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2D AND 3D COMPUTATIONAL MORPHOLOGY ON THE γ-NEIGHBORHOOD GRAPH

Remco Veltkamp
Leiden University, Department of Mathematics and Computer Science
P.O. Box 9512 2300 RA Leiden, The Netherlands
e-mail: veltkamp@hlerul5i.bitnet

ABSTRACT

This paper presents a computational morphology operation performed on a novel parameterized γ-neighborhood graph. In itself, this neighborhood graph is a unification of a range of geometrical graphs. The convex hull, the Delaunay triangulation, the Gabriel graph, the β-neighborhood graph and the complete graph are special instances.

By shrinking from the hull of the γ-neighborhood graph, exploiting the geometric information included in the γ-graph, a Hamilton polygon (2D) or a Hamilton polyhedron (3D) is determined, representing a boundary of a set of points. Shrinking from the γ-neighborhood graph succeeds in cases where shrinking from the Delaunay triangulation fails, either because the Delaunay triangulation contains no Hamilton polygon or polyhedron, or because the process gets locked.

The shrinking process is described as a mathematical morphology operation, where the power set of the input point set is the complete lattice whereupon the operation is performed, instead of a 2D pixel image.

Keywords: geometric modeling, computational geometry, computational morphology, neighborhood graph, Hamilton cycle, mathematical morphology.

INTRODUCTION

A boundary of a given set of points is often needed in order to be able to process this point set. In many applications in geometric modeling, computer graphics, computer vision, distance map image processing and pattern recognition, the input is a set of points from an object's boundary surface, while some boundary of the point set is indispensable for an unambiguous object model, or a visually appealing or realistic display. The boundary of a point set defines a corresponding shape, but conversely, the shape of a point set is an informal notion. When a computational geometrical structure is exploited to extract some shape from the input, it is referred to by Toussaint (1988) as a computational morphology task.

This paper addresses the problem of extracting a likely boundary from a set of points, that passes through all the points. The input is a set of coordinates of points in 2D or 3D. The points are assumed to lie on the boundary of an object without holes. In the 2D case, the output has to be a simple closed polygon, where the set of polygon vertices is the input point set. In graph theoretical terms the required polygon is called a Hamilton cycle. We will also use the term Hamilton polygon. In the 3D case, the output has to be a simple triangular closed polyhedron, where again the set of vertices is the input point set. We call this a Hamilton polyhedron. We will determine a boundary of the point set on the basis of the so called γ-neighborhood graph.

THE γ-NEIGHBORHOOD GRAPH

In the following we call the points from the input set either sites or vertices, in order to distinguish these points from arbitrary points. In the definition of the γ-neighborhood graph we use the following notation: for $k \geq 2$, $r(x_1, \ldots, x_k)$ denotes the radius of the smallest sphere through sites $x_1, \ldots, x_k$ in kD space. Thus for $k = 2$, $r(x_1, x_2)$ equals $d(x_1, x_2)/2$.

The neighborhood graph $\gamma(\gamma_0, \gamma_1)$ is defined for $-1 \leq \gamma_0, \gamma_1 \leq 1, |\gamma_0| \leq |\gamma_1|$. In kD space, the graph connects sites $x_1, \ldots, x_k$ with each other, if there is an empty neighborhood associated with these sites, that is defined by two kD spheres in the following way:

1. the radii of the spheres are $r(x_1, \ldots, x_k)/(1-|\gamma_0|)^{1/(k-1)}$ and $r(x_1, \ldots, x_k)/(1-|\gamma_1|)^{1/(k-1)}$,
2. if $\gamma_0\gamma_1 < 0$, the circumcenters lie on the same side of the plane through $x_1, \ldots, x_k$, if $\gamma_0\gamma_1 > 0$ the circumcenters lie on both sides of that plane,
3. if $\gamma_1 \leq 0$, we take the intersection of the two spheres, if $\gamma_1 \geq 0$, we take the union.

The neighborhood is called empty if no sites lie in its interior, except when a sphere of infinite radius is involved. Such a sphere is considered a half-space with its boundary through $x_1, \ldots, x_k$, and is called empty if no sites lie in the open half-space, nor inside or on the $(k-1)$D polygon through $x_1, \ldots, x_k$, that lies on the boundary of the half-space. For example in 2D, a half-plane through x_1 and x_2 is empty if the open half-plane is empty, and no sites lie on the line segment between x_1 and x_2 (but sites may lie on the boundary of the half-plane outside that segment).

Note that there can be two neighborhoods if γ_0 and γ_1 are both non-zero, and that the graph connects $x_1, \ldots, x_k$, as soon as one neighborhood is empty. Further, this definition is valid for $\gamma_0 = \gamma_1 = 0$. In that case, the two spheres coincide, their common circumcenter lies in the plane through $x_1, \ldots, x_k$, and the intersection equals the union.

For $k = 2$ the definition involves two sites and two circles, and $r(x_1, x_2)$ is scaled by factors $1/(1-|\gamma_0|)$ and $1/(1-|\gamma_1)|$. The planar graph $\gamma(\gamma_0, \gamma_1)$ reduces to well known geometric graphs for special values of γ_0 and γ_1:

- $\gamma_0 = \gamma_1 = 0$. The resulting neighborhood is the smallest circle through x_1 and x_2, also called the Gabriel neighborhood. $\gamma(0,0)$ is the Gabriel graph.
- $\gamma_0 = \gamma_1 = -1$. The intersection of the two half-planes yields the line through x_1 and x_2. If no three or more sites are collinear, then $\gamma(-1,-1)$ is the complete graph.
- $\gamma_0 = \gamma_1 = 1$. The union of the two half-planes gives the entire plane. If no three or more sites are collinear, $\gamma(1,1)$ is a void graph.
- $\gamma_0 = -1, \gamma_1 = 1$ and $\gamma_0 = 1, \gamma_1 = -1$. In both cases the two half-planes lie on the same side of the line through x_1 and x_2, and therefore coincide. The neighborhood is empty if all other sites lie on one side of the line through x_1 and x_2, or on the line, but outside the segment from x_1 to x_2. That occurs only if x_1 and x_2 lie on the convex hull. Therefore, $\gamma(-1,1)$ and $\gamma(1,-1)$ are the convex hull of the set of sites.
- $\gamma_0 = \gamma_1$. The graph $\gamma(\gamma_0, \gamma_0)$ reduces to the circle-based β-neighborhood graph, as introduced by Kirkpatrick and Radke (1985).

Figure 1 gives a graphical overview of the whole spectrum of planar neighborhoods.

In 3D space, the definition involves three sites and two spheres, and $r(x_1, x_2, x_3)$ is scaled by factors $1/(1-|\gamma_0|)^{1/2}$ and $1/(1-|\gamma_1|)^{1/2}$. Because the Gabriel neighborhood is most naturally generalized to kD as the smallest sphere trough two sites, whereas the kD $\gamma(0,0)$-neighborhood involves k sites, these neighborhoods do not coincide for $k > 2$. Further, it is not clear how to relate the higher dimensional $\gamma(\gamma_0, \gamma_0)$-neighborhood to the circle-based β-neighborhood, since Kirkpatrick and Radke (1985) do not tell how to generalize it to higher dimensions. The kD

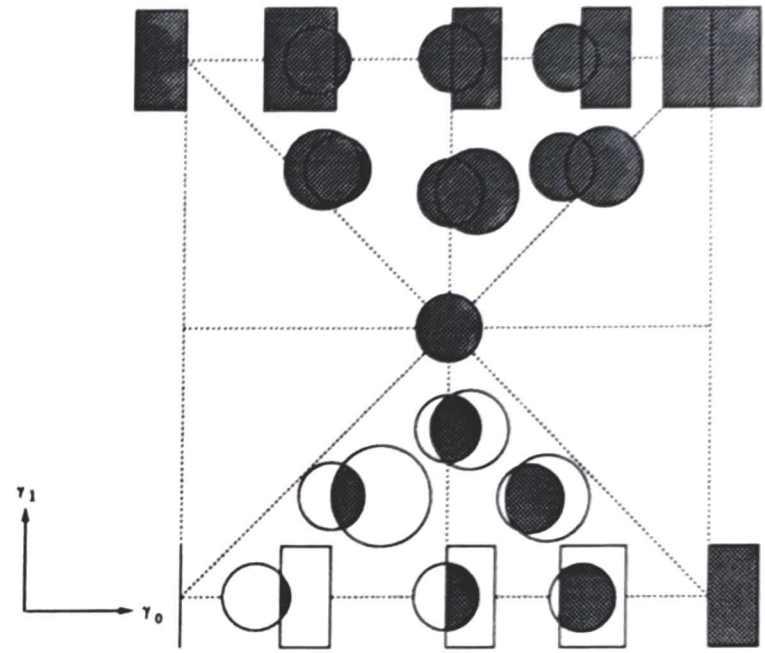

Figure 1. Overview of the spectrum of planar $\gamma(\gamma_0, \gamma_1)$-neighborhoods, $-1 \leq \gamma_0, \gamma_1 \leq 1$, and $-1 \leq \gamma_0/\gamma_1 \leq 1$ (rectangles denote half-spaces).

complete and the void graph however, result from the γ-graph if no $k+1$ or more sites lie in a $(k-1)$D plane, and the convex hull always equals $\gamma(-1,1)$ and $\gamma(1,-1)$. More general, $\gamma(\gamma_0, -\gamma_0) = \gamma(-\gamma_0, \gamma_0)$, and $\gamma(\gamma_0, \gamma_1) \subseteq \gamma(\gamma_2, \gamma_3)$, for $\gamma_0 \geq \gamma_2$ and $\gamma_1 \geq \gamma_3$, also in 2D.

So far we have considered fixed values of the γ-parameters, now we look at the largest values of the γ-parameters, for which the corresponding neighborhood is still empty. That is the value for which the sphere touches a $(k+1)$th site, or is either 1 or -1 if there is no such site. We define $\gamma([\gamma_0, \gamma_1], [\gamma_2, \gamma_3])$ to be the graph connecting sites $x_1, \ldots, x_k$ with each other, if the largest γ-parameter values for which the corresponding neighborhood is still empty, lie in $[\gamma_0, \gamma_1]$ and $[\gamma_2, \gamma_3]$ respectively.

The $\gamma([-1,1],[0,1])$-graph connects sites $x_1, \ldots, x_k$ in kD space if there are two spheres through these sites, of arbitrary radius, such that the union is empty. This is exactly a definition of the famous Delaunay triangulation, if no more than $k+1$ sites are cospherical. If there are more than $k+1$ cospherical sites, $\gamma([-1,1],[0,1])$ connects them all, whereas the Delaunay triangulation arbitrarily connects k sites, as long as the resulting $(k-1)$D-faces do not intersect.

COMPUTATIONAL MORPHOLOGY

In the following we use the general notion *boundary segment* for an edge lying on the boundary in 2D embedding space, and for a triangular face lying on the boundary in 3D. We employ *boundary simplex* for a triangle having a boundary segment in 2D, and for a tetrahedron having a boundary segment in 3D.

In order to obtain a Hamilton polygon or polyhedron, we take $\gamma([-1,1],[0,1])$, and successively remove simplices from the body of the object, by deleting one or more boundary segments from the hull (initially the convex hull). We go on shrinking the hull, until all vertices are included in the boundary. However, removing a simplex may not introduce an isolated vertex, dangling boundary segments, or a self-intersecting boundary. Thus, deletion is allowed only if:

1. the triangle or tetrahedron has one boundary segment, and the vertex opposite to that segment is not on the boundary,

2. or, (in 3D only) the tetrahedron has two boundary faces, and the edge opposite to the common edge of the boundary faces, does not lie on the boundary.

In order to select the simplex to remove, we associate to all current boundary simplices a value that is based on the γ-values of the boundary segments. However, we keep the sign of the γ-values of the boundary segments consistent with the following rule: if $-1 \leq \gamma < 0$, the center of the associated circle or sphere lies on the side of the boundary segment that is outside the current boundary, and if $0 < \gamma \leq 1$, the center lies on the side of the boundary segment that is inside the current boundary.

We first consider the 2D case, where only triangles with one boundary edge may be deleted. The selection of the triangle to remove is based on the attempt to, informally speaking, change the shape of the current boundary not too much, relative to the size of the triangle. Formally, we choose the triangle with the largest interior angle at the vertex opposite to the boundary edge. A second motivation for this selection criterion is that the interior vertex with the widest angle has the largest probability among the interior vertices of all the current boundary triangles to be seen, or sensed, by for example a laser range system.

Let us call the radius of the circle through the vertices of the triangle that we consider, R, the γ-value of the boundary edge corresponding to that triangle, γ, and the two vertices on the boundary, x_1 and x_2. We abbreviate $r(x_1, x_2)$ to r. The angle at the interior vertex, ϕ, increases as r/R increases, if γ is non-negative, and increases as $2 - r/R$ increases, if γ is non-positive. The exact relation is given by the sine rule: $r/R = \sin\phi$. By definition, r/R equals $1 - |\gamma|$, which is $1 - \gamma$ for a non-negative γ. Similarly, $2 - r/R$ expands to $1 + |\gamma|$, which equals $1 - \gamma$ for a non-positive γ. This results in the following selection rule:

> among all removable triangles, delete the one whose boundary edge has the largest value for $1 - \gamma$ (or equivalently, the smallest value for γ).

In 3D, both tetrahedra with one and with two boundary faces are removable, under the constraints as given above. We first consider the solid angle at the interior vertex of a tetrahedron having one boundary face. Analogous to the planar case, we call the radius of the sphere through the vertices of the tetrahedron, R, the γ-value of the boundary face that corresponds to that tetrahedron, γ, the vertices on the boundary, x_1, x_2, and x_3, and abbreviate $r(x_1, x_2, x_3)$ to r. Further, we call the interior vertex y.

To the best of my knowledge, there is no "3D sine rule", that relates the interior angle at y to the size of the opposite triangle. Nonetheless, we use an approximating measure for the solid angle ϕ at y, in a way similar to the 2D case. We take A/R^2 if γ is non-negative, where A is the area of the triangle (x_1, x_2, x_3). This measure is independent of R. For a fixed radius R, the value increases as A increases, suggesting a wider angle ϕ. From planar trigonometry we know that $A = d_1 d_2 d_3 r^2$, where $d_1 = d(x_2, x_3)/r$, and the other terms result from cyclic permutation. Because $R^2 = r^2/(1 - |\gamma|)$ by definition, A/R^2 expands to $d_1 d_2 d_3(1 - |\gamma|)$, which equals $d_1 d_2 d_3(1 - \gamma)$, since γ is non-negative. The value of $d_1 d_2 d_3$ expresses how thick triangle (x_1, x_2, x_3) is, relative to its size. The larger this value, the thicker the triangle, and the wider the angle ϕ, for a fixed value of γ. The value of $1 - \gamma$, ranging from 0 to 1, expresses how close vertex y lies to triangle (x_1, x_2, x_3), again relative to the triangle size. The larger this value, the closer y lies to the triangle and the wider angle ϕ, for a fixed value of A.

Similar to the 2D case, if γ is non-positive, we use $d_1 d_2 d_3(2 - (1 - |\gamma|))$ as approximating measure for ϕ. This expression equals $d_1 d_2 d_3(1 - \gamma)$. Again, for a fixed γ, the value increases as $d_1 d_2 d_3$ increases, giving a wider ϕ. Also, for a fixed value of $d_1 d_2 d_3$, a larger value of $1 - \gamma$ (which now lies between 1 and 2), results when y lies closer to the opposite triangle, giving a wider angle. Note that the measures for non-positive and non-negative γ coincide when γ is zero. If we only consider tetrahedra that have one face on the boundary, the resulting selection criterion is that we choose the one having the largest value of $d_1 d_2 d_3(1 - \gamma)$.

There are also tetrahedra that have two boundary faces. As stated before, such a tetrahedron is removable only if the edge opposite to the edge that is common to both boundary faces, does not lie on the boundary. Since all four vertices already lie on the boundary, deletion of the tetrahedron adds no points to the boundary. However, it can result in an extra boundary tetrahedron, and moreover, deletion of the tetrahedron gives the two vertices opposite to the boundary faces "more air", enlarging the probability that they are sensed from these directions. Because the solid angles at the two vertices bound two non-overlapping parts of space, it seems logical to take the sum of the values $d_1 d_2 d_3(1-\gamma)$ corresponding to both opposite boundary faces, as the criterion value for the tetrahedron. The selection rule that captures the criterion value for both the tetrahedra with one and with two boundary faces then becomes:

> delete the removable tetrahedron with the largest sum of the values $d_1 d_2 d_3(1-\gamma)$ that correspond to the boundary faces of the tetrahedron.

The method described so far does not always succeed. In the first place, the shrinking operation can get locked, although the initial graph does contain a Hamilton polygon or polyhedron. This happens when there are no more removable simplices, while not yet all the vertices are included in the boundary. Secondly, there exist non-Hamiltonian planar Delaunay triangulations (Dillencourt 1987), and thus $\gamma([-1,1],[0,1])$-graphs. So, methods based on shrinking from the Delaunay triangulation, as above, and Boissonnat (1984), do not guarantee success.

In both cases, the solution is to shrink from a $\gamma([-1,1],[\gamma_0,1])$-graph, for some $\gamma_0 < 0$. In such a graph more points are connected, forming more simplices. However, we have no longer a triangulation, or tetrahedralization, because the extra simplices overlap with those from $\gamma([-1,1],[0,1])$. When we delete a boundary simplex during the shrinking process, we must also delete all overlapping simplices, in order to keep the boundary valid. The extra simplices will have smaller interior angles at the vertex opposite to the boundary segment, than overlapping simplices from $\gamma([-1,1],[0,1])$, but they give more choice in the selection of the boundary segments to delete. For a γ_0 small enough, $\gamma([-1,1],[\gamma_0,1])$ will be Hamiltonian (the complete graph $\gamma([-1,1],[-1,1])$ always contains a Hamilton polygon or polyhedron), and locking of the shrinking process will not occur.

The example of the non-Hamiltonian Delaunay triangulation given by Dillencourt (1987) is two-connected, that is, between every two vertices there are at least two distinct paths in the graph. It is not yet known if there also exist three-connected non-Hamiltonian Delaunay triangulations. This is an interesting open question, since the 3D Delaunay triangulation is three-connected. However, also in 3D the shrinking process can get locked, in which case we use some $\gamma([-1,1],[\gamma_0,1])$, $\gamma_0 < 0$.

There is no way to tell for which value of γ_0 the graph $\gamma([-1,1],[\gamma_0,1])$ will be Hamiltonian, if the Delaunay triangulation does not already contain a Hamilton polygon or polyhedron. That value must be estimated by a human operator, or tried out automatically with several guesses.

MATHEMATICAL MORPHOLOGY

Although the potential of the theory of mathematical morphology is much more powerful, it is generally applied to pixel image transformations. While computational morphology operates on any suitable geometrical structure, for example the γ-graph in our application, mathematical morphology is defined on a complete lattice, a partial ordered set, such that every subset has a supremum and an infimum (Serra 1986).

If we call the set of input data points V, and let $\mathcal{L}$ be the power set of V (the set of all subsets of V), then $\mathcal{L}$ is a partial ordered set with the set inclusion as the partial order relation. Because every subset $\mathcal{K}$ of $\mathcal{L}$ has a supremum (the union of the elements of $\mathcal{K}$), and an infimum (the intersection of the elements of $\mathcal{K}$), $\mathcal{L}$ is a complete lattice. Vertices, edges,

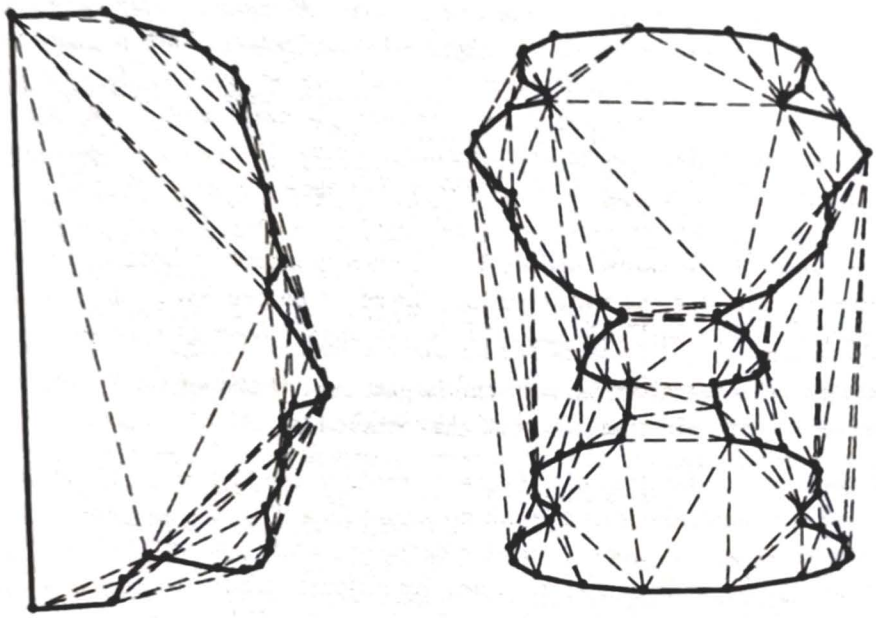

Figure 2. Planar example results: face and chalice. The found boundary is displayed with solid lines, $\gamma([-1,1],[0,1])$ with dashed lines.

triangles and tetrahedra are all elements of $\mathcal{L}$, so we can consider $\gamma([-1,1],[\gamma_0,1])$ as a subset of $\mathcal{L}$. Taking $\gamma([-1,1],[\gamma_0,1])$ as an initial graph G_0, and the convex hull as the initial boundary B_0, we can define the shrinking operation in the i-th iteration as $G_i = G_{i-1} \ominus \max(B_{i-1})$, $B_i = B_{i-1} \oslash \max(B_{i-1})$, where the operator max determines the boundary simplex having the largest criterion value, the operator $\ominus$ performs the deletion from the graph, and $\oslash$ performs the replacement of the boundary segments by new ones.

CONCLUSIONS

The γ-neighborhood graph is a unifying geometrical structure in computational geometry. It brings special structures such as the convex hull, Delaunay triangulation, Gabriel graph, β-neighborhood graph and the complete graph, together in a continuous spectrum of geometrical graphs. The application of the γ-neighborhood graph to computational morphology, provides in a tool that is more powerful than the Delaunay triangulation, which has been used before. The shrinking operation on the power set of the set of input points, is a nice example of mathematical morphology that performs transformations on a lattice that is not a 2D pixel image.

Figure 2 shows example results.

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