A robust affine invariant metric on boundary patterns

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Abstract

Affine invariant pattern metrics are useful for shape recognition. It is important that such a metric is robust for various defects. We formalise these types of robustness using four axioms. Then, we present the reflection metric. This is an affine invariant metric defined on $(n-1)$-dimensional complexes in $\mathbb{R}^n$. We prove that the reflection metric satisfies the four robustness axioms.

Keywords: pattern matching, shape recognition, affine invariance, metric.

1 Introduction

For any collection of patterns, subsets of a Euclidean space, there exist many fundamentally different metrics. In pattern matching and shape recognition it is often important that a metric is invariant under a transformation group. In practical applications it is desirable that a metric is robust for various defects caused by discretisation and unreliable feature detection.

Many shape recognition algorithms use a metric on simple closed curves. An important example is the Fréchet distance, see Alt and Godau [3]. Other pattern metrics for boundary curves are based on turning angle, see Cohen and Guibas [5], or normalised affine arc-length, see Huttenlocher and Kedem [10].

The Hausdorff metric is defined on the collection of all (non-empty) closed, bounded subsets of a metric space. Some algorithms are based on this metric, see [4], [11], [1]. However, the Hausdorff metric is not robust with respect to certain types of noise. For example, outliers, i.e. isolated points lying far away from the other points, can cause a dramatic increase in the Hausdorff distance. The Hausdorff metric is invariant for the group of isometries. The partial Hausdorff distance is a non-metric variant of the Hausdorff metric that is more robust for noise, see [12], [7]. The partial Hausdorff distance depends on a parameter estimating the amount of distortion.

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For solid patterns, robust affine invariant metrics exist. Examples include the normalised volume of symmetric difference, see Alt et al. [2], and the difference of normalised indicators, see Hagedoorn and Veltkamp [8].

Until now, little attention has been paid to affine invariant metrics on finite unions of curves in $\mathbb{R}^2$, or surfaces in $\mathbb{R}^3$. In this paper, we present the *reflection metric*. This metric is defined on finite unions of $(n-1)$-dimensional hypersurfaces in $\mathbb{R}^n$. The reflection metric is invariant under affine transformations. We show that the reflection metric is robust in many respects.

Section 2 discusses metrics and their invariance under transformation groups. If a metric is invariant for a transformation group, it induces a natural metric on the orbit space. We are interested in metrics on patterns, subsets of a Euclidean space. The orbit space of the pattern space under a transformation group corresponds to a collection of shapes. Section 3 presents axioms expressing four types of robustness of pattern metrics. Section 4 shows how invariant metrics can be constructed by mapping patterns to functions. Section 5 applies this technique in constructing the reflection metric. The reflection metric is defined on finite unions of $(n-1)$-dimensional surfaces and is invariant under affine transformations. We show that the reflection metric satisfies the four robustness axioms presented in Section 3.

2 Invariant metrics on patterns

Many pattern matching and recognition techniques are based on a similarity measure between patterns. A similarity measure is a function defined on pairs of patterns indicating the degree of resemblance of the patterns. It is desirable that such a similarity measure is a metric. Furthermore, a similarity measure should be invariant for the geometrical transformation group that corresponds to the matching problem. Below, we discuss metrics, and their invariance for transformation groups. After that, we show how an invariant metric on a collection of patterns leads naturally to a metric on shapes.

Let S be any set of objects. A *metric* on S is a function $d : S \times S \to \mathbb{R}$ satisfying the following two conditions for all $x, y, z \in S$:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(y, z) \leq d(x, y) + d(x, z)$.

A set S with a fixed metric d is called a metric space. Given two elements x and y of S, the value $d(x, y)$ is called the *distance* between x and y. Consider a weaker version of property (i):

(i)' $d(x, x) = 0$.

A function satisfying (i)' and (ii) is called a *semimetric*. Non-negativity and symmetry follow from (i)' and (ii). By identifying elements of S with zero distance, any semimetric induces a metric on the resulting partition.

A set of bijections G in S is a *transformation group* if $g^{-1}h \in G$ for all $g, h \in G$. A (semi)metric d on a set S is said to be *invariant* for the transformation group G acting on S if $d(g(x), g(y)) = d(x, y)$ for all $g \in G$ and $x, y \in S$.

The *orbit* of G passing through $x \in S$ is the set of images of x under G:

$$G(x) = \{\, g(x) \mid g \in G \,\}.$$

The orbits form a partition of S. The collection of all orbits is called the *orbit set*, denoted by S/G.

The following theorem shows that a semimetric invariant under a transformation group results in a natural semimetric on the orbit set. Rucklidge [13] used this principle to define a shape distance based on the Hausdorff distance.

Theorem 1 *Let G be a transformation group for a set S; let d be a semimetric on S invariant for G. Then $\tilde{d} : S/G \times S/G \to \mathbb{R}$ defined by*

$$\tilde{d}(G(x), G(y)) = \inf\{\, d(g(x), y) \mid g \in G \,\}.$$

is a semimetric.

proof. Property (i)': Trivial. Property (ii): For any $h, k \in G$:

$$\begin{aligned} d(kh^{-1}(y), z) &= d(h^{-1}(y), k^{-1}(z)) \\ &\leq d(x, h^{-1}(y)) + d(x, k^{-1}(z)) \\ &= d(h(x), y) + d(k(x), z). \end{aligned}$$

Using the previously derived inequality, we find for all $x, y, z \in S$:

$$\begin{aligned} \tilde{d}(G(y), G(z)) &= \inf\{\, d(g(y), z) \mid g \in G \,\} \\ &\leq \inf\{\, d(h(x), y) + d(k(x), z) \mid h, k \in G \,\} \\ &= \inf\{\, d(h(x), y) \mid h \in G \,\} + \inf\{\, d(k(x), z) \mid k \in G \,\} \\ &= \tilde{d}(G(x), G(y)) + \tilde{d}(G(x), G(z)). \end{aligned}$$

This finishes the proof. □

Let $\mathcal{P}$ be a fixed collection of subsets of $\mathbb{R}^n$. Any element of $\mathcal{P}$ is called a *pattern*. We call the collection $\mathcal{P}$ with a fixed metric d a *metric pattern space*. A collection of patterns $\mathcal{P}$ and a transformation group G determine a family of shapes $\mathcal{P}/G$. For a pattern $A \in \mathcal{P}$, the corresponding *shape* equals the orbit

$$G(A) = \{\, g(A) \mid g \in G \,\}.$$

The collection of all these orbits forms a *shape space*. If d is invariant for G, then Theorem 1 gives a semimetric $\tilde{d}$ on the shape space $\mathcal{P}/G$.

Shape recognition involves computing the similarity between two patterns independent of transformation. This exactly what the shape metric $\tilde{d}$ is good for. It determines the greatest lower bound of all $d(g(A), B)$ under transformations

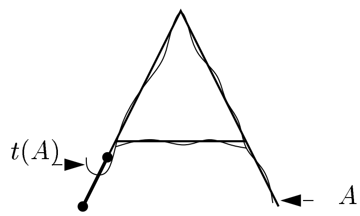

Figure 1: Deformation robust.

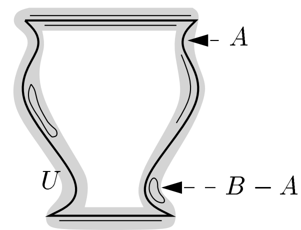

Figure 2: Blur robust.

$g \in G$, given two patterns A and B, resulting in a transformation-independent distance between the two corresponding shapes $G(A)$ and $G(B)$. If we have a robust, invariant metric on patterns, then we can perform shape recognition in a robust manner by using the shape metric. Next, we formalise four types of robustness.

3 Robustness axioms

In this section, we focus on "boundary patterns", i.e. boundaries of relatively compact subsets of $\mathbb{R}^n$. We introduce four axioms expressing robustness for what we call "deformation", "blur", "cracks" and "noise". Deformation robustness says that each point in a pattern may be moved a little bit without seriously affecting the value of the metric. Blur robustness says that new points may be added close to the original pattern. Crack robustness says that components of patterns may be broken up as long as the cracks are relatively thin. Noise robustness says that new small parts may be added to a pattern.

Let $\mathrm{C}^1(\mathbb{R}^n)$ be the group of C^1 diffeomorphisms acting on $\mathbb{R}^n$. Let $\mathcal{P}$ be a collection of compact sets equal to the boundary of a subset of $\mathbb{R}^n$.

A metric d on $\mathcal{P}$ is called *deformation robust* if it satisfies the following axiom:

Axiom 1 *For each $A \in \mathcal{P}$ and $\epsilon > 0$, there is a $\delta > 0$ s.t. $\|x - g(x)\| < \delta$ for all $x \in A$ implies $d(A, t(A)) < \epsilon$ for all $t \in \mathrm{C}^1(\mathbb{R}^n)$.*

Deformation robustness is equivalent to saying that for each pattern $A \in \mathcal{P}$, the map $t \mapsto t(A)$ with domain $\mathrm{C}^1(\mathbb{R}^n)$ and range $\mathcal{P}$ is continuous. Figure 1 shows the image of A under a transformation with a "small" δ, in the sense of Axiom 1.

We call a metric pattern space *blur robust* if the following holds:

Axiom 2 *For each $A \in \mathcal{P}$ and $\epsilon > 0$, an open neighbourhood U of A exists, such that $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$ satisfying $B - U = A - U$ and $A \subseteq B$.*

The axiom says that additions close to A do not cause discontinuities. Figure 2 shows a neighbourhood U of A in which parts of B occur that are not in A.

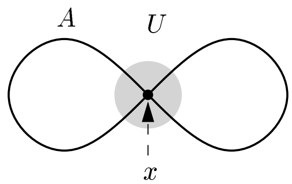

Figure 3: Crack robust.

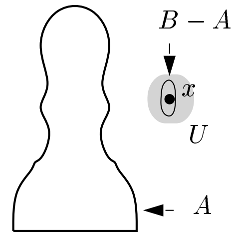

Figure 4: Noise robust.

A *crack* of A is a closed subset $R \subseteq A$ consisting entirely of limit points of $A - R$. This means that all open neighbourhoods of a point $x \in R$ intersect $A - R$. Cracks can be seen as parts of a pattern that can be "restored" after they have been removed from the pattern, by forming the closure of the remaining pattern. Changing a pattern in neighbourhoods of a crack may cause the pattern (or its complement) to become separated or connected. Figure 3 shows a pretzel, consisting of two topological 1-spheres glued together at a point x. The singleton set $R = \{ x \}$ is a crack of A.

We say $(X, \mathcal{P}, d)$ is *crack robust* if the next axiom holds:

Axiom 3 *For each $A \in \mathcal{P}$, each crack R of A, and $\epsilon > 0$, an open neighbourhood U of R exists such that $A - U = B - U$ implies $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$.*

The axiom says that applying changes to A within a small enough neighbourhood of a crack of A results in a pattern B close to A in pattern space. Whether the connectedness is preserved does not matter.

If the following axiom is satisfied, we call a metric pattern space *noise robust*:

Axiom 4 *For each $A \in \mathcal{P}$, $x \in X$, and $\epsilon > 0$, an open neighbourhood U of x exists such that $B - U = A - U$ implies $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$.*

This axiom says that changes in patterns do not cause discontinuities in pattern distance, provided the changes happen within small regions. By means of the triangle inequality, we obtain an equivalent axiom when neighbourhoods of finite point sets instead of singletons are considered.

Figure 4 shows a pattern A and a point x. Addition of noise $B - A$ within a neighbourhood U of x results in a new pattern B. Axiom 4 says that the distance between A and B can be made smaller by making U smaller.

4 Constructing invariant pattern metrics

In this section we show how affine invariant pattern metrics can be formed by mapping patterns to real-valued functions and computing a normalised difference between these functions. Affine invariance is essential many pattern matching and shape recognition tasks. Figure 5 shows two patterns superimposed on eachother on the left. On the right the images of the two patterns under an affine transformation are shown.

Let $\mathbf{I}(\mathbb{R}^n)$ be the vector space of real-valued Lebesgue-integrable functions on $\mathbb{R}^n$, with scalar multiplication and vector addition defined pointwise. Define the L^1 seminorm on $\mathbf{I}(\mathbb{R}^n)$:

$$|\mathbf{a}| = \int_{\mathbb{R}^n} |\mathbf{a}(x)|\, dx.$$

For a diffeomorphism $g \in \mathrm{C}^1(\mathbb{R}^n)$, let $D_g^x : \mathbb{R}^n \to \mathbb{R}^n$ be the derivative of g in x, a linear function. The Jacobi-determinant is the determinant of the derivative at a given point. We use $j_g(x) = \left|\det(D_g^x)\right|$, to denote the absolute value of the Jacobi-determinant of g in x.

For real-valued functions $\mathbf{a}, \mathbf{b} : \mathbb{R}^n \to \mathbb{R}$, let $\mathbf{a} \sqcap \mathbf{b}$ and $\mathbf{a} \sqcup \mathbf{b}$ denote the pointwise minimum and maximum, respectively. This notation is analogous to set intersection and union. Define the *normalised difference* of two functions with non-zero integrals by

$$\sigma_n(\mathbf{a}, \mathbf{b}) = \frac{|\mathbf{a} - \mathbf{b}|}{|\mathbf{a} \sqcup \mathbf{b}|}.$$

Lemma 1 *The normalised difference σ_n is a semimetric on the set of non-negative functions with non-zero integrals.*

proof. Property (i)': Trivial. Property (ii): Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be non-negative functions with non-zero integrals. We need to prove:

$$\frac{|\mathbf{b} - \mathbf{c}|}{|\mathbf{b} \sqcup \mathbf{c}|} \leq \frac{|\mathbf{a} - \mathbf{b}|}{|\mathbf{a} \sqcup \mathbf{b}|} + \frac{|\mathbf{a} - \mathbf{c}|}{|\mathbf{a} \sqcup \mathbf{c}|}.$$

Since $|\cdot|$ is a seminorm the inequality $|\mathbf{b} - \mathbf{c}| \leq |\mathbf{u} - \mathbf{b}| + |\mathbf{u} - \mathbf{c}|$ holds, implying:

$$\frac{|\mathbf{b} - \mathbf{c}|}{|\mathbf{b} \sqcup \mathbf{c}|} \leq \frac{|\mathbf{u} - \mathbf{b}|}{|\mathbf{c} \sqcup \mathbf{b}|} + \frac{|\mathbf{u} - \mathbf{c}|}{|\mathbf{b} \sqcup \mathbf{c}|}.$$

Choosing $\mathbf{u} = \mathbf{a} \sqcap (\mathbf{b} \sqcup \mathbf{c})$, both terms on the right side of this inequality can be bounded, obtaining the triangle inequality. We show it only for the first term, since the procedure for the second one is analogous.

$$\begin{aligned}
\frac{|\mathbf{u} - \mathbf{b}|}{|\mathbf{c} \sqcup \mathbf{b}|} &\leq \frac{|\mathbf{u} - \mathbf{b}|}{|\mathbf{u} \sqcup \mathbf{b}|} \\
&\leq \frac{|\mathbf{a} - \mathbf{u}| + |\mathbf{u} - \mathbf{b}|}{|\mathbf{a} - \mathbf{u}| + |\mathbf{u} \sqcup \mathbf{b}|} \\
&= \frac{|(\mathbf{a} - \mathbf{u}) + (\mathbf{u} - \mathbf{b})|}{|(\mathbf{a} - \mathbf{u}) + (\mathbf{u} \sqcup \mathbf{b})|} \\
&= \frac{|\mathbf{a} - \mathbf{b}|}{|\mathbf{a} \sqcup \mathbf{b}|}.
\end{aligned}$$

□

Let $\mathrm{CJ}(\mathbb{R}^n)$ be the subgroup of $\mathrm{C}^1(\mathbb{R}^n)$ consisting of those g for which the Jacobi-determinant $j_g(x)$ is constant in $x \in \mathbb{R}^n$. The next lemma shows that a large class of mappings from patterns to integrable functions result in invariant semimetrics based on the normalised difference σ_n.

Lemma 2 *Let $\mathcal{P}$ be a collection of subsets of $\mathbb{R}^n$. Let each $A \in \mathcal{P}$ define a unique function $\mathbf{n}_A : \mathbb{R}^n \to \mathbb{R}$ in $\mathbf{I}(\mathbb{R}^n)$. If $g \in \mathrm{CJ}(\mathbb{R})$ determines a number $\delta > 0$ such that*

$$\mathbf{n}_{g(A)}(g(x)) = \delta \mathbf{n}_A(x)$$

for all $A \in \mathcal{P}$ and $x \in \mathbb{R}^n$, then

$$\sigma_n(\mathbf{n}_{g(A)}, \mathbf{n}_{g(B)}) = \sigma_n(\mathbf{n}_A, \mathbf{n}_B)$$

for all $A, B \in \mathcal{P}$.

proof. Apply substitution of variables using the constant $j = j_g(x)$:

$$\begin{aligned}
&\sigma_n(\mathbf{n}_{g(A)}, \mathbf{n}_{g(B)}) \\
&= \frac{\left|\mathbf{n}_{g(A)} - \mathbf{n}_{g(B)}\right|}{\left|\mathbf{n}_{g(A)} \sqcup \mathbf{n}_{g(B)}\right|} \\
&= \frac{j\ \left|\mathbf{n}_{g(A)} \circ g - \mathbf{n}_{g(B)} \circ g\right|}{j\ \left|\mathbf{n}_{g(A)} \circ g \sqcup \mathbf{n}_{g(B)} \circ g\right|} \\
&= \sigma_n(\mathbf{n}_{g(A)} \circ g, \mathbf{n}_{g(B)} \circ g) \\
&= \sigma_n(\delta \mathbf{n}_A, \delta \mathbf{n}_B) \\
&= \sigma_n(\mathbf{n}_A, \mathbf{n}_B).
\end{aligned}$$

□

5 The reflection metric

In this section we define the reflection metric. First, we define the class of patterns on which this metric is defined. After that, we discuss the notion of visibility which is fundamental to the reflection metric. Using visibility, we construct functions from patterns, which lead to the definition of the reflection metric. The results from Section 4 ensure affine invariance of the metric. Finally, we show that the reflection metric satisfies the four robustness axioms from Section 3.

The axioms in Section 3 were defined in terms of a pattern collection $\mathcal{P}$ consisting of compact boundaries in $\mathbb{R}^n$. Let $\mathcal{R}^n$ be the patterns in $\mathcal{P}$ (not contained in any $(n-1)$-dimensional hyperplane) that are C^1-diffeomorphic

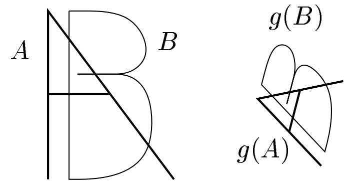

Figure 5: Affine invariance.

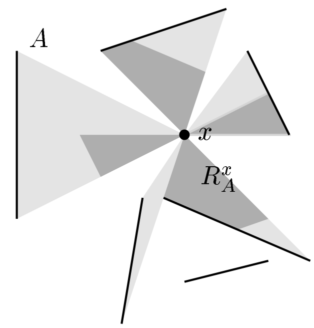

Figure 6: Reflected visibility.

to a properly joined union of closed $(n-1)$-simplices. Formally, we write each pattern $A \in \mathcal{R}^n$ as $A = \phi(\bigcup_{i=1}^k R_i)$, where $R_1, \ldots, R_k$ are properly joined closed $(n-1)$-simplices and $\phi \in \mathrm{C}^1(\mathbb{R}^n)$.

We use the notation $\overline{xy}$ for the open line segment connecting two distinct points $x, y \in \mathbb{R}^n$. We say that a point $y \in \mathbb{R}^n$ is visible (in A) from a point $x \in \mathbb{R}^n$ if $A \cap \overline{xy} = \varnothing$. For $A \in \mathcal{R}^n$ and $x \in \mathbb{R}^n$, the *visibility star* V^x_A is defined as the set of open line segments connecting points of A that are visible from x:

$$V^x_A = \bigcup \{ \overline{xa} \mid \ a \in A \text{ and } A \cap \overline{xa} = \varnothing \ \}.$$

We define the *reflection star* R^x_A by intersecting V^x_A with its reflection in x:

$$R^x_A = \{ x + v \in \mathbb{R}^n \mid \ x - v \in V^x_A \text{ and } x + v \in V^x_A \ \}.$$

Figure 6 shows the visibility star V^x_A and the corresponding reflection star R^x_A for a pattern $A \in \mathcal{R}^2$, and a point $x \in \mathbb{R}^2$.

Each pattern $A \in \mathcal{R}^n$ determines a function $\rho_A : \mathbb{R}^n \to \mathbb{R}$ given by $\rho_A(x) = \mathrm{vol}(R^x_A)$. Note that ρ_A is zero outside the convex hull of A. The definition of the reflection metric is based on the normalised difference, see Lemma 1.

Definition 1 *The* reflection metric $d_{\mathtt{R}}$ *for* $\mathcal{R}^n$ *is given by*

$$d_{\mathtt{R}}(A, B) = \sigma_n(\rho_A, \rho_B).$$

Lemma 2 tells us that the metric $d_{\mathtt{R}}$ is invariant under the group of affine transformations $\mathrm{Af}(\mathbb{R}^n)$.

Next, we prove the four robustness axioms from Section 3. Observe that for any two patterns $A, B \in \mathcal{R}^n$ and any point $x \in \mathbb{R}^n$:

$$\mathrm{vol}(R^x_A - R^x_B) \leq 2 \ \mathrm{vol}(V^x_A - V^x_B).$$

From this, we find that

$$|\rho_A(x) - \rho_B(x)| \leq 2 \ \mathrm{vol}((V^x_A - V^x_B) \cup (V^x_B - V^x_A)). \tag{1}$$

Thus, we can prove the first four axioms by bounding the change in the visibility star for deformation, blur, crack and noise.

The metric $d_{\mathtt{R}}$ is deformation robust. Let $A \in \mathcal{R}^n$ and $\epsilon > 0$ Using Eq 1, choose $\delta > 0$ small enough such that $\left|\rho_A - \rho_{t(A)}\right| < \epsilon \ |\rho_A|$ for all $t \in \mathrm{C}^1(\mathbb{R}^n)$ satisfying $\|x - t(x)\| < \delta$ for all $x \in A$. It follows that

$$d_{\mathtt{R}}(A, t(A)) \leq \frac{\left|\rho_A - \rho_{t(A)}\right|}{|\rho_A|}$$

for the same transformations t.

The reflection metric is blur robust. Let $A \in \mathcal{R}^n$ and $\epsilon > 0$ be given. Choose an open neighbourhood U of A with $\mathrm{vol}(U) < \delta$ for some given $\delta > 0$. Using Eq. 1, choose $\delta > 0$ small enough so that for all $B \in \mathcal{R}^n$ satisfying $B - U = A - U$ and $A \subseteq B$, the distance $d_{\mathtt{R}}(A, B)$ is smaller than ϵ.

The reflection metric is crack robust. Let $A \in \mathcal{R}^n$, R be a crack of A, and $\epsilon > 0$. By means of Eq. 1, we can choose a sufficiently small open neighbourhood U of the crack R such that $d_{\mathtt{R}}(A, B) < \epsilon$ for patterns $B \in \mathcal{R}^n$ satisfying $B - U = A - U$.

The reflection metric is noise robust. Let $A \in \mathcal{R}^n$ and $x \in \mathbb{R}^n$ be given. Using Eq. 1, we can choose an open neighbourhood U of x small enough such that $d_{\mathtt{R}}(A, B) < \delta$ for all $B \in \mathcal{R}^n$ satisfying $B - U = A - U$.

6 Conclusion

This paper has two contributions. First, our axioms describe four types of robustness, that are desirable in practical pattern recognition. Although the importance of robust similarity measures is recognised in literature, until now little serious attempts have been made to formalise them. Second, we provide an affine invariant metric, called the reflection metric, defined on "boundary patterns". This metric satisfies the four robustness axioms. Affine invariance is very useful since it allows patterns to be recognised even under scaling.

The next logical step in this research is the efficient computation of the reflection metric. This can be done using techniques from computational geometry. The reflection metric between two finite segment unions can be computed in $O((s+k)\log(s+k)+v)$ time, where s is the total number of segments, k is the total number of edges in both "visibility graphs", and v is the total number of vertices in both "reflection-visibility arrangements". For details see Hagedoorn and Veltkamp [9]. Another question is whether less complicated pattern metrics satisfy our axioms, perhaps invariant only for similarity transformations.

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