

GEOMETRIC MODELLING WITH α -COMPLEXES

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ABSTRACT

The shape of real objects can be so complicated, that only a sampling data point set can accurately represent them. Analytic descriptions are too complicated or impossible. Natural objects, for example, can be vague and rough with many holes. For this kind of modelling, α -complexes offer advantages over triangulations and hulls at little extra computational cost. Geometric and topological descriptions are well-formalised, with the flexibility to capture holes, up to a complete separation. Spatial distribution of the point set and the attachment of weights make “special modelling effects” possible. We explore in this paper the merits of geometric modelling with α -complexes, with the objective of evaluating their practical value. We discuss the α -complex as a model description and as a representation scheme. Varying the α -value is intuitive, but weighting can be tedious. We present a few strategies. We also show how to run FEM computations on α -complexes. α -Complexes form a useful addition to existing approaches and are applicable to a number of problems not (easily) handled by existing approaches.

KEYWORDS

geometric modelling, alpha shapes, representation scheme, weighting strategies.

NOMENCLATURE

E^d	=	d -dimensional euclidian space, $d \geq 0$
S	=	finite sampling data point set, $S \subset E^d$
W	=	set of real weights to attach to S
\tilde{S}	=	weighted sampling data point set, $\tilde{S} = S \otimes W$
\mathbf{x}	=	point in E^d
$\tilde{\mathbf{x}} \cong (\mathbf{x}, w)$	=	weighted point in $E^d \times W$
α	=	alpha value in $[0..∞)$
$\mathcal{C}_\alpha(\tilde{S})$	=	α -complex; $\mathcal{C}_\alpha(\tilde{S}) \subseteq \mathcal{T}(\tilde{S})$
$\mathcal{T}(\tilde{S})$	=	regular triangulation of point set \tilde{S}
k -face	=	simplex of dimension k , $-1 \leq k \leq d$

1. INTRODUCTION

Over the past couple of decades, engineers and designers came to understand how to unambiguously describe physical objects in terms of boundaries and volumes. Today, using predefined primitives, sweeps, extrusion and part-whole descriptions, fairly complex objects and assemblies can be created. Practical implementations exploiting these capabilities, however, target primarily at *engineering* objects and assemblies, with regular mathematical properties. As soon as *natural* or *measured* objects are to be modelled, many such practical approaches fall short. The more so if modelled objects of this type are to be submitted to some numerical analysis tool, e.g. for stress or tolerance analysis. Also, many of today’s modelling approaches are incapable of handling changing topologies. Finally, objects may get too complicated to find an analytical description, with the only remaining form of representation be-



Figure 1: example of an evolving zero-weight α -complex: left: $\alpha = 3.096634e + 01$, yielding a complex that is little more than just the point set with a few singular edges, developing into right: an overly fat ($\alpha = 1.805128e + 03$) complex obscuring all the fine details of the face and the neck, almost grown into the convex hull. The best α for the details of the face is approx. $\alpha = 1.444991e + 02$, the value of the centre picture. In this case, however, no single α -value exists that fits all regions in the model. Adding weights will have to compensate for this (data set by Silicon Graphics, $n = 2780$).

ing a sampling data point set. *Faceted polyhedral objects* are typical representatives. But similar problems are encountered in complex *motion planning, assembly planning, visibility analysis* and a wealth of other geometry-dominated problems.

This is exactly the field where α -complexes came up-front as an alternative, capable of modelling typical engineering as well as natural objects. α -Complex-modelling offers a machinery to study various appearances of the “shape” of a set of points. Roughly speaking, it may be seen as a triangulation of that point set with an upper bound on the edge length governed by a single parameter α . The effect is a sort of shrink-wrap of the object. The spatial occupancy by an α -complex is called an α -*shape* and an α -complex is a triangulated α -shape. If the partitioning of the underlying space is irrelevant, α -shape can also be read where α -complex is written. The sample data points can be attributed a *weight*; a real-valued magnitude expressing the dominance of that point over other points. In a neighbourhood of points with higher weights, the α -complex tends to develop at lower α -values, whereas negative weights discourage the α -complex to develop. Sometimes, a distinction is made between non-weighted and weighted α -complexes. Here, that distinction will not be made and α -complexes herein shall always be *weighted*. Of course, weights can all be chosen equal-valued, possibly zero. Weighting, on top of variation of α , makes “special modelling effects” possible, and is in fact the key to practical applications.

This paper is organised as follows. After this in-

roduction and a brief recapitulation of previous work, an informal introduction to α -complexes will be given, followed by the distinct steps of modelling with α -complexes. Next, weighting strategies will be introduced, both for “free form” modelling and for shape reconstruction. Then, a few geometric modelling case studies are discussed: free form shapes, reconstructed polyhedral engineering and natural objects, and dynamic and numerical problems. Eventually, the merits of modelling with α -complexes are evaluated, with regard to their practical value for the industrial and engineering community. The paper concludes with conclusions and suggestions for further research.

2. PREVIOUS WORK

α -Complexes are mainly due to Edelsbrunner (Edelsbrunner (1987), Edelsbrunner (1992), Edelsbrunner and Muecke (1994)), and build on the results of regular triangulations of weighted point sets (e.g., Lee (1991), Edelsbrunner and Shah (1996)). Studies on the geometry of spheres and balls are classical (e.g., Coolidge (1916), Brand (1947), Maxwell (1952)), with recent work in Schwerdtfeger (1979), Hahn (1994) and Boehm and Prautzsch (1994). Paoluzzi et al. (1993), pointed out the suitability of multi-dimensional simplicial complexes for geometric modelling. They did not encompass, however, non-regular complexes, like α -complexes. Edelsbrunner et al. (1998), described an application of α -shape-modelling to molecular geometric modelling and Gerritsen (1998) described applications in earth sciences.

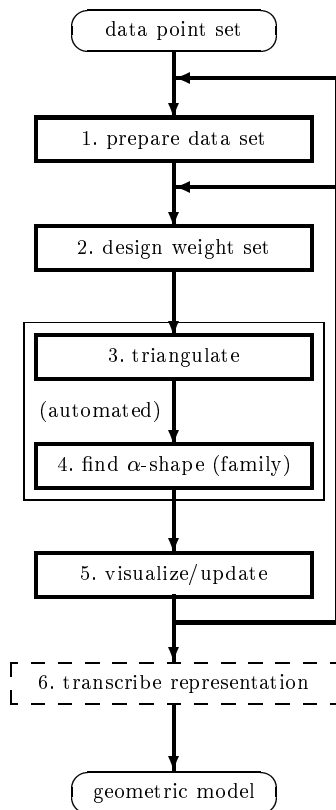


Figure 2: the sequence of basic steps involved in modelling with α -complexes, some steps may be void. Starting point is always a sampling data point set and the final result is always an α -complex, that may be transcribed into another representation. Dashed boxes represent optional steps. Triangulation and α -complex computation are steps with no or little human interaction. The modelling ingenuity is chiefly in the first steps.

Literature on weighting strategies is sparse; the theory of weighting is fairly well understood, particularly in conjunction to sphere-geometry (e.g., Aurenhammer (1987), Aurenhammer and Imai (1987), Edelsbrunner (1987), Edelsbrunner (1992)), but the use from a modellers’ perspective is faintly addressed (Aichholzer et al. (1999); Gerritsen (1998)). The mathematical foundation of representation schemes is mainly due to Requicha (1980). In more recent work, Kalay (1989) further works out a number of details of various representation schemes. Further, refer to Requicha (1983), Zeid (1991).

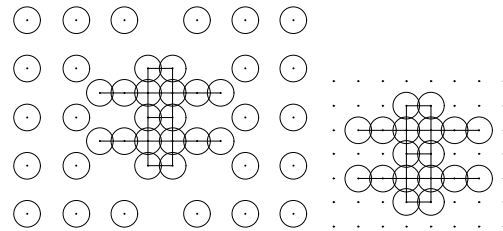


Figure 3: left: irregularly spaced but equally weighted points, right: regularly spaced, but unequally weighted points. Spheres represent weight; same size means same weight. Points with no spheres (right) have zero weight. The α -complex here is a 1-skeleton (“wire-frame”). Observe the similar and interchangeable effect of α and weight: $\mathcal{C}_{\alpha=w}(S \times W|_{w_i=0}) = \mathcal{C}_{\alpha=0}(S \times W|_{w_i=\alpha})$ Edelsbrunner (1992).

3. α -COMPLEXES

A formal description of α -complexes can be found in Edelsbrunner (1992), Edelsbrunner and Muecke (1994) and in Gerritsen et al. (2000). Less ponderously, an α -complex based on a point set S uniquely and unambiguously defines the object’s interior and exterior, including holes. Here, the term *holes* is used for a variety of topological features. Neither interior nor exterior need be connected and topologies may be manifold or non-manifold. The α -value reflects the distance over which neighbouring data points can connect. A low value of α yields an α -complex close to just the data point set, with no or few neighbours connected, a higher value of α results in an α -complex which matches or nearly matches the triangulation (see figure 1). With weights attached to the points, the evolution of an α -complex is not only determined by *proximity* (as reflected by α), but also by *dominance* (weights). In a neighbourhood of high weights, points tend to connect more easily than in neighbourhoods with lower weights. The α -complex and the eventual regular triangulation share faces (vertices, edges, triangles, tetrahedra, ...). Finally, a triangulation is for example also a *cellular decomposition*, paving the way to various “transcriptions”.

4. MODELLING STEPS

Modelling with α -complexes is a multi-step process, see figure 2. Basically, after the preparation of the *sampling data point set* S and the *weight set* W , the ordered Cartesian product $S \otimes W$ is triangulated. Under certain conditions, the use of transformations of S can be justified, to “precondition” the data. Transformations cause the spacing to alter and, as shown in figure 3, the effect is complementary to the effect of changing weights. This is not difficult to understand if one realizes that Euclidean distance and weight come together in the weighted distance underpinning weighted α -complexes. Then, sorted by α , the finite set of α -complexes (called: the α -family) is determined. As α grows, the α -complex grows into an ever greater subcomplex of the triangulation. Once identified, the appropriate α -complex may be converted into alternate representations. α -Complex modelling is best positioned by first considering the subdivision:

Problem 1; free form shaping: the α -complex representing the object is built from a modeller-generated, *synthetic* sampling data point set, thereby honouring geometric and topological constraints. Generally, the set of possible solutions to the problem is not limited to a single solution and changes to the sampling data point set S and/or the weight set W may be used to alter the shape.

Problem 2; shape reconstruction: the sampling data point set is now based on *observed* data. Generally, the observation will be subject to noise and errors. Although basically multiple solutions can be found, a single “best fit” can generally be identified according to optimisation of some cost function. The sampling data point set S is commonly left untouched (you don’t change observations) and control must therefore come from changes to weight set W .

Both classes of problems may be submitted to numerical analysis. Refer to figure 4.

The hypothesis is now that natural objects are not modelled using a mere free form shaping but using a combination of both free form shaping and shape reconstruction. Most likely, shaping will be feature- or knowledge-based, with geometry and topology resulting from variational ge-

ometry. α -Complexes with their *implicitly* ruled geometry and topology, are natural candidates for that purpose.

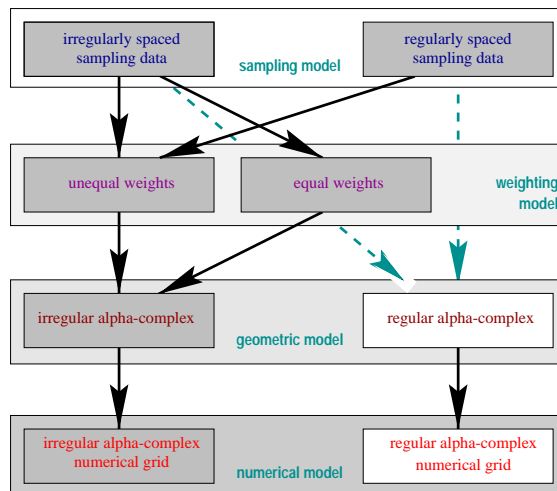


Figure 4: schematic view on the relationship between sampling data point set organisation, geometry and numerical model geometry. Gray-ed boxes show the path followed with α -complexes. Observe that both irregularly sampled data and regularly sampled data can be turned into an α -complex. For regularly spaced data samples (e.g., digital pictures), the use of unequal weights is a prerequisite.

5. WEIGHTING STRATEGIES

Both irregularly sampled data and regularly sampled data can be turned into an α -complex. For regularly spaced data samples (e.g., digital pictures), the use of (unequal) weights is essential to gain control over the geometry of the α -complex-model. With *weighting strategy*, we refer to the process of obtaining a weight set W such that for some $\alpha \in [0.. \infty)$ the resulting α -complex is fulfilling all the geometric and topological constraints imposed upon the object’s model.

5.1. UNSTRUCTURED WEIGHTING

Weight can be assigned on a per-point basis. This form of weighting will be referred to as *unstructured weighting*. The use of unstructured weighting, applicable to both regularly and irregularly spaced data, is of limited practical value. It is typically used in

combination with free-form shaping of simple shapes. To some extent, the painting (spraying) of weight can be regarded as unstructured weighting (cf. sect. 6.2). Unfortunately, such tools are still lacking for 3D. A best-practice solution would be the use of sliced data.

5.2. MASKING

With regularly spaced points, modelling effects have to be inflicted by weights. *Masking* is a weighting strategy for regularly spaced sampling point sets. Weights are thereby compiled by stacking the weight values of one or more basic weight *masks* on a common *canvas*. A canvas is a $2D$ - or $3D$ -regular grid representing a point set S , and the masks can be understood as overlay grids adding the weight. Stacking weights means accumulating the weight values of the corresponding grid nodes of the masks on the initially zero-weight canvas. The idea is similar to constructing a complex shape by merging a number of simple basic shapes, using boolean and math operations. Weight of different masks may be added, subtracted, negated, etc., as indicated in figure 5. Masks may also be translated relative to the canvas, rotated or scaled, as appropriate. Stacking is essentially a “pixel”-based logical or mathematical operation. Once the desired accumulated weight set has been constructed, the weighted α -complex, based on $S \otimes W$ can be constructed.

Standard masks can be collected in a library and a fitting tool may aid in finding the right combination for the desired effect. Masks may be based on standard weight functions, like the ones in table 1. These functions can be taken relative to a point, a line, a plane, etc. Arbitrary constrained weight functions can be compiled using for example Lagrange polynomials. Basically, masking is applicable to regularly shaped sampling data point sets of any model space dimension E^d , although tools become increasingly sparse for $d > 2$.

5.3. PHYSICAL PROPERTIES

Shape reconstruction is relevant when object models or object features are to be derived from

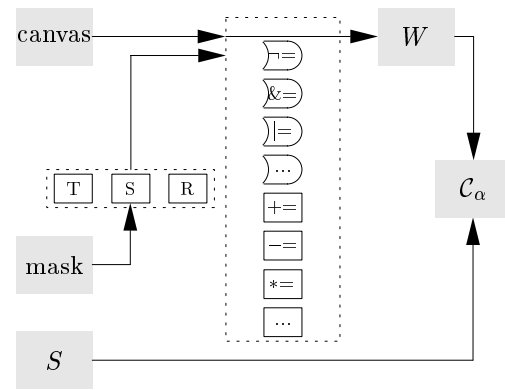


Figure 5: the masking technique; the aim is to compose a weight set W , that “highlights” the desired geometry (C_α) in the regularly spaced sampling data point set S . Starting point is an initially zero-weight canvas (regular grid) representing the sample points. One or more weight masks overlay the canvas, attributing weights to the sample points. Geometric transformations (T, S, R ; small dashed box) position the masks relative to the canvas. Logical and mathematical operations (large dashed box) allow to add up, subtract, negate, etc. the weights of masks to the canvas. Once the desired weight set W has been composed on the canvas, the Cartesian product $S \otimes W$ is input to C_α .

real-live counterparts. For example when shaping an artificial knee-cap for an individual patient. The quality of the observed geometry depends on a number of things. Among others:

- The observation technique.
- The “contrast” in the data.
- The threshold value chosen to discriminate “background” (embedding environment) from “foreground” (the target object).
- The suite of properties observed.

Often with real-live objects, the exact shape depends on the physical property being measured. An echo, an infra-red image, a PET-scan, they easily yield different geometries of the same object. We refer to such shapes as *property-ruled* shapes. Many natural objects “extracted” from observed data are in fact property-ruled, and a direct relationship can be established between

FUNCTION	DEFINITION
gamma(ζ)	$\Gamma_\gamma(\zeta) = \int_0^\infty \zeta^{\gamma-1} e^{-\zeta} d\zeta, \gamma > 0$
-ln(ζ)	$-\ln(\zeta) = \int_0^\zeta \eta^{-1} d\eta, \zeta > 0$
error(ζ)	$\text{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^\zeta e^{-\eta^2} d\eta$
normal(ζ)	$p(\zeta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\zeta-\mu)^2}{2\sigma^2}}$
sign(ζ)	$\text{sign}(\zeta) = \begin{cases} -1 & \text{for } \zeta < 0 \\ 0 & \text{for } \zeta = 0 \\ 1 & \text{for } \zeta > 0 \end{cases}$
block(ζ)	$p(\zeta) = \begin{cases} 0 & \text{for } \zeta \notin [a, b] \\ 1 & \text{for } \zeta \in [a, b] \end{cases}$

Table 1: natural weight functions exhibiting a “cut-off” effect. We may use the linear transform $\zeta = \beta_1 \xi + \beta_2$, with $\beta_1, \beta_2 \in \mathcal{R}$, to allow shifting and scaling of the “cut-off” point.

the weights and the measured property (e.g., sect. 6.3). Observed sampling data get blurred by noise and other undesired effects. If such features hamper the finding of the right α -complex, filtering away remote areas outside the region of interest is easily accomplished. Another problem with observed data is the translation of vector- and tensor-properties to the basically *omni-directional* (i.e., scalar) weights in the current concept.

5.4. NEIGHBOURHOOD WEIGHTING

The neighbourhood-based weighting scheme seeks to minimise the variance of weighted nearest neighbour distances, so that all parts of the α -complex develop within a well-controlled range of α . It can be applied to regularly and irregularly spaced data points. It is particularly useful for sampling data point sets with unequal spatial distributions in different regions or with too high a weighted distance variance. First, the global mean nearest neighbour distance of the entire sampling data point is determined. The idea is then to locally adapt weights such that the weighted distances in each neighbourhood are closest to this global mean. The effect is a minimisation of the overall variance, a homogenisation of the distances, by virtue of

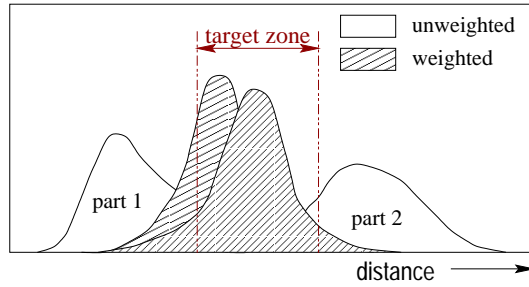


Figure 6: principle of homogenisation of weighted distances by neighbourhood weighting. Adding weight decreases the weighted distances. Graphs show the empirical distribution function of the weighted distances of different parts of an object. The α -value required for points to connect is related to the distances. Neighbourhood weighting seeks to modify the local distribution of weight such that a *target zone* can be found with an α that fits all parts of the model.

which a common α -value can be found, appropriate for each neighbourhood and hence for the entire α -complex. In practical applications, conflicts may occur and a global shift of the mean weighted distances may be needed in order to find a common α (see figure 6).

6. EXPERIMENTAL CASES

In this section, a number of α -complex-based geometric modelling cases, of both engineering objects and natural objects will be presented. Also, a few words will be spent on numerical modelling with α -complexes.

6.1. ENGINEERING OBJECTS

A goblet is a simple central axis circular symmetric object, as shown in figure 7. A first approximation by a zero-weight α -complex shows only few defects. A weight model based on weight function *block*(ζ) (cf. table 1 and figure 8) removes these defects, yielding a perfect goblet.

6.2. A RABBIT

A very simple example of a free-form natural object is given by the rabbit-like creature

of figure 10. The rabbit has been created by spraying coloured weight on a canvas. Sprayed weight can be homogenised for example using a “smearing”-filter. Approximately the same result could have been obtained (refer to figure 3) by spraying properly-spaced zero-weight data points.



Figure 8: single zone/single value weights, obtained by the $block(\zeta)$ -function (table 1). Observe the bolder dots in the zone from stem-to-cup. Left: front view, right: top view.



Figure 9: the effect of improper weight sets for the goblet α -complex model. Left: the negative stem-to-cup weight zone is too large. By the time the increasing α can override the negative weight, the entire zone will be bridged at once (observe the singular edges). Right: the negative weight zone is too small; the points at the inside bottom of the cup are able to connect, whereas the points from stem-to-cup are not. Compare to figure 7.

6.3. THE COMET WEST

A comet is an irregularly shaped natural object of frozen gas and rocky debris, orbiting around the sun. A comet has a focal bright approximately 10 km wide kernel, called the *nucleus*. When approaching the sun, a comet develops three tails: the bright *coma*, a large trailing cloud of diffuse material, the pale-blue *ion tail* of ionised plasma, and the yellowish *hydro-*

gen envelope, with hydrogens that escaped the comet’s gravity. The comet West¹ has been observed by various observers during its bright appearance in 1976. See figure 11, left picture.

The model of figure 11 was obtained by exploiting the effect demonstrated in figure 3. Rather than turning physical properties (represented here by white, yellow and blue of the tails) directly into weight, we augmented the 2-dimensional sampling point space S by a 3-dimensional colour space $R \times G \times B$. This results in a 5-dimensional hyper-space \tilde{S} . Generating an α -complex of \tilde{S} causes a “natural” clustering to appear, by colour value. Cluster membership, intersection and projection can be used to extract the region of the α -complex searched for. This approach works best for strong clusterings in property space. Notice that the colour determines the “direction”, but the intensity distribution (contrast) determines the distances within and between clusters. The clustering in this case is weak, but nevertheless the approach works.

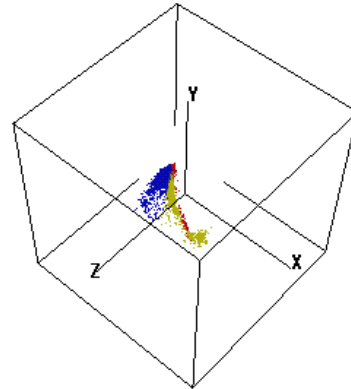


Figure 12: α -complex of the colour parameter space. X-axis represents red, Y-axis green and Z-axis blue. Multiple “clusters” are visible: a blue cluster (dark) representing the ion tail and the edges of the dust tail, a gray cluster along the main diagonal representing the dust tail and a rest cluster (nucleus, coma, dust tail).

6.4. MEDICAL MODELLING

Organ imaging is an important aid in clinical diagnosis. α -Complexes can be used in (multi-

¹named after the astronomer West, who first described his observation of the comet.



Figure 7: leftmost three figures show the zero-weight α -complex of a goblet data set ($n = 520$). Close examination shows extraneous faces in the stem-to-cup transition. Applying a non-zero weight set (figure 8) lifts this defect. The final result is shown at the right (data set obtained from GEOMVIEW DISTRIBUTION).



Figure 10: rabbit-like animal, free-form shaped by spraying “coloured” weight on a 400×400 regular rectangular canvas. RGB-colours are translated to normalised weight $w \in [-1..1]$: $w = \pm \frac{1}{2} \bar{R} \pm \frac{1}{3} \bar{G} \pm \frac{1}{6} \bar{B}$, where \bar{R}, \bar{G} and \bar{B} denote normalised colour components (gray-levels) for red (heavy) weight, green (medium) weight and blue (light) weight, resp. Left: blue-coloured S (observe sparsely sampled spots). Centre: α -complex of the blue-mask rabbit. Right: rabbit overlaid with a green weight mask covering the sparse locations; the 6 holes of the central figure have now vanished.

dimensional) modelling of organs. See for example Smets et al. (1990). Figure 13 shows an α -complex of the left and right shoulder blade (*scapula*). The picture shows a zero-weight α -complex, that captures the bones fairly well, except for relatively few singular triangles at the *glenoid cavity* (not visible). Weighting was required to turn the complex into a model good enough to be fed into an FEA-tool.

6.5. NUMERICAL MODELLING

An α -complex, in terms of numerical computing, is generally an *unstructured grid* and in practice, Finite Element Analysis (FEA) will be the analysis method to apply. The problem with α -complexes emerges from the fact that an α -complex may contain topological features that cannot be coped with by state-of-the-art FEA-codes, such as singular faces and unconnected parts. In addition, α -complexes often contain “slivers”: sharp tetrahedra with a



Figure 13: zero-weight α -complex of the left and right shoulder blade (data set by courtesy of Delft Univ. of Techn., dep. Elec.Eng., $n = 1717$).

very odd aspect-ratio. We developed a new approach to run FEA over α -complexes, that overcomes most of these problems. The framework of our method is to turn the tetrahedra of the α -complex model (“foreground material”) into



Figure 11: α -complex of the comet West. Left: as observed. Centre: nucleus (gray-scale weighting, with threshold: 0.4). Right: ion tail, obtained by extraction of the blue cluster points shown in figure 12.

tetrahedra FEM-elements. This α -complex of the object is embedded in the hosting regular triangulation (“background material”) that acts as the embedding environment (or: “bulk”). After applying suitable boundary and/or initial conditions, computations can be started.

Our method exploits the fact that an α -complex is a sub-complex of the triangulation (section 3.). A triangulation, which is a cellular decomposition *continuum*, can always be input to an FEA-package, and *boundary conditions* can be attached to its boundary faces. Sometimes, FEA-codes run tests with respect to tetrahedra aspect-ratio’s. Slivers will not pass such tests. Most slivers are developed when the α -shape approaches the convex hull, i.e., for the highest α -values. Therefore, rather than taking the bulk at the maximum α , we lower α somewhat to get rid of these slivers.

The approach is outlined below, an application can be found in Gerritsen (1998).

step 1: Compute the α -family and select an embedding background α -complex close to the triangulation. Assign background material properties to the elements of the embedding α -complex.

step 2: Compute the normals of all the triangles in the border of this complex, in order to find out how (directional) boundary conditions need to be attached.

step 3: Assign foreground material properties to the object model α -complex to analyse and “inject” this complex into the embedding background-complex. I.e., for each tetrahedron in the foreground α -complex, locate the corresponding tetrahedron in the embedding background complex, and flip

the background material properties into foreground properties.

step 4: Run the FEA-analysis and optimise the α -complex as appropriate.

7. EVALUATION

7.1. EVALUATION CRITERIA

For computer modelling purposes, different approaches in describing the geometry and topology of modelled objects exist (Kalay (1989), Zeid (1991), Taylor (1992)). Object descriptions can be qualified by a number of criteria. Apart from well-formedness, generality and completeness, the following criteria are considered:

Solidity: a convex polytope \mathcal{P} (e.g., a simplex), divides space into two regions: *interior* $Int \mathcal{P}$ and *exterior* $Ext \mathcal{P}$, separated by *boundary* $Bd \mathcal{P}$. Point location, for example, is feasible when this criterion is met.

Homogenous dimensionality: $Bd \mathcal{C}_\alpha$ shall be fully incident upon the interior $Int \mathcal{C}_\alpha$ and shall be composed of regular faces only.

Rigidity: the geometry of the modelled object shall not depend necessarily on its position, i.e., on its location or its orientation in space.

Continuity: the represented object shall not be composed of unconnected parts.

Closure: k -faces shall have no incidences to singular faces. For example, in the case of a triangulated object, every k -simplex shall be incident upon exactly $d + 1$ ($k - 1$)-

simplices. If so, k -faces can be represented as regularised set (r -sets).

Disjunct interiors: the interior $Int f_i$ and $Int f_j$ of any two k -faces f_i and f_j shall be disjunct.

Orientability: all k -faces shall be orientable.

Finite time and storage complexity: supporting finite time complexity of operations and finite storage complexity of the results.

Loosely formulated, a *representation scheme* is a way to describe real world objects in a symbolic notation. Widely used schemes are: Constructive Solid Geometry (CSG), boundary representation (B-rep), cell decomposition, spatial enumeration and voxel representation. In a sense, α -complexes can also be seen as a means of representation. Criteria for representation schemes are:

Uniqueness: the cardinality of all representations that describe the modelled object under this representation scheme.

Completeness: the richness of description, in support of operations, analysis and conversions.

Domain: the class of physical objects that can be represented and the class of valid representations that the scheme can produce.

Validity: the validity of the objects produced under this scheme.

These criteria will be applied to the evaluation of α -complexes below. Both as a model description and a representation scheme. There is not so much known about α -complexes in this regard. Some reasoning is possible, however, by comparing the α -complex model description to triangulations, simplicial complexes and cellular models, and the representation scheme to CSG-, B-rep and cellular decomposition representations. Table 2 summarizes the principle findings of this comparison.

7.2. MODEL DESCRIPTION

A model description based on an α -complex carries some characteristics of a:

- Part-whole description, with primitive instancing of k -simplices.
- Volumetric description, with a cellular complex built from d -simplices (possibly, after *regularisation*).
- (Faceted) boundary description, built from $(d - 1)$ -faces (*facets*) of the α -complex.

For α -complexes, with their internal voids and possibly singular faces, the solidity-criterion is not necessarily met. Provided that α is high enough to permit the forming of d -faces, removal of singular faces ensures homogenous dimensionality and closure. The exterior can be composed of disjunct parts, all but one bounded and one unbounded. Continuity of the interior cannot be guaranteed, moreover, α -complexes can scatter into many disjunct parts. This is a great advantage for object descriptions with a dynamic topology, like growing cracks, oxidation or eroding sand bodies. On the other hand, it complicates validity and consistency verifications on the model as well as further operations on the model. For many problems, the background-embedding approach of sect. 6.5. can be followed. For finite complexes, finite storage complexity can be shown to exist, as well as finite time complexity for operations upon them. To a great extent, characteristics are as with cellular decomposition. Faces of an α -complex are orientable. Voids usually have a “negative” orientation, for area, volume, etc. If $Bd C_\alpha$ (i.e., the corresponding α -shape) can be triangulated (tessellated with triangles), such that the boundary is closed and connected, then these two properties ensure well-formedness. But generally, continuity is not guaranteed and boundary is not connected, only on a part-by-part basis. The rigidity-criterion is obviously met: the α -complex is determined by distance and weight, not by position or orientation. Transformations that can be represented by a homeomorphic mapping do not change the α -complex topology. The interior of any two k -faces is always disjunct, as they are faces in the underlying triangulation. This also excludes self-intersection.

	FACETED B-REP	CSG	CELL DECOMP.	α -COMPLEX
solidity	yes	yes	yes	if regularized
continuity	yes	yes	yes	no, only part-by-part
rigidity	yes	yes	yes	yes
homo. dim.	yes	if regularized	yes	if regularized
closure	yes	if regularized	yes	if regularized
orientability	yes	yes	yes	yes
disjunct <i>Int</i>	yes	yes	yes	yes
finiteness	yes	yes	yes	yes
domain	rigid homogenous manifold complex shapes massive description	rigid homogenous (non-)manifold moderate complexity lean description	rigid homogenous manifold complex shapes massive description	rigid homogenous non-manifold very complex shapes massive description, reproducible from S
validity	euler rules, expensive	regularized, set-theoretic	validation expensive	euler + α -weighted metric, expensive
completeness	homogenous properties	homogenous properties	homo. prop. per cell	homo. prop. per face
uniqueness	no	no	yes	no

Table 2: α -Complexes compared to B-rep, CSG and cell decomposition. Top part: α -complex as model description, bottom part: α -complex as representation schemes

7.3. REPRESENTATION

In a sense, α -complexes can also be seen as a means of representation. First question to be raised (see figure 14) concerns the domain: can every physical object be represented by a sampling data point set? For the class of natural and engineering objects of interest in this paper, the answer to this question is assumed yes. A sampling data point set does not uniquely represent a physical object: many such sets can sample the same object, and one sampling point set can represent multiple physical objects. Data may also be added subset by subset, like with sliced data. The step to account for discretisation with finite-precision computer internal number representations, like in Worboys (1992), is not explicitly shown but assumed to be implicit in this step.

Once a data point set has been sampled, the next question is, can every point set be triangulated? Disregarding degenerate cases that can be perturbed all (finite and bounded) sampling data sets can be triangulated. Duplicates and *redundant* vertices, too close to dominant points must be dropped. Singular (minimum) objects, sampled by a single point, result in

a 0-triangulation of a single 0-face (vertex). Generally, a proper spatial distribution of the sampling points is required, in order to obtain a good-quality triangulation. It can be shown (Edelsbrunner and Muecke (1994)) that a unique relation exists between the sampling data point set, the α -complex and its underlying space. Variations in the α -value, even though constrained, may instantiate a finite family of models (e.g., figure 1) that cannot be thought of as equivalent classes, due to different topologies. α -Complexes cannot be traced back uniquely to sampling data point sets. But they can be uniquely reproduced from them, which makes an *unevaluated* description feasible, thus reducing storage requirements. With regard to the completeness; the data structure storing the α -complex can be easily augmented with auxiliary parameters, like material properties, central moments, etc. The simplex, the basic building block, allows for relatively simple computational schemes.

Conversion mapping transforms an α -complex representation of an object onto another such representation Q (figure 14). Its characteristics depend heavily on the target representation of the conversion. If the representation in \mathcal{C}_α and

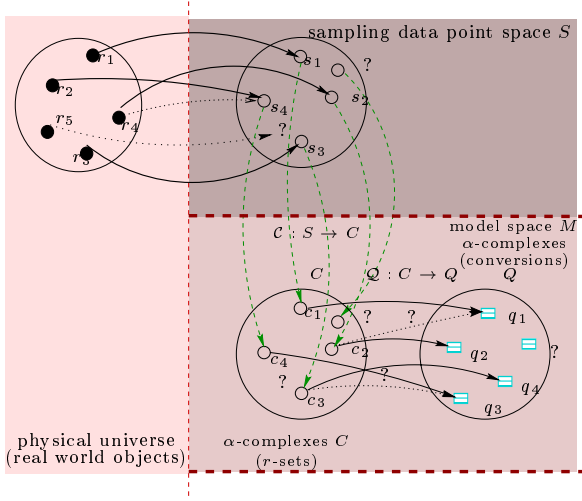


Figure 14: “chain” of representation-to-representation mappings in modelling with α -complexes. S is the sampling data point set, sampling the real world objects R . C is the set of α -complexes, Q the set of conversions. Dotted arrows and question mark-labelled elements refer to the issues addressed in the text.

in Q are *homeomorphic*, then a bijective relation can be shown to exist (Munkres (1984)) and conversion mapping is basically without loss of information. Topological relations existing in C_α are preserved under conversion. Examples of homeomorphic representations in C_α and Q are transcription of a non-degenerate regularised α -complex into a cellular decomposition. Multiple conversions may exist for a single α -complex and, depending on the type of the conversion, generally, multiple α -complexes may lead to the same conversion.

What sets out α -complex modelling from B-rep and CSG is the capability of handling a large number of cells, singular faces and disjunct parts. For example, inserting massive amounts of cells in a CSG-model is not very convenient. Unless regularised, singular k -faces will cause a CSG-representation to fail on an arbitrary α -complex. For example, when determining the union of a cellular volume with a singular point. An α -complex is not necessarily convex. As a result, boundary representations fail to describe the d -volume of an arbitrary α -complex. The representation that comes closest to what is needed to accurately and unambiguously describe α -complexes would be a part-whole description, built up using instantiated simplices.

Finally, the handling of holes will be addressed in greater detail. Every convex polyhedron can be triangulated (e.g., Munkres (1984)). Triangulations of a finite and bounded point set in E^d have only 1 unbounded cell; the unbounded exterior. When a (triangulated) object contains one or more voids, m say, there will be m additional (bounded) exteriors, completely surrounded by the interior. Notice that disjoint exteriors imply multiply connected interiors, and vice versa. Starting out with a triangulation (i.e., $C_\alpha \cong \mathcal{T}$), and assuming a monotonically decreasing value of α , voids may grow into pockets, pockets into cavities and/or handles and handles may grow into separations of the α -complex. Observe that during this process, the genus (number of handles) changes. We may even have that closure $Cl C_\alpha$ is not a separation, but $Int C_\alpha$ is. An α -shape (the underlying space of an α -complex) differs from the convex hull (the underlying space of the triangulation) by the total amount of space occupied by the various holes. In other words, for an arbitrary α -complex, the union of underlying space of the complex and the holes yields the convex hull. If the α -complex is identical to the triangulation, then the underlying space of the holes is a null-space.

The domain of the α -complex representation scheme is further expanded, if holes can be explicitly represented and treated as part in a bigger assembly. Therefore, define a *nil-object* as follows. A nil-object in Euclidian d -space E^d is an object represented by an α -complex containing the improper (-1) -dimensional simplicial face $\{\emptyset\}$ as its only face. This definition also holds for void space *within* an α -complex. *Nil-objects*, homeomorphic to an open d -ball, are geometrically and topologically conforming (fitting) to any neighbouring object. As a consequence, *nil-objects* can be inserted in between any two adjacent objects. Nil-objects map to background elements in a conversion mapping onto FEM-models. With this definition of a *nil-object*, consistent *spatial occupancy* of model space E^d and well-formedness of the modelled object are more easily obtained.

8. CONCLUSIONS

Relatively complex objects with many holes and separated parts can be modelled conveniently using α -complexes. No tedious description is required; just a sampling point set will do. Objects to be reconstructed can be sampled by a camera or measuring robot, for example, offering a suitable entry point for reverse engineering. Varying the α -value is intuitive, but the design of a proper weight set, an essential step in practice, can be cumbersome. Weighting has an *omni-directional* effect, ignoring tensor-like and vector-like phenomena. Considerable effort has gone into finding suitable weighting strategies. A few strategies have been developed and evaluated, but further research is still to be carried out on this subject. Occasionally, working with the accumulating effects of transformations and weight requires detailed knowledge of the underlying sampling data set and modelled object. Line-oriented data and data with strongly directional patterns were found to be ill-suited. Transformations can sometimes be used to precondition the spatial distribution.

Creating complex natural object models such as prostheses frequently takes a combined free form and shape reconstruction process, for which α -complexes with their implicitly ruled geometry and topology are natural candidates. It offers good possibilities for knowledge-based modelling and variational geometry. In shape reconstructions where the geometry depends on some physical phenomenon, finding a strong relation between properties and weight is usually the critical factor to obtain an intuitive form of modelling. With respect to natural objects, still more “realism” is anticipated by the use of α -complex-based object geometries with natural object texture maps.

The ability to handle singular faces and disjunct parts in object descriptions based on α -complexes is both a strength and a complication. An α -complex may violate the criteria of continuity, solidity, closure and homogeneous dimensionality, but regularisation and the background-embedding approach (sect. 6.5.) can be followed to remedy this. The α -complex as a representation scheme is unambiguous but not unique. Validation is generally expensive.

Generally, detailed models require vast amounts of data. For simple models ($\mathcal{O}(10^2..10^3)$ sample points), results show that implementations can generally be made fast enough for interactive use. For more complex and bigger problems ($\mathcal{O}(10^4..10^6)$ sample points), storage becomes increasingly critical. α -Complexes can be uniquely reproduced from point sets, which makes an unevaluated description feasible. This reduces persistent storage requirements. Finally, surface/surface and other intersections of faceted objects, though not addressed in this paper, remains a research area still suffering from great immaturity and concern (e.g, Zeid (1991), chp. 7), that may seriously hinder further applications of α -complexes. Further research is targeted at improving weighting strategies, such as neighbourhood weighting and anisotropic weighting, and tools.

Acknowledgements

Authors wish to thank the Netherlands Institute of Applied Geoscience TNO – *National Geological Survey* for their financial support.

References

- Aichholzer, O., Aurenhammer, F., Lee, D., Chen, D. Z., and Papadopoulou, E. (1999). Skew voronoi diagrams. *to appear in: International Journal of Computational Geometry & Applications*.
- Aurenhammer, F. (1987). Power diagrams: properties, algorithms and applications. *SIAM J. Comput.*, 16(1):78–96.
- Aurenhammer, F. and Imai, H. (1987). Geometric relations among Voronoi diagrams. In *Proc. 4th Sympos. Theoret. Aspects Comput. Sci.*, volume 247 of *Lecture Notes Comput. Sci.*, pages 53–65. Springer-Verlag.
- Boehm, W. and Prautzsch, H. (1994). *Geometric Concepts for Geometric Design*. A. K. Peters, Ltd., Wellesley, MA.
- Brand, L. (1947). *Vector and Tensor Analysis*. John Wiley & Sons, Inc., London, ninth printing edition.

- Coolidge, J. L. (1916). *A treatise on the circle and the sphere*. Oxford at the Clarendon Press, London.
- Edelsbrunner, H. (1987). *Algorithms in Combinatorial Geometry*. Springer-Verlag, Berlin-Heidelberg.
- Edelsbrunner, H. (1992). Weighted alpha shapes. Technical Report UIUCDCS-R-92-1760 (UIIU-ENG-92-1740), University of Illinois at Urbana-Champaign, Department of Computer Science, Urbana, Illinois.
- Edelsbrunner, H., Facello, M., and Liang, J. (1998). On the definition and the construction of pockets in macromolecules. *Discrete Applied Mathematics*, 88:83–102.
- Edelsbrunner, H. and Muecke, E. P. (1994). Three-dimensional alpha shapes. *ACM Transactions on Graphics*, 13:43–72.
- Edelsbrunner, H. and Shah, N. R. (1996). Incremental topological flipping works for regular triangulations. *Algorithmica*, 15:223–241.
- Gerritsen, B., Van der Werff, K., and Veltkamp, R. (2000). Geometric modelling with α -complexes. Technical Report UU-CS-2000-03, Utrecht University. <http://ftp.cs.uu.nl/pub/RUU/CS/techreps/CS-2000/2000-03.ps.gz>.
- Gerritsen, B. H. M. (1998). On the use of alpha complexes in subsurface modelling. In *Proc. ENSG Conference on 3D Modeling of Natural Objects, A Challenge for the 2000's*. ENSG, Nancy, France.
- Hahn, L.-s. (1994). *Complex Numbers & Geometry*. Spectrum. The Mathematical Association of America, Washington DC.
- Kalay, Y. E. (1989). *Modeling Objects and Environments*. Principles of Computer-Aided Design (Series Ed. Yehuda E. Kalay). John Wiley & Sons, New York.
- Lee, C. W. (1991). Regular triangulations of convex polytopes. In *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*, volume 4 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 443–456. AMS Press.
- Maxwell, E. (1952). *Elementary Coordinate Geometry*. Oxford at the Clarendon Press.
- Munkres, J. R. (1984). *Elements of Algebraic Topology*. Addison-Wesley, Redwood, CA.
- Paoluzzi, A., Bernardini, F., Cattani, C., and Ferrucci, V. (1993). Dimension-independent modelling with simplicial complexes. *ACM Transactions on Graphics*, 12(1):56–102.
- Requicha, A. (1980). Representation of rigid solid objects. In Encarna cao, J., editor, *Computer Aided Design; Modelling, systems engineering, CAD systems*. *CREST Advanced Course, Darmstadt, September 1980*, Lecture Notes in Computer Science (series eds. G. Goos and J. Hartmanis). Springer-Verlag, Berlin-Heidelberg.
- Requicha, A. A. G. (1983). Toward a theory of geometric tolerancing. *Int. J. Robot. Res.*, 2(4):45–60.
- Schwerdtfeger, H. (1979). *Geometry of complex numbers; Circle geometry, Moebius transformation, non-euclidean geometry*. Dover Publications, Inc., New York, dover edition.
- Smets, G., de Groof, M., Nuyts, J., van der Meulen, D., Suetens, P., Marchal, G., and Oosterlinck, A. (1990). Interpretation of 3d medical scenes. In Freeman, H., editor, *Machine Vision for Three-Dimensional Scenes*, pages 163–193. Academic Press, Inc., Boston.
- Taylor, D. L. (1992). *Computer-Aided Design*. Addison-Wesley series in Mechanical Engineering. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts.
- Worboys, M. F. (1992). A generic model for planar geographical objects. *International Journal of Geographical Information Systems*, 6(5):353–372.
- Zeid, I. (1991). *CAD/CAM Theory and Practice*. McGraw-Hill Series in Mechanical Engineering (series eds. Jack P. Holman and John R. Lloyd). McGraw-Hill, Inc., New York.