand side is always zero by means of $\mathrm{d}\omega = 0$. The second term must therefore be zero, which means, by using Poincaré's lemma, that locally, we have (3.13) for some function H.

Poisson brackets: Given a symplectic manifold (M, ω), we consider the space of real functions defined on M. Using the inverse of the symplectic form, we can define Poisson brackets, in local coordinates,

$$\{f, g\} \equiv \frac{\partial f}{\partial x^\mu}\,\omega^{\mu\nu}(x)\,\frac{\partial g}{\partial x^\nu} \ . \tag{3.15}$$

It is clear from the definition that $\{f, g\} = -\{g, f\}$ since $\omega^{\mu\nu}$ is antisymmetric. When we consider f as an hamiltonian, we have that

$$\mathcal{L}_{X_f}(g) = \{g, f\} \ . \tag{3.16}$$

Here, the Lie-derivative acts on functions in the usual way: in local coordinates, $\mathcal{L}_X(g) = X^\mu\partial_\mu g$.

Exercise 3.1: Show that the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \ , \tag{3.17}$$

is satisfied as a consequence of the closure of the symplectic form, $\mathrm{d}\omega = 0$.

As a consequence, all symplectic manifolds are Poisson manifolds (manifolds equipped with a Poisson bracket). Notice that the opposite need not be true. This is because on a Poisson manifold, the matrix $\omega^{\mu\nu}$, defining the Poisson bracket, need not be invertible.

We end this chapter with the following important theorem:

Theorem: Symplectic manifolds are almost complex.

Proof: We assume that the manifold is Riemannian, so there exists a globally defined metric g that we write in local coordinates as

$$g = g_{\mu\nu}(x)\,\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} \ . \tag{3.18}$$

On a symplectic manifold we define a matrix A from

$$\omega(X,Y) = g(AX,Y) \implies A_{\mu}{}^{\nu}(x) = \omega_{\mu\rho}(x) g^{\rho\nu}(x) \ , \tag{3.19}$$

where $g^{\mu\nu}$ is the inverse metric. A can be seen as a linear operator acting on the tangent space T_xM. It is easy to show that this operator is anti-hermitian with respect to the metric:

$$g(AX,Y) = \omega(X,Y) = -\omega(Y,X) = -g(AY,X) = -g(X,AY) \ . \tag{3.20}$$

The operator $AA^{\dagger} = -A^2$ is hermitian and positive definite with respect to g and can therefore be diagonalised with positive eigenvalues on the diagonal. We can therefore take the square-root of this matrix, and its inverse. This allows us to define

$$J = (\sqrt{AA^{\dagger}})^{-1} A \ . \tag{3.21}$$

We then find that

$$J^2 = (AA^{\dagger})^{-1} A^2 = -\mathbb{I} \ , \tag{3.22}$$

where we have used again that $A^{\dagger} = -A$. The tensor J is globally defined, since g and ω are. Hence, J defines an almost complex structure. QED.

Definition: An almost complex structure J is said to be compatible with the symplectic form ω if for all vector fields X, Y we have

$$\omega(JX,JY) = \omega(X,Y) \ , \qquad \omega(X,JX) > 0 \ . \tag{3.23}$$

Corollary: The almost complex structure J in (3.21) is compatible with ω.

<u>Proof</u>: Notice first that $J^\dagger = -J$ and $AJ = JA$. Straightforward computation then yields

$$\omega(JX, JY) = g(AJX, JY) = g(JAX, JY) = g(AX, -J^2Y) = \omega(X, Y) \ , \tag{3.24}$$

for any two vector fields X and Y. Secondly, we have

$$\omega(X, JX) = g(AX, JX) = g(-JAX, X) = g(\sqrt{AA^\dagger}X, X) > 0 \ . \tag{3.25}$$

This completes the proof. QED.

4 Kahler manifolds

To introduce the notion of a Kähler manifold, we first need to define the concept of a Hermitian metric.

Definition: Let M be a complex manifold, with Riemannian metric g and complex structure J. If g satisfies

$$g(JX, JY) = g(X, Y) \ , \tag{4.1}$$

for any two vector fields X and Y, then g is said to be a *Hermitian metric*. The pair (M, g) is called a Hermitian manifold.

Remark: Similarly, if (M, J) is an almost complex manifold, with a metric satisfying (4.1), then g is called an almost Hermitian metric, and (M, g, J) is an almost Hermitian manifold. In this chapter, and the remainder of these lectures, we will only focus on Hermitian manifolds.

Lemma: Holomorphic vector fields Z, W are orthogonal with respect to a Hermitian metric. Similarly for anti-holomorphic vector fields.

<u>Proof</u>: Holomorphic vector fields Z, by definition, satisfy $JZ = iZ$. We then find

$$g(Z, W) = g(JZ, JW) = -g(Z, W) \ , \tag{4.2}$$

hence $g(Z, W) = 0$. Similarly, $g(\bar{Z}, \bar{W}) = 0$, and so the only nonzero elements are of the form $g(Z, \bar{W})$.

Remark: Hermiticity is a condition on the metric, not on the manifold. In local coordinates, a Hermitian metric satisfies

$$g_{\mu\nu} = J_\mu{}^\rho J_\nu{}^\sigma g_{\rho\sigma} \ . \tag{4.3}$$

Exercise 4.1: Show that on a Hermitian manifold, the Hermitian metric can be written in local complex coordinates z^a (and complex conjugate $\bar{z}^a$) as

$$g = g_{a\bar{b}}(z,\bar{z}) \Big(\mathrm{d}z^a \otimes \mathrm{d}\bar{z}^b + \mathrm{d}\bar{z}^b \otimes \mathrm{d}z^a \Big) \ , \tag{4.4}$$

where $g_{a\bar{b}}$ determine the components of a $2n \times 2n$ hermitian matrix representation of the metric g.
Solution: There is various ways of showing this. The easiest is perhaps to use the canonical form of the complex structure from (1.22) and use it in (4.5). From that it follows that $g_{ab} = g_{\bar{a}\bar{b}} = 0$. Furthermore, symmetry of the metric implies $g_{\bar{b}a} = g_{a\bar{b}}$, and under complex conjugation we have $(g_{a\bar{b}})^* = g_{\bar{a}b} = g_{b\bar{a}}$. The desired result then follows immediately.

Theorem 4.1: A complex manifold (M, J) always admits a Hermitian metric.

Proof: If g is any Riemannian metric on M, one can define

$$h(X,Y) \equiv \frac{1}{2}\Big(g(X,Y) + g(JX,JY) \Big) \ . \tag{4.5}$$

It is clear that this metric satisfies (4.1), and h is positive definite when g is. QED

Corollary: A symplectic manifold always admits an almost Hermitian metric. This follows immediately from the fact that symplectic manifolds are almost complex. So one can use the almost complex structure (3.21) in (4.5).

Definition of the fundamental form: Let (M, J, g) be a Hermitian manifold. We can define the fundamental two-form ω as

$$\omega(X,Y) \equiv g(JX,Y) \ . \tag{4.6}$$

That this is a two-form follows from

$$\omega(X,Y) = g(JX,Y) = g(J^2X,JY) = -g(X,JY) = -g(JY,X) = -\omega(Y,X) \ . \tag{4.7}$$

In local real coordinates, the components of the fundamental form are

$$\omega_{\mu\nu}(x) = J_\mu{}^\rho(x)\, g_{\rho\nu}(x) \ , \tag{4.8}$$

and $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Sometimes, the fundamental form is also called the Kähler form. We will not use this terminology unless the manifold itself is Kähler, see below.

Exercise 4.2: Show that the fundamental form can be written in local complex coordinates as

$$\omega = 2ig_{a\bar{b}}\, \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^b \; . \tag{4.9}$$

Hence the fundamental form is of type $(1,1)$.
Solution: This follows directly from using (1.22) and (4.4) in (4.8).

Corollary: The fundamental form is non-degenerate.

Proof: This can be seen most easily in local coordinates. We define ($g^{\mu\nu}$ is the inverse metric),

$$\omega^{\mu\nu} \equiv -g^{\mu\rho} J_\rho{}^\nu = J_\rho{}^\mu g^{\rho\nu} \; , \tag{4.10}$$

where the second step follows from the fact that g is Hermitian (see exercise below). One can easily check that this defines the inverse, in the sense of $\omega^{\mu\rho}\omega_{\rho\nu} = \delta^\mu_\nu$. QED.

Exercise 4.3: Show that for a Hermitian metric g, the inverse metric satisfies

$$g^{\mu\rho} J_\rho{}^\nu = -J_\rho{}^\mu g^{\rho\nu} \; . \tag{4.11}$$

Solution: This follows from the fact that the hermiticity relation (4.5) can be written in matrix form as $g = JgJ^t$, or $J = -gJ^t g^{-1}$. Multiplying from the left with the inverse metric then yields $g^{-1}J = -J^t g^{-1}$, which is the desired result.

Corollary: The fundamental form ω is compatible with J, in the sense of (3.23).

Proof: This follows straightforwardly from

$$\omega(JX, JY) = g(J^2X, JY) = g(J^3X, J^2Y) = g(JX, Y) = \omega(X, Y) \; , \tag{4.12}$$

where after the second equality, we have used the fact that g is Hermitian. Moreover, we have

$$\omega(X, JX) = g(JX, JX) = g(X, X) > 0 \; . \tag{4.13}$$

QED.

Definition of a Kähler manifold: Let M be a complex manifold with Hermitian metric g and fundamental two-form ω. If ω is closed,

$$\mathrm{d}\omega = 0 \; , \tag{4.14}$$

then M is called a Kähler manifold, g the Kähler metric, and ω the Kähler form.

Remark: When (M, g, J) is an almost Hermitian manifold, with closed fundamental two-form ω, then M is called *almost Kähler*.

Example 1: All complex manifolds of real dimension 2 are Kähler. This follows from the fact that complex manifolds are Hermitian, and that any two-form ω on a two-dimensional manifold is closed.

Clearly, all Kähler manifolds are also symplectic, since the Kähler form is closed and non-degenerate. The opposite need not be true, but we have the following theorem:

Theorem 4.2: Let (M, ω, J) be a symplectic manifold with a compatible, integrable complex structure. Then M is Kähler.

Proof: To proof this, we need to construct a Hermitian metric whose fundamental form is precisely the symplectic form ω. We define a metric

$$g(X, Y) \equiv -\omega(JX, Y) \ . \tag{4.15}$$

This metric is symmetric since ω and J are compatible,

$$g(Y, X) = -\omega(JY, X) = -\omega(J^2Y, JX) = \omega(Y, JX) = -\omega(JX, Y) = g(X, Y) \ . \tag{4.16}$$

The metric is also positive definite,

$$g(X, X) = -\omega(JX, X) = \omega(X, JX) > 0 \ , \tag{4.17}$$

as a consequence of the compatibility between J and ω. Finally, the metric is also Hermitian since

$$g(JX, JY) = -\omega(J^2X, JY) = -\omega(J^3X, J^2Y) = -\omega(JX, Y) = g(X, Y) \ . \tag{4.18}$$

Finally, that ω is the fundamental form for g follows from

$$g(JX, Y) = -\omega(J^2X, Y) = \omega(X, Y) \ . \tag{4.19}$$

The fundamental form is closed because the symplectic form is closed. This shows that M is Kähler. QED.

The Kähler potential: The fact that the fundamental form is closed has important consequences. In a given patch U_i, we can choose local complex coordinates for which we have (4.9). We then compute

$$\mathrm{d}\omega = i(\partial_c g_{a\bar b})\,\mathrm{d}z^c\wedge \mathrm{d}z^a\wedge \mathrm{d}\bar z^b + i(\partial_{\bar c} g_{a\bar b})\,\mathrm{d}\bar z^c\wedge \mathrm{d}z^a\wedge \mathrm{d}\bar z^b\ , \tag{4.20}$$

which consists of a (2,1) part and an (1,2) part. These parts must separately vanish,

$$\partial_c g_{a\bar b} - \partial_a g_{c\bar b} = 0\ ,\qquad \partial_{\bar c} g_{a\bar b} - \partial_{\bar b} g_{a\bar c} = 0\ . \tag{4.21}$$

This implies that, locally in the patch U_i, there must exist a function $K_i(z,\bar z)$, called the *Kähler potential* such that

$$g_{a\bar b} = \partial_a\partial_{\bar b} K_i\ . \tag{4.22}$$

Similarly the Kähler form can locally be written as

$$\omega = i\partial\bar\partial K_i\ . \tag{4.23}$$

Using $\mathrm{d} = \partial+\bar\partial$ and $\mathrm{d}^2 = \partial^2 = \bar\partial^2 = 0$ on a complex manifold, it follows that $\partial\bar\partial = -\frac{1}{2}\,\mathrm{d}\,(\partial-\bar\partial)$, and hence the Kähler form would be locally exact. However, the Kähler form need not be globally exact because the Kähler potential is only defined in the patch U_i. Similarly, the metric is globally defined, but only of the form (4.22) in each U_i. On the overlap of two patches $U_i\cup U_j$, the functions K_i and K_j need not be equal to each other, but can be related by a *Kähler transformation*,

$$K_i(z,\bar z) = K_j(z,\bar z) + f_{ij}(z) + \bar f_{ij}(\bar z)\ , \tag{4.24}$$

where $f_{ij}(z)$ is a holomorphic function. Notice that adding holomorphic or anti-holomorphic functions to the Kähler potential does not change the metric.

Example 2: The complex projective space $\mathbb{C}P^n$ is a Kähler manifold.

We remind from (2.14) that $\mathbb{C}P^n$ is a complex manifold, and we can choose coordinates in the neighborhoods U_j,

$$U_j = \{z^a; a=1,...,n+1|z^j\neq 0\}\ ,\qquad \zeta^a_{[j]}\equiv\frac{z^a}{z^j}\ , \tag{4.25}$$

for fixed j. Define now

$$K_j\equiv\log\Big(\sum_{a=1}^{n+1}|\zeta^a_{[j]}|^2\Big)\ . \tag{4.26}$$

On the overlap $U_j\cup U_k$, we have (2.15) and hence

$$K_j = K_k - \log\,\zeta^j_{[k]} - \log\,\bar\zeta^j_{[k]}\ , \tag{4.27}$$

and therefore (4.24) is satisfied. This means we can define a globally defined metric

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K_j = \partial_a \partial_{\bar{b}} K_k \ , \tag{4.28}$$

and similarly for the Kähler form. The Kähler metric with Kähler potential (4.26) is called the Fubini-Study metric. It remains to be shown that this metric is positive definite.

Exercise 4.4: Show that the Fubini-Study metric is positive definite.
Solution: It suffices to work in the patch $j = n + 1$. The Kähler potential is then

$$K = \log \Big(1 + z^1 \bar{z}^1 + ... + z^n \bar{z}^n \Big) \ . \tag{4.29}$$

Straightforward computation of the metric components yields

$$K_{a\bar{b}} = \frac{1}{(1 + |\vec{z}|^2)^2} \Big(\delta_{a\bar{b}} (1 + |\vec{z}|^2) - z_a \bar{z}_b \Big) \ , \tag{4.30}$$

where now $a = 1, ..., n$ and $\vec{z} = (z^1, ..., z^n)$. Positive definiteness then follows from the Schwarz inequality.

Remark: It can be shown that $\mathbb{C}P^n$ is compact.

Definition: Let (M, g) be any Riemannian manifold. In local real coordinates x^μ, we define the Christoffel symbols as

$$\Gamma_{\mu\nu}{}^{\rho} \equiv \frac{1}{2} g^{\rho\sigma} \Big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \Big) \ . \tag{4.31}$$

They form the components of a torsion free connection ∇, called the Levi-Civita connection, which preserves the metric, $\nabla g = 0$.

Remark: In local coordinates, the connection defines a covariant derivative on tensors. E.g., for a $(1, 1)$ tensor T, we have

$$\nabla_\mu T_\nu{}^\rho \equiv \partial_\mu T_\nu{}^\rho + \Gamma_{\mu\sigma}{}^\rho T_\nu{}^\sigma - \Gamma_{\mu\nu}{}^\sigma T_\sigma{}^\rho \ . \tag{4.32}$$

These covariant derivatives are the components of the tensor $\nabla_X T$, where $X = \partial / \partial x^\mu$.

Lemma: In complex coordinates z^a and $\bar{z}^a$ on a Kähler manifold, the only non-vanishing components of the Christoffel symbols are

$$\Gamma_{ab}{}^c = (\partial_a g_{b\bar{d}}) g^{\bar{d}c} \ , \qquad \Gamma_{\bar{a}\bar{b}}{}^{\bar{c}} = (\Gamma_{ab}{}^c)^* \ . \tag{4.33}$$

Furthermore, one has the property

$$\Gamma_{ab}{}^{b} = \partial_a[\log\sqrt{\det g}] \ . \tag{4.34}$$

Proof: This follows from straightforward computation, where one uses Hermiticity ($g_{ab} = g_{\bar{a}\bar{b}} = 0$) and Kählarity $\partial_a g_{b\bar{c}} = \partial_b g_{a\bar{c}}$ (and complex conjugate). Furthermore, we have

$$\Gamma_{ab}{}^{b} = (\partial_a g_{b\bar{c}}) g^{\bar{c}b} = \frac{1}{2}\mathrm{Tr}\,[g^{-1}(\partial_a g)] = \frac{1}{2}\partial_a[\log \det g] \ , \tag{4.35}$$

which proves (4.34). QED.

Theorem 4.3: A Hermitian manifold (M, g, J) is Kähler if and only if the complex structure satisfies

$$\nabla J = 0 \ , \tag{4.36}$$

where ∇ is the Levi-Civita connection associated with g.

Proof: We find it most convenient to give the proof in local coordinates. First, suppose that $\nabla J = 0$ on a Hermitian manifold. This means that their components are covariantly constant, $\nabla_\mu J_\nu{}^\sigma = 0$. Since also the metric components are covariantly constant, we have that the components of the fundamental form satisfy

$$\nabla_\mu \omega_{\nu\rho} = 0 \ . \tag{4.37}$$

But this implies $\mathrm{d}\omega = 0$. Hence M is Kähler.
Now suppose M is Kähler, and we want to prove that $J_\mu{}^\rho$ is covariantly constant. This is most easily checked using the complex coordinates in which J takes its canonical form. Using the above Lemma, we have

$$\nabla_a J_b{}^c = \partial_a J_b{}^c + \Gamma_{ad}{}^c J_b{}^d - \Gamma_{ab}{}^d J_d{}^c = 0 \ , \tag{4.38}$$

where we have used also that $J_a{}^b = i\delta_a^b$. Similarly for the complex conjugate. Remains to be checked that also $\nabla_{\bar{a}} J_b{}^c = 0$, and its complex conjugate. Using the above Lemma again, this is straightforward. QED.

Definition of the Riemann curvature: The curvature R of a connection ∇ is a rank $(1,3)$ tensor with components, in local coordinates,

$$R_{\mu\nu\rho}{}^\sigma \equiv \partial_\mu \Gamma_{\nu\rho}{}^\sigma - \Gamma_{\mu\rho}{}^\tau \Gamma_{\nu\tau}{}^\sigma - (\mu \leftrightarrow \nu) \ . \tag{4.39}$$

Exercise 4.5: Show that on a Kähler manifold we have

$$R_{abc}{}^{d} = R_{abc}{}^{\bar{d}} = R_{ab\bar{c}}{}^{d} = R_{ab\bar{c}}{}^{\bar{d}} = 0 \ , \tag{4.40}$$

and complex conjugate. Show that the only non-vanishing components are given by

$$R_{\bar{a}bc}{}^{d} = -R_{b\bar{a}c}{}^{d} = \partial_{\bar{a}}\Gamma_{bc}{}^{d} \ , \qquad R_{a\bar{b}\bar{c}}{}^{\bar{d}} = -R_{\bar{b}a\bar{c}}{}^{\bar{d}} = \partial_{a}\Gamma_{\bar{b}\bar{c}}{}^{\bar{d}} \ . \tag{4.41}$$

Notice that these components are symmetric in b and c.

Theorem 4.4: Let (M, g) be a Kähler manifold of complex dimension n. Then the holonomy group of M with respect to the Levi-Civita connection is contained in $U(n)$. (Without proof, but one can show that this is a consequence of (4.41), given that the Lie-algebra of the holonomy group is generated by the curvature tensor (as a consequence of the celebrated Ambrose-Singer theorem)).

The Ricci-tensor and Ricci-form of a Kähler manifold:

The Ricci-tensor on a Riemannian manifold is defined by

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} \ , \tag{4.42}$$

and is symmetric in its indices. On a Kähler manifold, this takes a particularly simple form. The components with both holomorphic indices are zero, and similarly for both anti-holomorphic indices,

$$R_{ab} = R_{\bar{a}\bar{b}} = 0 \ , \tag{4.43}$$

while the non-vanishing components are

$$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} \, [\log \sqrt{\det g}] \ , \tag{4.44}$$

where we have used (4.34). From this, we can define the *Ricci-form*

$$\mathcal{R} \equiv i R_{a\bar{b}} \, \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^b = i \, \partial\bar{\partial} \log \sqrt{g} \ , \tag{4.45}$$

where we denoted $g = \det g$ for brevity. This implies in particular that the Ricci-form is closed,

$$\mathrm{d}\mathcal{R} = 0 \ . \tag{4.46}$$

Using $\partial\bar{\partial} = -\frac{1}{2}\mathrm{d}(\partial - \bar{\partial})$, it follows that the Ricci-form is locally exact. However, it is not globally exact since $\sqrt{g}$ is not a coordinate scalar, but tranforms as a density (i.e. like the inverse of $\mathrm{d}^{2n}x$, such that $\mathrm{d}^{2n}x \sqrt{g}$ is a volume form).

Definition of the first Chern class:

The Ricci-form defines a cohomology class,

$$c_1 = \left[\frac{1}{2\pi} \mathcal{R} \right] . \tag{4.47}$$

The theory of Chern classes is beyond the scope of these lectures, so it is not clear what the meaning of (4.47) is beyond its definition. In particular, the Chern class tells something about the manifold, not about the metric. In fact, one can show that c_1 is a topological invariant, that is, invariant under deformations of the metric. The first Chern class is often used to define Calab-Yau manifolds. We return to this issue in the next chapter.

5 Calabi-Yau manifolds

The study of Calabi-Yau (CY) manifolds has been very intense over the last two decades. One of the reasons is its appearance in string theory, where it serves as the manifold to compactify string theory from ten to four dimensions (our three spatial dimensions and one time). CY manifolds appear in all even dimensions, but in string theory mainly threefolds (of real dimension six) appear. The literature on CY spaces is enormous and advanced, and in this chapter we only give a first taste of its geometrical properties. Important properties like the Hodge structure and harmonic forms of CY manifolds are not discussed here since it requires a separate course, see e.g. [4] for an excellent review.

There are many ways to define what is a CY manifold, and all these definitions are equivalent. In these lectures, we have defined certain classes of manifolds by stating that they admit certain tensors or forms with certain properties. In the same spirit, we define CY manifolds:

Definition: A compact Kähler manifold of complex dimension n is called *Calabi-Yau* if it admits a nowhere vanishing holomorphic n-form Ω. In local complex coordinates z^a and $\bar{z}^a; a = 1, ..., n$, we can write

$$\Omega = \Omega_{a_1 \cdots a_n}(z)\, \mathrm{d}z^{a_1} \wedge \cdots \mathrm{d}z^{a_n} \ , \tag{5.1}$$

where the coefficients are holomorphic functions. E.g. on a threefold, we have

$$\Omega = \Omega_{abc}(z)\, \mathrm{d}z^a \wedge \mathrm{d}z^b \wedge \mathrm{d}z^c \ . \tag{5.2}$$

Theorem: The holomorphic form Ω is closed.

Proof: We use that CY manifolds are complex, and hence $\mathrm{d} = \partial + \bar{\partial}$. Since Ω is holomorphic, we have

$$\bar{\partial}\Omega = 0 \ . \tag{5.3}$$

Now, Ω is also a $(n,0)$-form, so $\partial\Omega$ would be an $(n+1,0)$ form. But since n is the complex dimension, there cannot be $(n+1,0)$ forms, hence $\partial\Omega = 0$. We conclude that

$$\mathrm{d}\Omega = \partial\Omega = \bar{\partial}\Omega = 0 \ . \tag{5.4}$$

This proves the theorem. QED.

Corollary: Ω is unique up to constant rescalings.

Proof: The proof goes as follows. Suppose there is another holomorphic and globally defined n-form $\tilde{\Omega}$ which is nowhere vanishing. Since the components are totally antisymmetric in n indices, they must be proportional to the permutation symbol, see (5.7) below. Hence there must be a non-singular holomorphic function $h(z)$ such that

$$\tilde{\Omega}(z) = h(z)\,\Omega(z) \ . \tag{5.5}$$

But on a compact complex manifold there cannot be a globally defined holomorphic function, except for the constant function. This is a generalization of the fact that for a holomorphic function in one complex variable, its modulus $|h(z)|^2 \equiv h(z)\bar{h}(\bar{z})$ cannot have a maximum or minimum, so h must be constant. We conclude therefore that h in (5.5) is constant, and hence it shows that the holomorphic n-form is unique up to overall rescalings.

Remark: Ω is also co-closed, and therefore it is harmonic.

Theorem: Calabi-Yau manifolds have vanishing first Chern class (4.47), $c_1 = 0$.

Proof: By definition, the CY has a nowhere vanishing holomorphic n form Ω (5.1). We can define from it a real and positive function

$$||\Omega||^2 \equiv \frac{1}{n!}\,\Omega_{a_1\cdots a_n}(z)\,g^{a_1\bar{b}_1}\cdots g^{a_n\bar{b}_n}\,\bar{\Omega}_{\bar{b}_1\cdots\bar{b}_n}(\bar{z}) \ . \tag{5.6}$$

Since $\Omega_{a_1\cdots a_n}$ is totally antisymmetric in all its n indices, and is maximal (that is, the indices run over n values), it must be proportional to the permutation symbol. So within the given coordinate patch, we can write

$$\Omega_{a_1\cdots a_n}(z) = f(z)\,\epsilon_{a_1\cdots a_n} \ , \tag{5.7}$$

with $f(z)$ a holomorphic and nowhere vanishing function. Plugging this into (5.6), we find

$$||\Omega||^2 = |f|^2(\sqrt{g})^{-1} \ . \tag{5.8}$$

Since Ω is nowhere vanishing, we can write this as

$$\sqrt{g} = \frac{|f|^2}{||\Omega||^2} \ , \tag{5.9}$$

Since CY manifolds are Kähler, we can plug this equation into the general expression of the Ricci-form (4.45). This yields

$$\mathcal{R} = -i\partial\bar{\partial} \log\left(||\Omega||^2\right) \ . \tag{5.10}$$

Since Ω is globally defined, we conclude that the Ricci-form is exact and hence trivial in cohomology. We conclude that therefore the first Chern class is zero, $c_1 = 0$. QED.

Remark: The opposite theorem also holds, i.e. on a $2n$-dimensional compact Kähler manifold with vanishing first Chern class, one can define a nowhere vanishing holomorphic n-form, so the two properties are equivalent. One can prove this by using Yau's theorem, stating that every CY admits a Ricci-flat metric. Most often, in the literature, one defines a CY manifold as a compact Kähler manifold with vanishing first Chern class. A proper understanding of all this requires some more advanced mathematics than treated in these lectures. See e.g. [4] for more information.

Example 1: "CY_1". Elliptic curves: the torus.

The simplest examples of CY manifold arise in real dimension two. We have already seen that any two-dimensional orientable manifold is complex. In fact, it follows immediately that they are also Kähler, since the fundamental two-form is automatically closed in two dimensions. Hence all Riemann surfaces are Kähler, and we consider the compact ones since this appears in the definition of CY manifolds. The existence of a nowhere vanishing and globally defined one-form puts further restrictions on the Riemann surface. In fact, only the genus one surface (a torus) admits a globally defined holomorphic one-form. They can be represented as an algebraic curve in two complex variables x and y satisfying

$$y^2(x) = x^3 + ax + b \ , \tag{5.11}$$

where a and b are real constants. The unique (up to rescalings) holomorphic one-form on the torus can be written as

$$\Omega = \frac{\mathrm{d}x}{2y(x)} \ . \tag{5.12}$$

The factor of 2 is purely conventional. Defining $f(x,y) \equiv y^2 - x^3 - ax - b$, we have

$$\Omega = \frac{\mathrm{d}x}{\partial f/\partial y} = -\frac{\mathrm{d}y}{\partial f/\partial x} \ . \tag{5.13}$$

It can be shown that this one-form is globally defined and nowhere vanishing (see exercise below).

One can actually rewrite (5.11) as a hypersurface in complex projective space $\mathbb{C}P^2$. Indeed, start with three homogeneous coordinates z_0, z_1, z_2 on $\mathbb{C}P^2$, one can define, in the patch where $z_0 \neq 0$, inhomogeneous coordinates

$$x = \frac{z_1}{z_0} \ , \qquad y = \frac{z_2}{z_0} \ , \tag{5.14}$$

such that (5.11) becomes

$$F(z_0, z_1, z_2) \equiv z_2^2 z_0 - z_1^3 - a z_1 z_0^2 - b z_0^3 = 0 \ . \tag{5.15}$$

This is a cubic polynomial equation in three complex variables, and defines a particular hypersurface in $\mathbb{C}P^2$. We furthermore have that $F(z_0, z_1, z_2) = z_0^3 f(x,y)$, and so we might have used F instead of f in (5.13) since the difference is an overall constant rescaling with z_0.

Exercise 5.1: Use the description of $\mathbb{C}P^2$ to show that Ω, as in (5.12), is globally defined and nowhere vanishing. Do this by constructing the holomorphic forms locally in the different patches, similar to (5.13), and show that on the overlap of two patches, the forms agree.

Solution: Denote first $\Omega = \Omega_0$ since the inhomogeneous coordinates x and y are defined only in the patch construct a holomorphic form in the patch U_1 in which $z_1 \neq 0$. To do so, we first write $F(z_0, z_1, z_2) = z_1^3(\tilde{x}\tilde{y}^2 - 1 - a\tilde{x}^2 - b\tilde{x}^3) \equiv z_1^3 g(\tilde{x}, \tilde{y})$ where $\tilde{x} = z_0/z_1$ and $\tilde{y} = z_2/z_1$. We then construct

$$\Omega_1 \equiv -\frac{\mathrm{d}\tilde{x}}{\partial g/\partial \tilde{y}} = -\frac{\mathrm{d}\tilde{x}}{2\tilde{x}\tilde{y}} = \frac{\mathrm{d}\tilde{y}}{\tilde{y}^2 - 2a\tilde{x} - 3b\tilde{x}^2} = \frac{\mathrm{d}\tilde{y}}{\partial g/\partial \tilde{x}} \ . \tag{5.16}$$

On the overlap $U_0 \cap U_1$ we have $\tilde{x} = 1/x$ and $\tilde{y} = y/x$, and we compute

$$\Omega_1 = \Omega_0 \ . \tag{5.17}$$

This means we have extended the one-form correctly over the patches U_0 and U_1. The third patch U_2 is the one in which $z_2 \neq 0$, so we define $x' = z_0/z_2; y' = z_1/z_2$ and write $F(z_0, z_1, z_2) = z_2^3(x' - y'^3 - a x'^2 y' - b x'^3) \equiv z_2^3 h(x', y')$. We the define the holomorphic one-form

$$\Omega_2 \equiv \frac{\mathrm{d}x'}{\partial h/\partial y'} = -\frac{\mathrm{d}x'}{3y'^2 + a x'^2} \ . \tag{5.18}$$

On the overlap $U_0 \cap U_2$ we have $x' = 1/y$ and $y' = x/y$, and so

$$\Omega_2 = \frac{\mathrm{d}y}{3x^2 + a} = \frac{\mathrm{d}x}{2y} = \Omega_0 \ . \tag{5.19}$$

One can repeat this also on the overlap $U_1 \cap U_2$ to finally show that Ω is extended globally into a holomorphic one-form, and nowhere vanishing.

Exercise 5.2: Consider the hypersurface in $\mathbb{C}P^2$ defined by

$$F(z_0, z_1, z_2) = z_0^n + z_1^n + z_2^n = 0 \ , \tag{5.20}$$

for some positive integer n. Using inhomogeneous coordinates x and y in the patch where $z_0 \neq 0$, we can write this as $F(z_0, z_1, z_2) = z_0^n f(x, y)$ with $f(x, y) = x^n + y^n + 1$, and we can use this function f to define a holomorphic one-form like in (5.13). Show now that this one-form is only globally defined when $n = 3$.
Solution: First of all, we compute from

$$\Omega_0 = \frac{1}{n}\frac{\mathrm{d}x}{y^{n-1}} = -\frac{1}{n}\frac{\mathrm{d}y}{x^{n-1}} \ , \tag{5.21}$$

where $x = z_1/z_0$ and $y = z_2/z_0$ are coordinates on the patch U_0 in $\mathbb{C}P^2$ in which $z_0 \neq 0$. We can repeat this we define coordinates $\tilde{x} = z_0/z_1$ and $\tilde{y} = z_2/z_1$, write $F(z_0, z_1, z_2) = z_1^n(1 + \tilde{x}^n + \tilde{y}^n) = z_1^n f(\tilde{x}, \tilde{y})$, and write a holomorphic form

$$\Omega_1 = -\frac{1}{n}\frac{\mathrm{d}\tilde{x}}{\tilde{y}^{n-1}} \ . \tag{5.22}$$

On the overlap $U_0 \cap U_1$, we have $\tilde{x} = 1/x$ and $\tilde{y} = y/x$, and find

$$\Omega_1 = x^{n-3}\,\Omega_0 \ . \tag{5.23}$$

and Ω_0 only if $n = 3$. We can repeat this for the third patch in which $z_2 \neq 0$ and find the same result. In this way, we have defined the holomorphic one-form globally, and it vanishes nowhere.

Remark: Any cubic polynomial in $\mathbb{C}P^2$ leads to a Calabi-Yau. A generic cubic polynomial can be written as

$$\sum_{i,j,k} a_{ijk} z_i z_j z_k = 0 \ , \tag{5.24}$$

where a_{ijk} is complex and symmetric in its three indices. Such an object contains 10 complex independent parameters. However, we can act with the general linear group $GL(3, \mathbb{C})$ on the variables z_i which leaves the form of the polynomial (5.24) invariant. The dimension of $GL(3,\mathbb{C})$ is 9, hence we can eliminate 9 out of the 10 parameters in the a_{ijk}. We therefore end up with just one complex parameter, which is also called *modulus*. It is known that a torus has one complex modulus, often denoted by τ. In the equation for the elleptic curve, we have traded τ for two real parameters a and b.

The construction of explicit examples of CY manifolds in higher dimensions is non-trivial, and a classification is at present unknown. The method of constructing CY manifolds as hypersurfaces in complex projective space turns out to be also applicable in higher dimensions, and a large class of them are constructed and studied in this way. We illustrate this (rather briefly) in the case of four and six dimensions below.

Example 2: "CY_2". $K3$. Fermat's quartic.

In four dimensions, Calabi-Yau manifolds with the additional requirement of being simply connected, are also called $K3$ surfaces. The condition of simply connectedness rules out the four-torus T^4 as a $K3$ surface. There is an important theorem stating that any two $K3$ surfaces are diffeomorphic to each other (we will not attempt to proof this here). This implies that all topological properties of $K3$ surfaces can be obtained by considering one example. A class of $K3$'s, but certainly not all of them, can be constructed as hypersurfaces in $\mathbb{C}P^3$. The simplest example is to consider Fermat's quartic, i.e the hypersurface defined as

$$F(z_0,z_1,z_2,z_3) \equiv z_0^4+z_1^4+z_2^4+z_3^4=0\ . \tag{5.25}$$

Clearly this defines a complex two-dimensional submanifold. We could have replaced the fourth power by some other positive integer n. However, the previous example taught us that only for a particular value of n can define a nowhere vanishing holomorphic two-form on it. This value is fixed to be $n=4$. We introduce again inhomogeneous coordinates, in the patch $z_0\neq 0$,

$$y_1=\frac{z_1}{z_0}\ ,\qquad y_2=\frac{z_2}{z_0}\ ,\qquad y_3=\frac{z_3}{z_0}\ , \tag{5.26}$$

and similarly for the other patches. The nowhere vanishing holomorphic two-form in this patch takes the form

$$\Omega=\frac{\mathrm{d}y_1\wedge\mathrm{d}y_2}{\partial f/\partial y_3}=\frac{\mathrm{d}y_2\wedge\mathrm{d}y_3}{\partial f/\partial y_1}=\frac{\mathrm{d}y_3\wedge\mathrm{d}y_1}{\partial f/\partial y_2}\ , \tag{5.27}$$

with $F=z_0^4 f(y_1,y_2,y_3)$ and $f(y_1,y_2,y_3)=y_1^4+y_2^4+y_3^4+1$.

Remark: Similarly to the case of elliptic curves, a class of $K3$ surfaces can be represented by a general quartic polynomial equation

$$\sum_{i,j,k,l} a_{ijkl} z_i z_j z_k z_l = 0 \ . \tag{5.28}$$

The completely symmetric tensor a_{ijkl} contains 35 independent parameters [1], but acting with $GL(4, \mathbb{C})$ on the z_i, with dimension 16, one only has $35-16 = 19$ independent moduli.

Example 3: "CY_3". The quintic.

Calabi-Yau threefolds (dimension six) are perhaps most interesting because of its connection to string theory. Metrics on Calabi-Yau manifolds have not been constructed, and usually CY manifolds are constructed as compact complex submanifolds in $\mathbb{C}P^n$. Such submanifolds can be described as the zero locus of algebraic equations. A generic theory of this construction is beyond the scope of these lectures, but we mention one example, *the quintic*. This is a CY threefold which is the complex submanifold of $\mathbb{C}P^4$ described as the locus of the equation

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \ . \tag{5.29}$$

These coordinates define five complex variables in $\mathbb{C}^5$, but we projectivize to $\mathbb{C}P^4$ by introducing inhomogeneous coordinates (e.g. in the patch in which $z_5 \neq 0$),

$$y_k \equiv \frac{z_k}{z_5} \ , \qquad k = 1, ..., 4 \ . \tag{5.30}$$

In these coordinates, (5.29) can be written as

$$y_1^5 + y_2^5 + y_3^5 + y_4^5 = -1 \ . \tag{5.31}$$

This solution of this equation is a complex three-dimensional compact space and we can eliminate e.g. y_4 in terms of the others. This manifolds turns out to be CY. To show this, one can define a holomorphic three-form

$$\Omega = \frac{1}{y_4^4} \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3 \ . \tag{5.32}$$

That this form is globally defined and nowhere vanishing is a non-trivial exercise that relies on the fact that we took polynomials of fifth degree in (5.29). Again, any quintic polynomial

$$\sum_{ijklm} a_{ijklm} z_i z_j z_k z_l z_m = 0 \ , \tag{5.33}$$

[1] A symmetric tensor with r indices running over n values has $\begin{pmatrix} n+r-1 \\ r \end{pmatrix}$ independent components. In our case, we have $r = n$, so this means $\frac{(2n-1)!}{n!(n-1)!}$.

yields a Calabi-Yau threefold. This time we have 126 parameters, but the action of $GL(5,\mathbb{C})$ eliminates 25 of them, leading to 101 *complex structure moduli*. More information on Calabi-Yau threefolds can be found in [4].

6 Hyperkahler manifolds

In this chapter, we introduce a special class of Kähler manifolds, called *hyperkähler manifolds* that allow for more than one complex structure. They are based on the algebra of quaternions.

Quaternions: Quaternions are elements of a vector space $\mathbb{H}$. This is a four-dimensional vector space over the real numbers and is isomorphic to $\mathbb{R}^4$. The basis elements of $\mathbb{H}$ are denoted by $1, i, j, k$. On $\mathbb{H}$,we define a multiplication that acts on the basis elements as

$$i^2 = j^2 = k^2 = ijk = -1 \ , \tag{6.1}$$

and 1 is the identity operation in $\mathbb{H}$. In particular, (6.15) implies $ij = k, jk = i$ and as a consequence $ik = -j$. Clearly, this multiplication is not commutative. Any element $q \in \mathbb{H}$ can be decomposed as

$$q = a\,1 + b\,i + c\,j + d\,k \ , \tag{6.2}$$

where $(a, b, c, d) \in \mathbb{R}^4$.

Since the multiplication is not commutative, we cannot find a representation of the quaternions in terms of real and imaginary numbers. The simplest representation is based on four-by-four matrices, and a particular choice is

$$i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ , \qquad j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \ , \qquad k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \ , \tag{6.3}$$

and the basis element 1 is just the four-by-four identity matrix. In this way, the quaternionic structure acts on $\mathbb{R}^4$ as a linear operator by matrix multiplication.

Another representation is given by complex two-by-two matrices, based on the Pauli matrices:

$$i = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix} \ , \qquad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad k = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix} \ , \tag{6.4}$$

where $\imath = \sqrt{-1}$ is the imaginary unit.

We can extend the quaternionic structure for higher dimensional vector spaces $\mathbb{H}^n$, which is isomorphic to $\mathbb{R}^{4n}$. We can define a quaternionic structure by linear operators I, J, K acting on $\mathbb{R}^{4n}$ and satisfying the relations

$$I^2 = J^2 = K^2 = IJK = -\mathbb{I} \ , \tag{6.5}$$

and one can think of I, J, K as $4n \times 4n$ matrices with $\mathbb{I}$ the identity matrix. We can rewrite this equation by introducing the vector notation $\vec{J} = (I, J, K)$. The components of this vector $J_i; i = 1, 2, 3$ then satisfy the quaternionic algebra

$$J_i J_j = -\delta_{ij}\mathbb{I} + \epsilon_{ijk} J_k \ , \tag{6.6}$$

where ϵ_{ijk} is the permutation symbol with $\epsilon_{123} = 1$.

Exercise 6.1: Consider a linear combination of the quaternionic structure

$$\mathcal{J} \equiv a_1 J_1 + a_2 J_2 + a_3 J_3 \ , \tag{6.7}$$

with $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$. Show that $\mathcal{J}^2 = -\mathbb{I}$ when

$$|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2 = 1 \ . \tag{6.8}$$

This is the equation for a two-sphere S^2, and for every point on S^2, we have an operator squaring to minus the identity. On $\mathbb{H}^n$, we therefore say that one has *a two-sphere of complex structures*. The quaternionic structure is then only defined up to rotations on S^2.

These considerations allow us to introduce a quaternionic structure for a $4n$ dimensional manifold M, by letting the operators $\vec{J}$ act on the tangent space T_pM which is isomorphic to $\mathbb{R}^{4n}$. It is not guaranteed however that $\vec{J}$ varies smoothly over M, even up to a rotation of the $\vec{J}$. When it does, the manifold has an additional property, and not all $4n$ dimensional manifolds need not to allow for this.

Definition: A manifold $(M, \vec{J})$ of dimension $4n$ that admits a globally defined quaternionic structure satisfying (6.6) is called an *almost quaternionic manifold*.

Remark: By globally defined, we mean non-singular at every point p and varying smoothly on M. The variation can also allow for a rotation of the quaternionic structure, $\vec{J} \to R\vec{J}$; $R \in SO(3)$, when we vary $p \in M$. If there is no such rotation, then the manifold is called *almost hypercomplex*, with an almost hypercomplex structure. Almost hypercomplex manifolds are therefore almost complex, but almost quaternionic manfiolds need not be almost complex. A well-known example is the four-sphere S^4 which does not admit an almost complex structure, but it does admit a quaternionic structure.

Definition: When the hypercomplex structure on an almost hypercomplex manifold is integrable, in the sense of a vanishing Nijenhuis tensor for each of the J_i, then the manfold is called *hypercomplex*. Hypercomplex manifolds are thus in particular complex manifolds.

Examples: Clearly, $\mathbb{H}$ and $\mathbb{R}^4$ are hypercomplex, with the hypercomplex structure given by (6.3). The four-torus T^4 is hypercomplex. The product manifold $S^1 \times S^3$ is also hypercomplex. (Without proof).

Remark: There also the possibility to define a quaternionic manifold. This arise when the Nijenhuis tensor for the quaternionic structure on an almost quaternionic manifold satisfies a particular property. S^4 is a quaternionic manifold. We will not discuss these spaces in our lectures.

We now proceed in a similar way as for Kähler manifolds.

Definition: When a metric g on a hypercomplex manifold satisfies

$$g(J_i X, J_i Y) = g(X, Y)\ , i = 1, 2, 3\ , \tag{6.9}$$

for any two vector fields X and Y, we call g a hyper-Hermitian metric.

Theorem: A hypercomplex manifold $(M, \vec{J})$ always admits a hyper-Hermitian metric.

Proof: The proof is similar as for hermitian metrics on complex manifolds. If g is any Riemannian metric on M, we can define

$$h(X, Y) \equiv \frac{1}{4}\Big[g(X, Y) + g(IX, IY) + g(JX, JY) + g(KX, KY)\Big]\ . \tag{6.10}$$

Using the quaternionic algebra, it is then straightforward to check that h is hyper-Hermitian. QED.

Fundamental forms: Let $(M, g, \vec{J})$ be hypercomplex with a hyper-Hermitian metric. We can then define a triplet of fundamental two-forms as

$$\vec{\omega}(X, Y) \equiv g(\vec{J}X, Y)\ . \tag{6.11}$$

Definition of a hyperkähler manifold: Let M be a hypercomplex manifold with a hyper-Hermitian metric g and a triplet of fundamental forms $\vec{\omega}$. When the fundamental forms are closed,

$$\mathrm{d}\vec{\omega} = 0 \ , \tag{6.12}$$

the manifold M is called *hyperkähler*. Notice that hyperkähler manifolds are in particular Kähler.

Theorem: A hypercomplex manifold $(M, \vec{J})$ with hyper-Hermitian metric g is hyperkähler if and only if the complex structures are covariantly constant:

$$\nabla I = \nabla J = \nabla K = 0 \ , \tag{6.13}$$

where ∇ is the Levi-Civita connection.

Proof: First, suppose the complex structures satisfy $\nabla \vec{J} = 0$. Then, it follows that all three ω are closed because, in local coordinates,

$$\vec{\omega}_{\mu\nu} = \vec{J}_\mu{}^\rho g_{\rho\nu} \ , \tag{6.14}$$

implies $\nabla_\mu \omega_{\nu\rho} = 0$ which in its turn implies $\mathrm{d}\omega = 0$. Conversely, we have that a hyperkähler manifold is Kähler with respect to any of the given complex structures I, J and K, so we can repeat the construction in the proof of Theorem 4.4 in each complex structure.

Exercise 6.2: Let M be a hyperkähler manifold, endowed with three integrable complex structures $\vec{J} = (J_1, J_2, J_3) = (I, J, K)$ satisfying the quaternionic algebra relations

$$I^2 = J^2 = K^2 = IJK = -\mathbb{I} \ . \tag{6.15}$$

A hyperkähler manifold admits, by definition, a triplet of closed fundamental two-forms $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$

$$\vec{\omega}(X, Y) \equiv g(\vec{J}X, Y) \ , \tag{6.16}$$

where g is a hyper-Hermitian metric. Now define

$$\omega_+ \equiv \frac{1}{2}(\omega_2 - i\omega_3) \ , \qquad \omega_- \equiv \frac{1}{2}(\omega_2 + i\omega_3) \ . \tag{6.17}$$

- Show that $\omega_+(IX, Y) = i\omega_+(X, Y)$ and $\omega_-(IX, Y) = -i\omega_-(X, Y)$ on any two vector fields X and Y.
- Use these properties to show that ω_+ is of type $(2, 0)$ and ω_- of type $(0, 2)$, with respect to the projection operators $P^\pm = \frac{1}{2}(\mathbb{I} \mp iI)$.

- Give, up to an overall normalization, the hyperkähler two-forms $\omega_\pm$ and ω_1 on $\mathbb{R}^4 \cong \mathbb{C}^2$ in terms of the complex Cartesian coordinates z^1 and z^2.

<u>Solution</u>

- On any two vector fields X and Y, we have

$$\omega_+(X,Y) = \frac{1}{2}\big(g(JX,Y) - ig(KX,Y)\big) \ . \tag{6.18}$$

  We then compute

$$\omega_+(IX,Y) = \frac{1}{2}\big(g(KX,Y) + ig(JX,Y)\big) = i\omega_+(X,Y) \ , \tag{6.19}$$

  and similarly $\omega_-(IX,Y) = -i\omega_-(X,Y)$.

- This follows from the properties

$$\omega_+(P^-X,Y) = \omega_+(X,P^-Y) = 0 \ , \qquad \omega_+(P^+X,Y) = \omega_+(X,P^+Y) = \omega_+(X,Y) \ .$$

  These properties imply $\omega_+(P^-X,P^-Y) = \omega_+(P^+X,P^-Y) = \omega_+(P^-X,P^+Y) = 0$ and $\omega_+(P^+X,P^+Y) = \omega_+(X,Y)$, i.e. ω_+ is $(2,0)$. Similarly ω_- is $(0,2)$, as follows from complex conjugation.

- The two-forms can be constructed as

$$\omega_+ = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \ , \quad \omega_- = \overline{\omega_+} \ , \quad \omega_1 = i\Big(\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 + \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2\Big) \ ,$$

  up to an overall normalization.

Theorem: Hyperkähler manifolds are Ricci-flat, that is, the Ricci-tensor for the Levi-Civita connection vanishes.

Exercise 6.3: Proof this theorem. This is a non-trivial exercise, but with the following hint it becomes tractable: use the fact that for integrable complex structures, we have

$$0 = [\nabla_\mu, \nabla_\nu]\vec{J}_\rho{}^\sigma = -R_{\mu\nu\rho}{}^\tau \vec{J}_\tau{}^\sigma + R_{\mu\nu\tau}{}^\sigma \vec{J}_\rho{}^\tau \ . \tag{6.20}$$

One can show this identity as a separate exercise (the second equality holds in fact for any rank $(1,1)$ tensor). As a second step, one can use the quaternionic algebra and sum over the indices ν and τ to construct the Ricci tensor out of the Riemann curvature.

Corollary: Hyperkähler manifolds have vanishing first Chern class. As a consequence, all compact hyperkähler manifolds are Calabi-Yau manifolds.

Remark: The opposite need not be true, of course. Even when the Calabi-Yau is $4n$-dimensional, it need not be hyperkähler. For $n = 1$, this statement happens to be true when the Calabi-Yau is simply connected, i.e. a $K3$ surface. Simply connected CY_n-folds have holonomy contained in $SU(n)$, and for $n = 1$ we have $SU(2) = Sp(1)$. The statement is then true because of the following theorem:

Theorem: The holonomy group for the Levi-Civita connection of a $4n$-dimensional hyperkähler manifold is contained in the symplectic group $Sp(n, \mathbb{H}) = USp(2n, \mathbb{C})$. (Without proof).

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