A robust affine invariant metric on boundary patterns

Michiel Hagedoorn* Remco C. Veltkamp
Department of Computer Science, Utrecht University
Padualaan 14, 3584 CH Utrecht, The Netherlands
mh@cs.uu.nl Remco.Veltkamp@cs.uu.nl

Abstract

Affine invariant pattern metrics are useful for shape recognition. It is important that such a metric is robust for various defects. We formalise these types of robustness using four axioms. Then, we present the reflection metric. This is an affine invariant metric defined for the large family of "boundary patterns". A boundary pattern is a finite union of $(n-1)$-dimensional algebraic surface patches in $\mathbb{R}^n$. Such a pattern may have multiple connected components. We prove that the reflection metric satisfies the four robustness axioms.

Keywords: pattern matching, shape recognition, affine invariance, metric.

1 Introduction

For any collection of patterns, subsets of a Euclidean space, there exist many fundamentally different metrics. In pattern matching and shape recognition it is often important that a metric is invariant under a given transformation group. In practical applications it is desirable that a metric is robust for various defects caused by discretisation and unreliable feature detection.

Many shape recognition algorithms use a metric on simple closed curves. An important example is the Fréchet distance, see Alt and Godau [3]. Other pattern metrics for curves are based on turning angle, see Cohen and Guibas [5], or normalised affine arc-length, see Huttenlocher and Kedem [11].

The Hausdorff metric is defined on the collection of all (non-empty) closed, bounded subsets of a metric space. Some algorithms are based on this metric, see [4], [12], [1]. However, the Hausdorff metric is not robust with respect to certain types of noise. For example, outliers, i.e. isolated points lying far away from the other points, can cause a dramatic increase in the Hausdorff distance. The Hausdorff metric is invariant for the group of isometries. The partial Hausdorff

*supported by Philips Research

distance is a non-metric variant of the Hausdorff metric that is more robust for noise, see [13], [9]. The partial Hausdorff distance depends on a parameter estimating the amount of distortion.

For solid patterns, robust affine invariant metrics exist. Examples include the normalised volume of symmetric difference, see Alt et al. [2], and the difference of normalised indicators, see Hagedoorn and Veltkamp [10].

Until now, little attention has been paid to affine invariant metrics on finite unions of curve segments in $\mathbb{R}^2$, or surfaces patches in $\mathbb{R}^3$. In this paper, we present the *reflection metric*. This metric is defined on finite unions of $(n-1)$-dimensional surface patches in $\mathbb{R}^n$. The reflection metric is invariant under affine transformations. We show that the reflection metric is robust in many respects.

Section 2 discusses metrics and their invariance under transformation groups. If a metric is invariant for a transformation group, it induces a natural metric on the orbit space. We are interested in metrics on patterns, subsets of a Euclidean space. The orbit space of the pattern space under a transformation group corresponds to a collection of shapes. Section 3 presents axioms expressing four types of robustness of pattern metrics. Section 4 shows how invariant metrics can be constructed by mapping patterns to functions. Section 5 applies this technique in constructing the reflection metric. The reflection metric is defined on finite unions of $(n-1)$-dimensional surface patches and is invariant under affine transformations. We show that the reflection metric satisfies the four robustness axioms presented in Section 3.

2 Invariant metrics on patterns

Many pattern matching and recognition techniques are based on a similarity measure between patterns. A similarity measure is a function defined on pairs of patterns indicating the degree of resemblance of the patterns. It is desirable that such a similarity measure is a metric. Furthermore, a similarity measure should be invariant for the geometrical transformation group that corresponds to the matching problem. Below, we discuss metrics, and their invariance for transformation groups. After that, we show how an invariant metric on a collection of patterns leads naturally to a metric on shapes.

Let S be any set of objects. A *metric* on S is a function $d : S \times S \to \mathbb{R}$ satisfying the following two conditions for all $x, y, z \in S$:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(y, z) \leq d(x, y) + d(x, z)$.

A set S with a fixed metric d is called a metric space. Given two elements x and y of S, the value $d(x, y)$ is called the *distance* between x and y. Consider a weaker version of property (i):

(i)' $d(x, x) = 0$.

A function satisfying (i)' and (ii) is called a *semimetric*. Non-negativity and symmetry follow from (i)' and (ii). By identifying elements of S with zero distance, any semimetric induces a metric on the resulting partition.

A set of bijections G in S is a *transformation group* if $g^{-1}h \in G$ for all $g, h \in G$. A (semi)metric d on a set S is said to be *invariant* for the transformation group G acting on S if $d(g(x), g(y)) = d(x, y)$ for all $g \in G$ and $x, y \in S$. The *orbit* of G passing through $x \in S$ is the set of images of x under G:

$$G(x) = \{\, g(x) \mid g \in G \,\}.$$

The orbits form a partition of S. The collection of all orbits is called the *orbit set*, denoted by S/G.

The following theorem shows that a semimetric invariant under a transformation group results in a natural semimetric on the orbit set. Rucklidge [14] proves this principle for a shape distance based on the Hausdorff distance. Here, we present a simpler proof for semimetrics in general.

Theorem 1 *Let G be a transformation group for a set S; let d be a semimetric on S invariant for G. Then $\tilde{d} : S/G \times S/G \to \mathbb{R}$ defined by*

$$\tilde{d}(G(x), G(y)) = \inf\{\, d(g(x), y) \mid g \in G \,\}.$$

is a semimetric.

proof. Property (i)': Trivial. Property (ii): For any $h, k \in G$:

$$\begin{aligned} d(kh^{-1}(y), z) &= d(h^{-1}(y), k^{-1}(z)) \\ &\leq d(x, h^{-1}(y)) + d(x, k^{-1}(z)) \\ &= d(h(x), y) + d(k(x), z). \end{aligned}$$

Using the previously derived inequality, we find for all $x, y, z \in S$:

$$\begin{aligned} \tilde{d}(G(y), G(z)) &= \inf\{\, d(g(y), z) \mid g \in G \,\} \\ &\leq \inf\{\, d(h(x), y) + d(k(x), z) \mid h, k \in G \,\} \\ &= \inf\{\, d(h(x), y) \mid h \in G \,\} + \inf\{\, d(k(x), z) \mid k \in G \,\} \\ &= \tilde{d}(G(x), G(y)) + \tilde{d}(G(x), G(z)). \end{aligned}$$

This finishes the proof. □

Let $\mathcal{P}$ be a fixed collection of subsets of $\mathbb{R}^n$. Any element of $\mathcal{P}$ is called a *pattern*. We call the collection $\mathcal{P}$ with a fixed metric d a *metric pattern space*. A collection of patterns $\mathcal{P}$ and a transformation group G determine a family of shapes $\mathcal{P}/G$. For a pattern $A \in \mathcal{P}$, the corresponding *shape* equals the orbit

$$G(A) = \{\, g(A) \mid g \in G \,\}.$$

The collection of all these orbits forms a *shape space*. If d is invariant for G, then Theorem 1 gives a semimetric $\tilde{d}$ on the shape space $\mathcal{P}/G$. Shape recognition

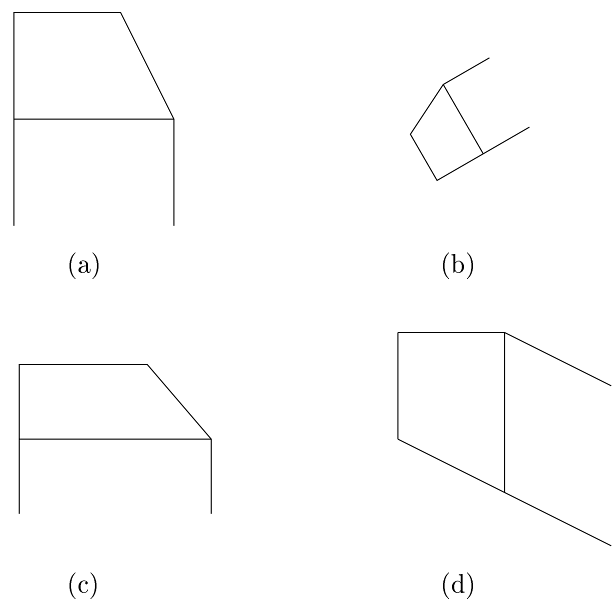

Figure 1: Pattern images under various transformation groups.

involves computing the similarity between two patterns independent of transformations. This is exactly what the shape metric $\tilde{d}$ is good for. It determines the greatest lower bound of all $d(g(A), B)$ under transformations $g \in G$, given two patterns A and B, resulting in a transformation-independent distance between the two corresponding shapes $G(A)$ and $G(B)$.

Figure 1 shows a two-dimensional pattern (a) and transformed images (b), (c), and (d). The group of similarity transformations consists of translations combined with rotation and scaling. Pattern (b) is the image of a similarity transformation. Therefore, we say (a) and (b) belong to the same similarity shape. Consider the group of translations combined with scaling along both axes. Under this transformation group, pattern (c) belongs to the same shape as (a). Pattern (d) belongs to the same affine shape as pattern (a). Each of the two previously mentioned transformation groups are subgroups of the affine transformations. Therefore, an affine invariant metric can be used to measure the distance between shapes, under any of the previously mentioned transformation groups. In Section 4, we discuss techniques that can be used to derive affine invariant pattern metrics. We apply this technique to construct a robust, affine invariant metric in Section 5. But first, in the next section, we discuss what is meant with the robustness of a metric on patterns.

3 Robustness axioms

In this section, $\mathcal{P}$ denotes some collection of patterns, each of which is the boundary of a relatively compact subset of $\mathbb{R}^n$. We introduce four axioms expressing robustness for what we call "deformation", "blur", "cracks" and "noise". Deformation robustness says that each point in a pattern may be moved a little bit without seriously affecting the value of the metric. Blur robustness says that new points may be added close to the original pattern. Crack robustness says that parts of patterns may be joined or broken up as long as the cracks are relatively thin. Noise robustness says that small pattern-parts may be added or deleted. The types of robustness correspond to various effects occurring in binary patterns obtained using image processing techniques, such as, for example, edge detection. If a metric satisfies the robustness axioms, then its value is not severely affected by these effects. In fact, the axioms say that the value of a robust metric is continuous in the amount of distortion caused by unreliable feature detection.

Let T be the group of all homeomorphisms $t : \mathbb{R}^n \to \mathbb{R}^n$ such that both $t^{-1}(A)$ and $t(A)$ are element of $\mathcal{P}$ for each $A \in \mathcal{P}$. A metric d on $\mathcal{P}$ is called *deformation robust* if it satisfies the following axiom:

Axiom 1 *For each $A \in \mathcal{P}$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\|a - t(a)\| < \delta$ for all $a \in A$ implies $d(A, t(A)) < \epsilon$ for all $t \in T$.*

Deformation robustness is equivalent to saying that for each pattern $A \in \mathcal{P}$, the map $t \mapsto t(A)$ with domain T and range $\mathcal{P}$ is continuous. Figure 2 shows the image of A under a transformation with a "small" δ, in the sense of Axiom 1. In words, deformation robustness says that changing the position of each point of a pattern a little bit resulting in a deformed pattern, does not affect the distance of that pattern to any other pattern very much. In practical applications of geometric pattern matching, deformation is typically a result of inaccurate feature detection or numerical roundoff errors.

We call a metric pattern space *blur robust* if the following holds:

Axiom 2 *For each $A \in \mathcal{P}$ and $\epsilon > 0$, an open neighbourhood U of A exists, such that $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$ satisfying $B - U = A - U$ and $A \subseteq B$.*

The axiom says that additions close to A do not cause discontinuities. Figure 3 shows a neighbourhood U of A in which parts of B occur that are not in A. Blur robustness corresponds to the vagueness of boundaries in images, and the resulting unreliability of edge detection techniques. A typical result is the detection of multiple edges, instead of one edge, somewhere in an image. Under a metric satisfying the blur robustness axiom the distance between patterns does not change significantly if "double edges" are added to the patterns.

A *crack* of A is a closed subset $R \subseteq A$ consisting entirely of limit points of $A - R$. This means that all open neighbourhoods of a point $x \in R$ intersect $A - R$. Cracks are parts of a pattern that can be "restored" after they have been removed from the pattern, by forming the closure of the remaining pattern. A

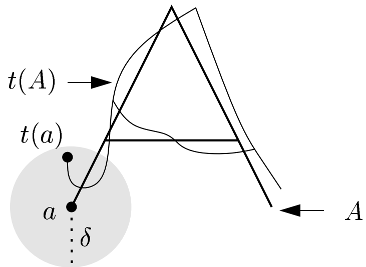

Figure 2: Deformation robust.

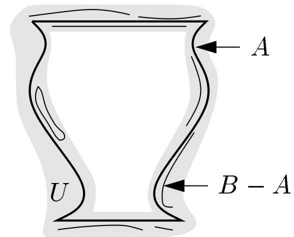

Figure 3: Blur robust.

crack need not be a finite set. Changing a pattern in neighbourhoods of a crack may cause the pattern (or its complement) to become separated or connected. Figure 4 shows a pretzel, consisting of two topological 1-spheres glued together at a point x. The singleton set R is a crack of A. A metric that is used to match feature patterns obtained using image processing techniques should use geometric information, but should ignore the connectivity of the feature patterns as much as possible. The reason for this is that the topological structure of images cannot be extracted reliably using image processing techniques. For example an edge detector, might, instead of the "real edge", detect two edges with a small gap between them. The crack robustness axiom says that the introduction of such gaps should not affect distances between patterns significantly. Conversely, the axiom also says that connecting two edges separated by a small gap should not matter much.

We say a metric d is *crack robust* if the next axiom holds:

Axiom 3 *For each $A \in \mathcal{P}$, $\epsilon > 0$, and each crack R of A, an open neighbourhood U of R exists such that $B - U = A - U$ implies $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$.*

The axiom says that applying changes to A within a small enough neighbourhood of a crack of A results in a pattern B close to A in pattern space. Whether the connectedness is preserved does not matter.

If the following axiom is satisfied, we call a metric pattern space *noise robust*:

Axiom 4 *For each $A \in \mathcal{P}$ $\epsilon > 0$, and each $x \in \mathbb{R}^n$, an open neighbourhood U of x exists such that $B - U = A - U$ implies $d(A, B) < \epsilon$ for all $B \in \mathcal{P}$.*

This axiom says that changes in patterns do not cause discontinuities in pattern distance, provided the changes happen within small neighbourhoods. By means of the triangle inequality, we obtain an equivalent axiom when neighbourhoods of finite point sets instead of singletons $\{x\}$ are considered. The noise robustness axiom says that introducing or removing small parts in a pattern, although these changes may happen far from the other pattern points, does not influence distances between patterns much. The Hausdorff distance, as an example, does not satisfy this axiom. Additions far away from the orignal pattern, no matter

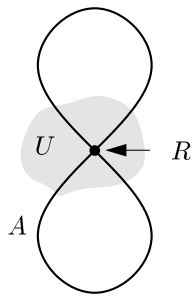

Figure 4: Crack robust.

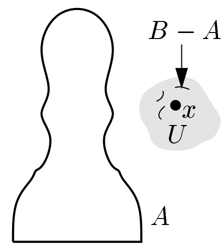

Figure 5: Noise robust.

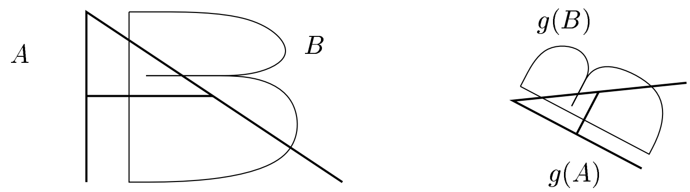

Figure 6: Affine invariance.

how small, change the values of the Hausdorff distance significantly. Strangely, the Hausdorff distance is used extensively in pattern recognition.

Figure 5 shows a pattern A and a point x. Addition of noise $B - A$ within a neighbourhood U of x results in a new pattern B. Axiom 4 says that the distance between A and B can be made smaller by making U smaller.

4 Constructing invariant pattern metrics

In this section we show how affine invariant pattern metrics can be formed by mapping patterns to real-valued functions and computing a normalised difference between these functions. Affine invariance is essential in many pattern matching and shape recognition tasks. Figure 6 shows two patterns superimposed on eachother on the left. On the right the images of the two patterns under an affine transformation are shown.

Let $\mathbf{I}(\mathbb{R}^n)$ be the vector space of real-valued Lebesgue-integrable functions on $\mathbb{R}^n$, with scalar multiplication and vector addition defined pointwise. Define the L^1 seminorm on $\mathbf{I}(\mathbb{R}^n)$:

$$\|\mathbf{a}\| = \int_{\mathbb{R}^n} |\mathbf{a}(x)|\, dx.$$

Let $C^1(\mathbb{R}^n)$ be the group of order 1 diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$. For a diffeomorphism $g \in C^1(\mathbb{R}^n)$, let $D_g^x : \mathbb{R}^n \to \mathbb{R}^n$ be the derivative of g in x, a linear function. The Jacobi-determinant is the determinant of the derivative at a given point. We use $j_g(x) = \left|\det(D_g^x)\right|$, to denote the absolute value of the Jacobi-determinant of g in x.

For real-valued functions $\mathbf{a}, \mathbf{b} : \mathbb{R}^n \to \mathbb{R}$, let $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{a} \vee \mathbf{b}$ denote the pointwise minimum and maximum, respectively. Define the *normalised difference* of two non-negative functions with non-zero integrals by

$$\sigma_n(\mathbf{a}, \mathbf{b}) = \frac{||\mathbf{a} - \mathbf{b}||}{||\mathbf{a} \vee \mathbf{b}||}.$$

Lemma 1 *The normalised difference σ_n is a semimetric on the set of non-negative functions with non-zero integrals.*

proof. Property (i)': Trivial. Property (ii): Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be non-negative functions with non-zero integrals. We need to prove:

$$\frac{||\mathbf{b} - \mathbf{c}||}{||\mathbf{b} \vee \mathbf{c}||} \leq \frac{||\mathbf{a} - \mathbf{b}||}{||\mathbf{a} \vee \mathbf{b}||} + \frac{||\mathbf{a} - \mathbf{c}||}{||\mathbf{a} \vee \mathbf{c}||}.$$

Since $||\cdot||$ is a seminorm the inequality $||\mathbf{b} - \mathbf{c}|| \leq ||\mathbf{u} - \mathbf{b}|| + ||\mathbf{u} - \mathbf{c}||$ holds, implying:

$$\frac{||\mathbf{b} - \mathbf{c}||}{||\mathbf{b} \vee \mathbf{c}||} \leq \frac{||\mathbf{u} - \mathbf{b}||}{||\mathbf{c} \vee \mathbf{b}||} + \frac{||\mathbf{u} - \mathbf{c}||}{||\mathbf{b} \vee \mathbf{c}||}.$$

Choosing $\mathbf{u} = \mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c})$, both terms on the right side of this inequality can be bounded, obtaining the triangle inequality. We show it only for the first term, since the procedure for the second one is analogous.

$$\begin{aligned}
\frac{||\mathbf{u} - \mathbf{b}||}{||\mathbf{c} \vee \mathbf{b}||} &\leq \frac{||\mathbf{u} - \mathbf{b}||}{||\mathbf{u} \vee \mathbf{b}||} \\
&\leq \frac{||\mathbf{a} - \mathbf{u}|| + ||\mathbf{u} - \mathbf{b}||}{||\mathbf{a} - \mathbf{u}|| + ||\mathbf{u} \vee \mathbf{b}||} \\
&= \frac{||(\mathbf{a} - \mathbf{u}) + (\mathbf{u} - \mathbf{b})||}{||(\mathbf{a} - \mathbf{u}) + (\mathbf{u} \vee \mathbf{b})||} \\
&= \frac{||\mathbf{a} - \mathbf{b}||}{||\mathbf{a} \vee \mathbf{b}||}.
\end{aligned}$$

□

Let $CJ(\mathbb{R}^n)$ be the subgroup of $C^1(\mathbb{R}^n)$ consisting of those g for which the Jacobi-determinant is constant as a function of $\mathbb{R}^n$. The next lemma shows that a large class of mappings from patterns to integrable functions result in invariant semimetrics based on the normalised difference σ_n.

Lemma 2 *Let $\mathcal{P}$ be a collection of subsets of $\mathbb{R}^n$. Let each $A \in \mathcal{P}$ define a unique function $\mathbf{n}_A : \mathbb{R}^n \to \mathbb{R}$ in $\mathbf{I}(\mathbb{R}^n)$. If $g \in \mathrm{CJ}(\mathbb{R})$ determines a number $\delta_g > 0$ such that*

$$\mathbf{n}_{g(A)}(g(x)) = \delta \mathbf{n}_A(x)$$

for all $A \in \mathcal{P}$ and $x \in \mathbb{R}^n$, then

$$\sigma_n(\mathbf{n}_{g(A)}, \mathbf{n}_{g(B)}) = \sigma_n(\mathbf{n}_A, \mathbf{n}_B)$$

for all $A, B \in \mathcal{P}$.

proof. Apply substitution of variables using the constant $j = j_g(x)$:

$$\begin{aligned}
&\sigma_n(\mathbf{n}_{g(A)}, \mathbf{n}_{g(B)}) \\
&= \frac{\|\mathbf{n}_{g(A)} - \mathbf{n}_{g(B)}\|}{\|\mathbf{n}_{g(A)} \vee \mathbf{n}_{g(B)}\|} \\
&= \frac{j\,\|\mathbf{n}_{g(A)} \circ g - \mathbf{n}_{g(B)} \circ g\|}{j\,\|\mathbf{n}_{g(A)} \circ g \vee \mathbf{n}_{g(B)} \circ g\|} \\
&= \sigma_n(\mathbf{n}_{g(A)} \circ g, \mathbf{n}_{g(B)} \circ g) \\
&= \sigma_n(\delta_g \mathbf{n}_A, \delta_g \mathbf{n}_B) \\
&= \sigma_n(\mathbf{n}_A, \mathbf{n}_B).
\end{aligned}$$

□

5 The reflection metric

In this section we define the reflection metric. First, we define the class of patterns on which this metric is defined. After that, we discuss the notion of visibility which is fundamental to the reflection metric. Using visibility, we construct functions from patterns, which lead to the definition of the reflection metric. The results from Section 4 ensure affine invariance of the metric. Finally, we show that the reflection metric satisfies the four robustness axioms from Section 3. From here on, we assume that the dimension n is at least 2.

The axioms in Section 3 were defined in terms of an abstract pattern collection $\mathcal{P}$ consisting of boundaries of relatively compact subsets of $\mathbb{R}^n$. In this section we choose $\mathcal{P} = \mathcal{R}^n$, defined as follows. Each element of $\mathcal{R}^n$ is a finite union of $(n-1)$-dimensional algebraic surface patches in $\mathbb{R}^n$ that is not contained in an $(n-1)$-dimensional hyperplane. In two dimensions, $n = 2$, each pattern in $\mathcal{R}^n$ is a finite union of algebraic curve segments.

We use the notation $\overline{xy}$ for the straight open line segment connecting two distinct points $x, y \in \mathbb{R}^n$. We say that a point $y \in \mathbb{R}^n$ is visible (in A) from a point $x \in \mathbb{R}^n$ if $A \cap \overline{xy} = \varnothing$. For $A \in \mathcal{R}^n$ and $x \in \mathbb{R}^n$, the *visibility star* V_A^x is

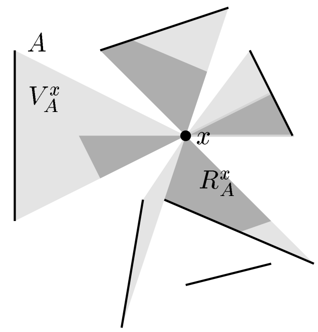

Figure 7: Reflected visibility.

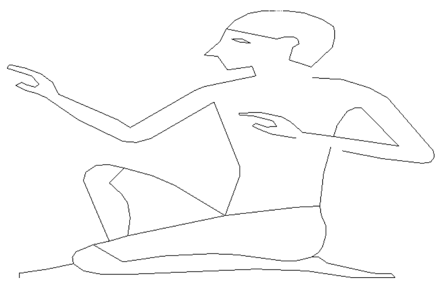

Figure 8: A pattern $A \in \mathcal{R}^2$.

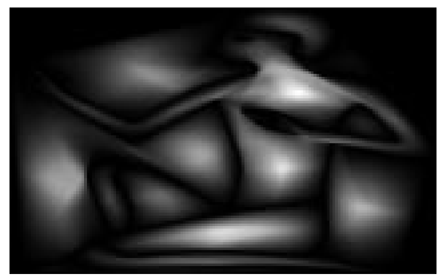

Figure 9: The function ρ_A.

defined as the set of open line segments connecting points of A that are visible from x:

$$V_A^x = \bigcup \{ \, \overline{xa} \mid \ a \in A \text{ and } A \cap \overline{xa} = \varnothing \, \}.$$

We define the *reflection star* R_A^x by intersecting V_A^x with its reflection in x:

$$R_A^x = \{ \, x + v \in \mathbb{R}^n \mid \ x - v \in V_A^x \text{ and } x + v \in V_A^x \, \}.$$

Figure 7 shows the visibility star V_A^x and the corresponding reflection star R_A^x for a pattern $A \in \mathcal{R}^2$, and a point $x \in \mathbb{R}^2$.

Each pattern $A \in \mathcal{R}^n$ determines a function $\rho_A : \mathbb{R}^n \to \mathbb{R}$ given by $\rho_A(x) = \mathrm{vol}(R_A^x)$. Note that ρ_A is zero outside the convex hull of A. Figure 8 shows a two-dimensional pattern A consisting of a finite number of line segments. Figure 9 shows the corresponding function $\rho_A : \mathbb{R}^2 \to \mathbb{R}$ represented as a grey-scale image, in which black corresponds with value 0. The example pattern is the hieroglyphic "A1" obtained from the hieroglyphica sign list, see [7].

The definition of the reflection metric is based on the normalised difference, see Lemma 1.

Definition 1 *The* reflection metric $d_{\mathtt{R}}$ *for* $\mathcal{R}^n$ *is given by*

$$d_{\mathtt{R}}(A, B) = \sigma_n(\rho_A, \rho_B).$$

Lemma 2 tells us that the metric $d_{\mathtt{R}}$ is invariant under the group of affine transformations $\mathrm{Af}(\mathbb{R}^n)$.

Next, we prove the four robustness axioms from Section 3. Observe that for any two patterns $A, B \in \mathcal{R}^n$ and any point $x \in \mathbb{R}^n$:

$$\mathrm{vol}(R_A^x - R_B^x) \leq 2\ \mathrm{vol}(V_A^x - V_B^x).$$

From this, we find that

$$|\rho_A(x) - \rho_B(x)| \leq 2\ \mathrm{vol}((V_A^x - V_B^x) \cup (V_B^x - V_A^x)). \tag{1}$$

Thus, we can prove the first four axioms by bounding the change in the visibility star for deformation, blur, crack and noise.

The metric $d_{\mathtt{R}}$ is deformation robust. Let $A \in \mathcal{R}^n$ and $\epsilon > 0$. Using Eq 1, choose $\delta > 0$ small enough such that $\|\rho_A - \rho_{t(A)}\| < \epsilon\ \|\rho_A\|$ for all transformations $t \in T$ satisfying $\|x - t(x)\| < \delta$ for all $x \in A$. It follows that

$$d_{\mathtt{R}}(A, t(A)) \leq \frac{\|\rho_A - \rho_{t(A)}\|}{\|\rho_A\|}$$

for the same transformations t.

The reflection metric is blur robust. Let $A \in \mathcal{R}^n$ and $\epsilon > 0$ be given. Choose an open neighbourhood U of A with $\mathrm{vol}(U) < \delta$ for some given $\delta > 0$. Using Eq. 1, choose $\delta > 0$ small enough so that for all $B \in \mathcal{R}^n$ satisfying $B - U = A - U$ and $A \subseteq B$, the distance $d_{\mathtt{R}}(A, B)$ is smaller than ϵ.

The reflection metric is crack robust. Let $A \in \mathcal{R}^n$, R be a crack of A, and $\epsilon > 0$. By means of Eq. 1, we can choose a sufficiently small open neighbourhood U of the crack R such that $d_{\mathtt{R}}(A, B) < \epsilon$ for patterns $B \in \mathcal{R}^n$ satisfying $B - U = A - U$.

The reflection metric is noise robust. Let $A \in \mathcal{R}^n$ and $x \in \mathbb{R}^n$ be given. Using Eq. 1, we can choose an open neighbourhood U of x small enough such that $d_{\mathtt{R}}(A, B) < \delta$ for all $B \in \mathcal{R}^n$ satisfying $B - U = A - U$.

6 Experimental results

In this section, we present experimental results obtained using a computer program that computes the reflection metric. Our test inputdata consists of hieroglypics, represented as unions of line segments. The experiment compares the robustness of the Hausdorff distance and the reflection metric.

We compare the behaviour of the Hausdorff metric and the reflection metric under increasing amounts of distortion. Figure 10 shows patterns $A_1, \ldots, A_5$

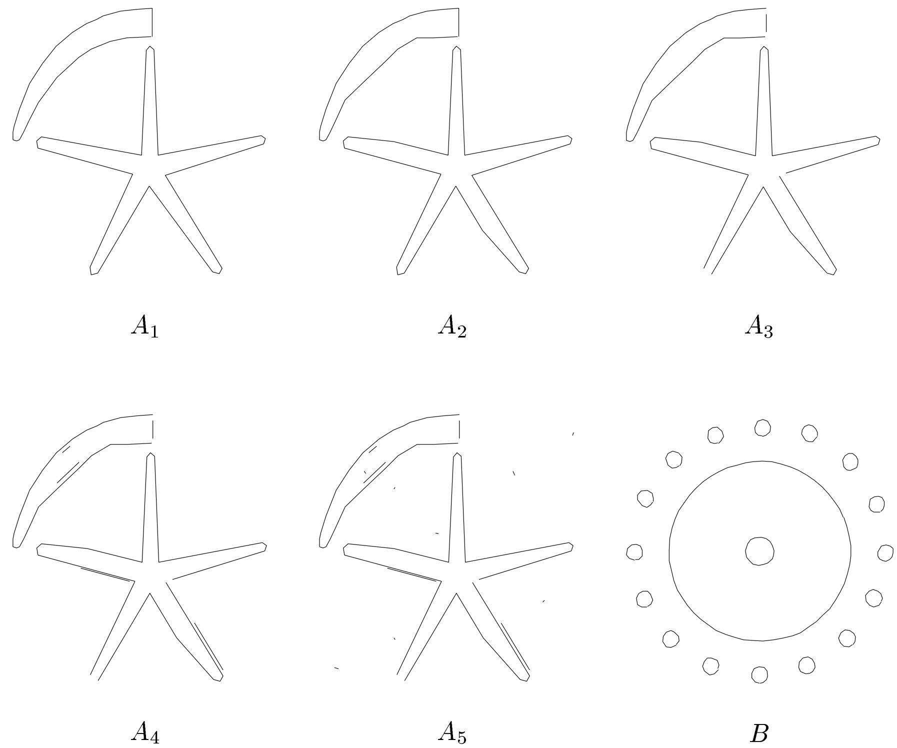

Figure 10: Increasingly distorted patterns.

and B. Pattern A_1 a hieroglyphic (coded "N13" in [7]). Patterns $A_2, \ldots, A_5$ are increasingly distorted versions of A_1. Pattern A_2 is a slightly deformed image of A_1. Pattern A_3 is a version of A_2 in which parts have been connected and separated at small cracks. Pattern A_4 is a blurred version of A_3. Pattern A_5 is A_4 with noise. Pattern B (coded "N55")is another hieroglypic than A_1, not similar to A_1 for the human eye.

We computed a square containing all the patterns. After that we scaled all patterns such that the bounding box had diameter 1. This causes the values of the hausdorff distances to lie between 0 and 1, just like the reflection distances (which of course do not change under this scaling). Then, we computed the distances between any two patterns in $A_1, \ldots, A_5, B$. We did this using both the Hausdorff distance and the reflection distance. Tables 1 and 2 are distance matrices for the Hausdorff distance and the reflection metric, respectively.

Under both the Hausdorff metric and the reflection metric, the distance from the original increases as the pattern becomes more distorted. That is, $d_{\mathtt{H}}(A_1, A_i)$ and $d_{\mathtt{R}}(A_1, A_i)$ increase as i increases. Since both metrics are robust for the first three types of distortion, we see that the first three increases in distance are relatively small for both metrics. The fourth step, adding noise, has far more impact on the Hausdorff metric as on the reflection metric. Table 1 indicates

$d_{\mathtt{H}}$	A_1	A_2	A_3	A_4	A_5	B
A_1	0	0.0158114	0.0158114	0.0158114	0.29	0.240832
A_2	0.0158114	0	0.0141421	0.0158114	0.29	0.240832
A_3	0.0158114	0.0141421	0	0.0158114	0.29	0.240832
A_4	0.0158114	0.0158114	0.0158114	0	0.29	0.240832
A_5	0.29	0.29	0.29	0.29	0	0.135093
B	0.240832	0.240832	0.240832	0.240832	0.135093	0

Table 1: Hausdorff distance matrix for $A_1, \ldots, A_5, B$.

$d_{\mathtt{R}}$	A_1	A_2	A_3	A_4	A_5	B
A_1	0	0.0624949	0.089861	0.096348	0.133712	0.790727
A_2	0.0624949	0	0.0370037	0.0502051	0.0884063	0.792415
A_3	0.089861	0.0370037	0	0.0141517	0.0630621	0.795813
A_4	0.096348	0.0502051	0.0141517	0	0.0521414	0.796604
A_5	0.133712	0.0884063	0.0630621	0.0521414	0	0.792002
B	0.790727	0.792415	0.795813	0.796604	0.792002	0

Table 2: Reflection distance matrix for $A_1, \ldots, A_5, B$.

that the Hausdorff distance between A_1 and B is smaller than the Hausdorff distance $d_{\mathtt{H}}$ between A_1 and A_5. This example illustrates that the Hausdorff distance is not noise robust. For the human eye, A_5 is quite similar to A_1, while A_1 and B seem to have no visual similarities at all. The reflection metric $d_{\mathtt{R}}$ agrees with this, since $d(A_1, A_5)$ is much smaller than $d(A_1, B)$ in Table 2.

Thus, both the Hausdorff distance and the reflection metric are robust for deformation, crack, and blur. However, when noise is added, the Hausdorff distance, not being noise robust, can produce undesirable results. The noise robustness of the reflection metric is confirmed by the test results.

7 Conclusion

This paper has two contributions. First, we presented four general axioms expressing robustness for metrics defined on boundaries of relatively compact subsets of $\mathbb{R}^n$. Although the importance of robust similarity measures is recognised in literature, until now little serious attempts have been made to formalise them. Second, we presented the reflection metric, which is an affine invariant pattern metric defined on finite unions of $(n-1)$-dimensional algebraic surface patches in $\mathbb{R}^n$. We proved that the four robustness axioms hold for the reflection metric. The experimental results in Section 6 indicate that the reflection metric is also robust in practice. It compares favorably with the Hausdorff distance, if noise is added to the patterns.

The next logical step in this research is the efficient computation of the reflection metric. This can be done using techniques from computational geometry. The two-dimensional version of the reflection metric can be computed in $O(r(s_A + s_B))$ time for two separate collections of line segments with sizes s_A and s_B, where r is the complexity of the overlay of two *reflection partitions*, and the time needed to integrate a quotient of polynomials of maximum degree d over a triangle is $\Theta(d)$. For details, see [8].

More interesting than just computing the reflection metric for pairs of patterns, is computing the *minimum* reflection metric under a set of affine transformations. This way, the distance between affine shapes can be computed. However, this minimisation is a difficult non-convex optimisation problem. Another question is whether less complicated affine invariant pattern metrics on boundary patterns exist, satisfying our axioms.

References

[1] O. Aichholzer, H. Alt, and G. Rote. Matching shapes with a reference point. In *Int. J. of Computational Geometry and Applications*, volume 7, pages 349–363, August 1997.

[2] H. Alt, U. Fuchs, G. Rote, and G. Weber. Matching convex shapes with respect to the symmetric difference. In *Algorithms ESA '96, Proc. 4th Annual European Symp. on Algorithms, Barcelona, Spain, September '96*, pages 320–333. LNCS 1136, Springer, 1996.

[3] H. Alt and M. Godeau. Computing the Fréchet distance between two polygonal curves. *Int. J. of Computational Geometry & Applications*, pages 75–91, 1995.

[4] L. P. Chew, M. T. Goodrich, D. P. Huttenlocher, K. Kedem, Jon M. Kleinberg, and Dina Kravets. Geometric pattern matching under Euclidean motion. In *Fifth Canadian Conference on Computational Geometry*, pages 151–156, 1993.

[5] S. D. Cohen and L. J. Guibas. Partial matching of planar polylines under similarity transformations. In *Eight Annual ACM-SIAM Symp. on Discrete Algorithms*, pages 777–786, January 1997.

[6] E.T. Copson. *Metric spaces*. Cambridge University Press, 1968.

[7] N. Grimal, J. Hallof, and D. van der Plas. Hieroglyphica, sign list. http://www.ccer.theo.uu.nl/hiero/hiero.html, 1993.

[8] M. Hagedoorn, M.H. Overmars, and R. C. Veltkamp. New visibility partitions with applications in affine pattern matching. Technical Report UU-CS-1999-21, Utrecht University, 1999. http://www.cs.uu.nl/docs/research/publication/TechRep.html.

[9] M. Hagedoorn and R. C. Veltkamp. A general method for partial point set matching. In Jean-Daniel Boissonnat, editor, *Proc. 13th Annual ACM Symp. Computational Geometry*, pages 406–408. ACM Press, 1997.

[10] M. Hagedoorn and R. C. Veltkamp. Reliable and efficient pattern matching using an affine invariant metric. *International Journal of Computer Vision*, 31(2, 3):103–115, 1999.

[11] D. P. Huttenlocher and K. Kedem. Computing the minimum Hausdorff distance for point sets under translation. In *Proc. 6th Annual ACM Symp. Computational Geometry*, pages 340–349, 1990.

[12] D. P. Huttenlocher, K. Kedem, and M. Sharir. The upper envelope of Voronoi surfaces and its applications. *Discrete and Computational Geometry*, 9:267–291, 1993.

[13] D.P. Huttenlocher, K. Kedem, and W.J. Rucklidge. Comparing images using the Hausdorff distance. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 15:850–863, 1993.

[14] W. Rucklidge. *Efficient Visual Recognition Using the Hausdorff Distance.* Number 1173 in Lecture Notes in Computer Science. Springer-Verlag, 1996.