Corrections and Comments
to
Methods and Applications of Singular Perturbations

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August 10, 2006

Contents

1 Corrections

In this section a list of errors will be presented. Note that typos are not listed here.

1.1 Two-point Boundary Value Problems, chapter 5

p. 46, line 2 f.a. should read

$$\psi_0(0) = \alpha - \phi_0(0),$$

1.2 The Continuation Method, chapter 10

- p. 145, line 7 f.b. should read

  $$\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \cdots = f(t, x_0 + \varepsilon x_1 + \varepsilon^2 \cdots, \varepsilon),\ a(\varepsilon) = a_0 + \varepsilon a_1 + \varepsilon^2 \cdots,$$

- p. 146, line 12 f.a. should read

  $$\ddot{x}_1 + x_1 = -2\sin t,\ x_1(0) = 0,\ \dot{x}_1(0) = 0,$$

1.3 Averaging and Timescales, chapter 11

- p.166, line 4 f.b. should read

  $$-\dot{r}\sin(t+\psi) - \dot{\psi} r\cos(t+\psi) = \varepsilon f(r\cos(t+\psi), -r\sin(t+\psi)).$$

  Likewise, p.166, line 2 f.b. should read

  $$\dot{r} = -\varepsilon \sin(t+\psi) f(r\cos(t+\psi), -r\sin(t+\psi)),\quad (11.2)$$

1.4 Advanced Averaging, chapter 12

- In the middle of page 217 there is a cut-and-paste error in the formula. The sentence starting on line 17 f.a. should read (in the notation of section 12.5.1): Following the construction of the second-order approximation of the preceding section, we have with $f^0(x) = 0$ the equation

  $$\dot{v} = \varepsilon^2 f_1^0(v) + \varepsilon^2 g^0(v),\ v(0) = x(0).$$

1.5 Appendices, chapter 15

- p. 280, line 11 f.b. should start with
  We conclude that $\psi_0(\xi) = \alpha$.

- The exact solution on p. 282, line 7 f.b. should read

  $$\phi_\varepsilon(x) = (1+\alpha)\left(1 - x + \left(\frac{1+\beta}{1+\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}} x\right)^{\frac{\varepsilon}{\varepsilon-1}} - 1.$$

- p. 294, line 8 f.a., the righthand side should read

  $$= \sum_{0}^{N(\varepsilon)} \cdots + \sum_{N(\varepsilon)+1}^{\infty} \cdots,$$

2 Comments

A few readers comment on the absence of discussion of prerequisite knowledge. For instance looking for 'asymptotic stability' in the index, one is referred to p. 109, but there is no definition on this page. It is just the first page where this concept is being used.
In general, we suppose basic knowledge of concepts in ODEs and PDEs as can be found in books like

- W.E. Boyce and R.C. DiPrima, Introduction to Differential Equations, John Wiley, from 1970 on, there are many editions.
- F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, Springer, 2000.
- R. Haberman,Elementary Applied Partial Differential Equations, Prentice-Hall, 1998.
- W.A. Strauss, Partial Differential Equations, an introduction, John Wiley, 1992.

2.1 Sections 5.1-2, 7.1 Matching without the subtraction trick

Matching of boundary layer and regular expansions can be carried out in various ways. We used the subtraction trick in chapters 5 and 7. By subtracting the regular expansion from the solution, we can match the boundary layer expansion in the resulting equation against zero. The multiple timescales expansion by O'Malley, described in chapter 8, uses essentially the same procedure.
On the other hand, in example 6.5 we have a case where the subtraction trick is not so useful. To handle such cases we formulated a slightly more general matching rule in section 6.2. We have here an equation (ODE) on a 1-dimensional domain with supposedly a boundary layer on the lefthand side $x = 0$. If the boundary layer variable is indicated by ξ, the first term of the boundary layer expansion by $\psi_0(\xi)$, the first term of the regular expansion by $\phi_0(x)$, the matching rule is

$$\lim_{\xi\to\infty} \psi_0(\xi) = \lim_{x\to 0} \phi_0(x).$$

Note that in section 15.4, we discuss further generalisation of this matching rule.

We will analyse now three problems, using this matching rule. Of course the final results will be the same as obtained before.

Section 5.1, boundary layers at two endpoints

We consider the problem on $[0,1]$

$$\varepsilon L_1\phi + b_0(x)\phi = f(x), \quad x \in (0,1)$$

with boundary values $\phi_\varepsilon(0) = \alpha, \phi_\varepsilon(1) = \beta$. For the operator L_1, we write

$$L_1 = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$$

with, as in section 5.1,

$$a_2(x) > 0, b_0(x) < 0, \quad x \in [0,1].$$

For the regular expansion, we find as a first term

$$\phi_0(x) = f(x)/b_0(x).$$

Introducing near $x = 0$ the local variable

$$\xi = \frac{x}{\delta(\varepsilon)}, \quad \delta(\varepsilon) = o(1),$$

and writing $\phi_\varepsilon(\delta\xi) = \phi^*(\xi)$, we find

$$L^*\phi^* = \frac{\varepsilon}{\delta^2}a_2(\delta\xi)\frac{d^2\phi^*}{d\xi^2} + \frac{\varepsilon}{\delta}a_1(\delta\xi)\frac{d\phi^*}{d\xi} + \varepsilon a_0(\delta\xi)\phi^* + b_0(\delta\xi)\phi^* = f(\delta\xi).$$

A significant degeneration L_0^* arises if $\delta(\varepsilon) = \sqrt{\varepsilon}$, which yields

$$L_0^*\phi_0^* = a_2(0)\frac{d^2\phi_0^*}{d\xi^2} + b_0(0)\phi_0^* = f(0).$$

Putting $\omega_0 = (-b_0(0)/a_2(0))^{\frac{1}{2}}$, the solution is

$$\phi_0^*(\xi) = Ae^{-\omega_0\xi} + Be^{\omega_0\xi} + f(0)/b_0(0).$$

Using our more general matching rule we find $B = 0$. The boundary condition at $x = 0$ yields

$$A = \alpha - f(0)/b_0(0).$$

The boundary layer expansion near $x = 0$ and the regular expansion have a common part $f(0)/b_0(0)$. Subtracting this, we find as first order contribution

$$\phi_0^*(\xi) + \phi_0(x) - f(0)/b_0(0).$$

We have still to add the contribution from the boundary layer near $x = 1$; the computation runs along the same lines.

Section 5.2, boundary layer at one endpoint

The equation of section 5.1 is modified to

$$\varepsilon L_1\phi + b_1(x)\frac{d\phi}{dx} + b_0(x)\phi = f(x), b_1(x) < 0, \quad x \in (0,1).$$

As in section 5.2, we find for the first term of the regular expansion

$$\begin{aligned} g(x) &= \textstyle\int_0^x \frac{b_0(t)}{b_1(t)}\,dt, \\ \phi_0(x) &= Ae^{-g(x)} + e^{-g(x)} \textstyle\int_0^x e^{g(t)} f(t)\,dt. \end{aligned}$$

Because of the choice of signs of $a_2(x), b_1(x)$, we know from section 5.2 that we have a boundary layer near $x = 1$; the regular expansion extends to $x = 0$, so $A = \alpha$. Introduce the local variable

$$\eta = \frac{1-x}{\delta(\varepsilon)}, \quad \delta(\varepsilon) = o(1).$$

The equation becomes with $\phi_\varepsilon(1-\delta\eta) = \phi^*(\eta)$,

$$\begin{aligned} L^*\phi^* &= \frac{\varepsilon}{\delta^2}a_2(1-\delta\eta)\frac{d^2\phi^*}{d\eta^2} - \frac{\varepsilon}{\delta}a_1(1-\delta\eta)\frac{d\phi^*}{d\eta} + \varepsilon a_0(1-\delta\eta)\phi^* \\ &\quad -\frac{b_1(1-\delta\eta)}{\delta}\frac{d\phi^*}{d\eta} + b_0(1-\delta\eta)\phi^* = f(1-\delta\eta). \end{aligned}$$

A significant degeneration L_0^* arises if $\delta(\varepsilon) = \varepsilon$, yielding

$$L_0^*\phi_0^* = a_2(1)\frac{d^2\phi_0^*}{d\eta^2} - b_1(1)\frac{d\phi_0^*}{d\eta} = 0.$$

The solution is

$$\phi_0^*(\eta) = B + Ce^{-\omega\eta}, \omega = -b_1(1)/a_2(1).$$

The (more general) matching condition produces

$$B = \phi_0(1).$$

The boundary condition at $x = 1$ ($\eta = 0$) yields

$$C = \beta - \phi_0(1).$$

The *difference* with the problem with two boundary layers is that $\phi_0(x)$ and $\phi_0^*(\eta)$ have no common terms near $x = 1$ ($\eta = 0$). So the first order contribution to the asymptotic expansion is

$$\phi_0(x) + \phi_0^*(\eta).$$

Section 7.1, elliptic problem for the circle

Consider the domain $D = \{x, y | x^2 + y^2 < 1\}$ and the equation

$$\varepsilon \Delta \phi - \phi = f(x, y)$$

with boundary condition $\phi|_\Gamma = \theta(x, y)$ on the boundary circle Γ.
The first term of the regular expansion is

$$\phi_0(x, y) = -f(x, y).$$

Expecting a boundary layer near Γ, we introduce radius-angle coordinates (see section 7.1)

$$x = (1 - \rho) \cos \alpha, y = (1 - \rho) \sin \alpha$$

to find the equation

$$\varepsilon \left(\frac{\partial^2 \phi}{\partial \rho^2} - \frac{1}{1 - \rho} \frac{\partial \phi}{\partial \rho} + \frac{1}{(1 - \rho)^2} \frac{\partial^2 \phi}{\partial \alpha^2} \right) - \phi = f((1 - \rho) \cos \alpha, (1 - \rho) \sin \alpha).$$

We introduce a boundary layer variable along Γ by

$$\xi = \frac{\rho}{\varepsilon^\nu}.$$

Transforming x, y to ρ, α and introducing ξ, ϕ transforms to $\phi^*(\xi, \alpha)$ with equation

$$L^* \phi^* = \varepsilon^{1-2\nu} \frac{\partial^2 \phi^*}{\partial \xi^2} - \frac{\varepsilon^{1-\nu}}{1 - \varepsilon^\nu \xi} \frac{d\phi^*}{d\xi} + \frac{\varepsilon}{(1 - \varepsilon^\nu \xi)^2} \frac{\partial^2 \phi^*}{\partial \alpha^2} - \phi^* = f(.,.),$$

where $f(.,.) = f((1 - \varepsilon^\nu \xi) \cos \alpha, (1 - \varepsilon^\nu \xi) \sin \alpha)$. The differential operator has a significant degeneration L_0^* if $\nu = \frac{1}{2}$ so that $\xi = \rho / \varepsilon^{\frac{1}{2}}$ and then

$$L_0^* \phi_0^* = \frac{\partial^2 \phi_0^*}{\partial \xi^2} - \phi_0^* = f(\cos \alpha, \sin \alpha).$$

The solution is

$$\phi_0^*(\xi, \alpha) = A(\alpha) e^{-\xi} + B(\alpha) e^{+\xi} - f(\cos \alpha, \sin \alpha).$$

Matching produces

$$B(\alpha) = 0$$

and applying the boundary condition

$$A(\alpha) = \theta(\cos \alpha, \sin \alpha) + f(\cos \alpha, \sin \alpha).$$

Subtracting the common part, the first order part of the asymptotic expansion becomes (in mixed coordinates)

$$\phi_0(x, y) + \phi_0^*(\xi, \alpha) + f(\cos \alpha, \sin \alpha).$$

2.2 Chapter 8: Canards

In this chapter, one of the topics is the exponential closeness of solutions to stable slow manifolds. A theorem is formulated and proved in Section 15.7. In the cases that the slow manifold becomes unstable, we have the possibility of "exponential sticking". The solution continues for an $O(1)$ time along the slow manifold that has become unstable and jumps after that away, for instance to another slow manifold. Following Pontrjagin, Neishtadt (reference [142] in the book) calls this "delay of stability loss".
This delay- or sticking process is closely connected to the so-called *canard* phenomenon. We shall use the following description.
Canard solutions are bounded solutions that, starting near an attracting normally hyperbolic slow manifold, cross a singularity of the system of differential equations and follow for an $O(1)$ time a normally hyperbolic repelling slow manifold.
Note that, depending on the dimension of the problem and the nature of the singularity, the description usually has to be more specific; see also the references.
The first example of canard behaviour was found by the Strassbourg group working in nonstandard analysis for a perturbed van der Pol-equation; see for details and references [2]. In this first example in the literature, the singularity crossed is a fold point. The reason to omit this interesting topic from our book was that the analysis of this problem is very technical.
Fortunately, we have now a simple example of a canard, see [8] for details and many more references. In this example,the *logistic canard*, we consider a transition through a transcritical singularity, that can be calculated explicitly.

Example 2.1
Consider the modified logistic equation for $y \geq 0$

$$\varepsilon \dot{y} = x(t)y - y^2, \tag{1}$$

in which the growth rate $x(t)$ can take positive and negative values. We can solve the equation for general continuous $x(t)$ and $y(0) = y_0 > 0$:

$$y(t) = \frac{e^{\frac{1}{\varepsilon}\int_0^t x(s)ds}}{\frac{1}{y_0} + \frac{1}{\varepsilon}\int_0^t e^{\frac{1}{\varepsilon}\int_0^s x(u)du} ds}. \tag{2}$$

If $x(t)$ is T-periodic, we can write

$$x(t) = a + f(t)$$

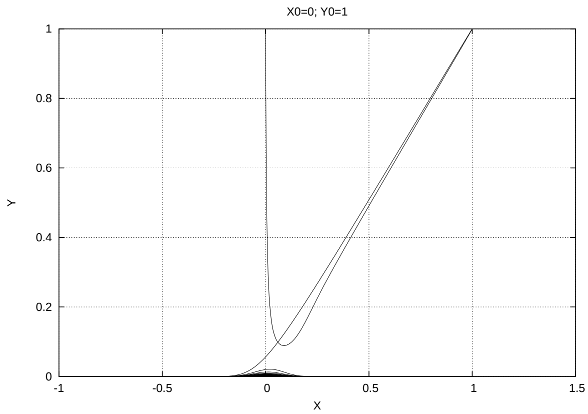

Figure 1: Solution approaching the periodic solution in $y = 0$ of the modified logistic equation (1) with $x(t) = \sin t, x(0) = 0, y(0) = 1, \varepsilon = 0.01$. To the right of the y-axis, $y = 0$ corresponds with an unstable slow manifold.

with a a real constant and $f(t)$ a zero average continuous function. In the cases that $x(t)$ is quasi- or almost-periodic we can write similar expressions. Assuming that $x(t)$ changes sign and is T-periodic, we have two cases.
Case 1: $a \leq 0$. We state that $\lim_{t\to\infty} y(t) = 0$. In the extended (x, y) system, the only periodic solution is the trivial solution $x = x(t), y = 0$. If $a < 0$ the proof is simple: multiplying with $\exp(-at/\varepsilon)$ produces a bounded numerator and a monotonically increasing denominator. A similar reasoning applies if $a = 0$. In this case $y(t)$ also decreases exponentially; this can be seen by estimating a lower bound of the integral in the denominator by restricting to the time-intervals where $f(t)$ is positive. In Fig. 1 we took $x(t) = \sin t$ for an illustration.

Case 2: $a > 0$ with $a + \min_{0\leq t\leq T} f(t) < 0$. In this case a canard-like periodic solution exists. Several proofs are possible, but a simple one runs as follows.
Assuming that $y(0) = y(T) = y_0(> 0)$, we obtain from the solution (2) the expression

$$y_0 = \frac{e^{\frac{aT}{\varepsilon}} - 1}{\frac{1}{\varepsilon}\int_0^T e^{\frac{1}{\varepsilon}\int_0^s (a+f(u))du} ds}.$$

The righthand side being positive, we have constructed a positive solution for y_0 and so a positive periodic solution.
For more explicit choices of $x(t)$ we can show that y_0 can be exponentially

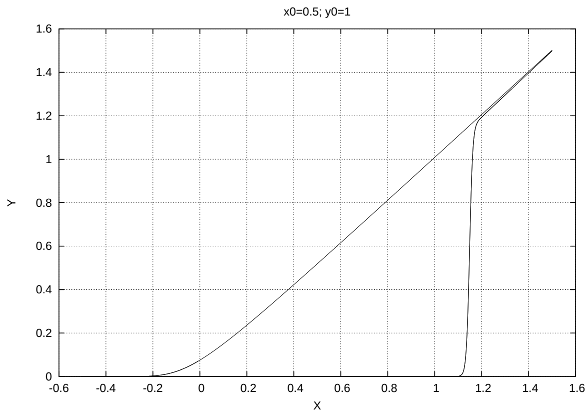

Figure 2: Canard-like periodic solution of the modified logistic equation (1) with $x(t) = 0.5+\sin t, \varepsilon = 0.01$. To the right of the y-axis, $y = 0$ corresponds with an unstable slow manifold.

small and can be $O(1)$. Consider for instance the case $x(t) = 0.5 + \sin t$, see fig. 2.

This canard-like behaviour of the solutions of the modified logistic equation persists if the growth rate function $x(t)$ is quasi- or almost-periodic. Most of the results of the periodic case carry over, although the arguments are more complicated, even if we have an almost-periodic function with a spectrum that is not bounded away from zero (see [1]).

References

[1] T. Bakri, *On a modified logistic equation*, (2006), to be published.

[2] W. Eckhaus, *Relaxation oscillations including a standard chase on French ducks*, in *Asymptotic Analysis II*, Lecture Notes in Mathematics 985 (1983) pp. 449-494.

[3] M. Krupa and P. Szmolyan, *Relaxation oscillation and canard explosion*, J. Diff. Equations 174, (2001) pp. 312-368.

[4] M. Krupa and P. Szmolyan, *Extending slow manifolds near transcritical and pitchfork singularities*, Nonlinearity 14, (2001) pp. 1473-1491.

[5] Yu.A. Kuznetsov, S. Muratori and S. Rinaldi, *Homoclinic bifurcations in slow-fast second order systems*, Nonlinear Analysis, Theory, Methods & Applications 25, (1995) pp. 747-762.

[6] P. Szmolyan and M. Wechselberger, *Relaxation oscillations in* $\mathbb{R}^3$, J. Diff. Equations 200, (2004) pp. 69-104.

[7] A.I. Neihstadt, *Asymptotic investigation of the loss of stability by an equilibrium as a pair of imaginary eigenvalues slowly cross the imaginary axis*, Usp. Mat. Nauk 40 (1985) pp. 190-191.

[8] Ferdinand Verhulst, *Periodic Solutions and Slow Manifolds*, (2006), to be published in Int. J. Bifurcation and Chaos.

2.3 Chapter 10: The continuation method

- It is noted on p. 148 that a certain Jacobian J_ε should not vanish for periodic solutions to exist. In these problems we can expand in a convergent series

  $$J_\varepsilon = J_0 + \varepsilon J_1 + \cdots + \varepsilon^n J_n + \cdots .$$

  In the simplest case $J_0 \neq 0$. If $J_0 = 0$, we should compute J_1. If J_1 does not vanish we are near to having demonstrated existence of a periodic solution. What we still have to show is that the result is robust. Compare for this problem the theory of perturbations of matrices in the appendix section 15.3.

- Also on p. 148 line 10 f.a. the term 'periodicity conditions' is used. Often these are referred to as 'secularity conditions' (better would be 'nonsecularity conditions', but this is unusual).

2.4 Sections 11.4 and 12.7: Timescales,

The concept of *timescale* refers to an interval of time in the asymptotic sense, for instance of length $O(1)$, $O(1/\varepsilon)$, etc. *Time (or timelike) variables* play a part in the asymptotic aproximations as for instance, $t, \varepsilon t, \varepsilon^2 t$, etc.
The discussion of the use of averaging (or other normal form methods) on one hand, and multiple-timescales methods on the other hand, produces many responses, sometimes emotional ones. This applies in particular to the conclusion on page 223:

> "It should be clear by now that the method of multiple timescales for initial value problems of ordinary differential equations is not suitable for research problems except in the case of simple problems that are accessible also to first-order averaging. For other problems, this method presupposes the presence of certain timescales, a knowledge we simply do not have."

We have in the Mathieu equation a real-life example (12.27) to illustrate this. We will add a few observations here.

Consider a perturbation problem of the form

$$\dot{x} = \varepsilon f(t, x, \varepsilon),$$

where x is n-dimensional, f can be expanded in a Taylor series with respect to ε. We may add an initial value to this ordinary differential equation. Normalisation, averaging or multiple timing all lead to a first order normal form (or averaged) equation of the form

$$\dot{y} = \varepsilon u_1(y),$$

where x and y are connected by a near-identity transformation. A second order approximation is obtained from the equation

$$\dot{y} = \varepsilon u_1(y) + \varepsilon^2 u_2(y),$$

and in general an n-th order approximation from

$$\dot{y} = \varepsilon u_1(y) + \varepsilon^2 u_2(y) + \cdots + \varepsilon^n u_n(y).$$

At all levels of approximation we can extract ε, put $\tau = \varepsilon t$, and obtain an equation of the form

$$\frac{dy}{d\tau} = u_1(y) + \varepsilon^2 u_2(y) + \cdots + \varepsilon^{n-1} u_n(y).$$

According to the Poincaré expansion theorem (section 10.1) we can simply obtain a Taylor expansion for the solution of the form

$$y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 \cdots . \qquad (3)$$

Here $y_0(\tau)$ is the solution of the equation $dy_0/d\tau = u_1(y_0)$ with suitable initial value. The higher order terms are determined by linear equations in the usual way (see again section 10.1, page 145). The approximation, obtained

from this expansion, is valid on the timescale 1 in τ, which implies validity on the timescale $1/\varepsilon$ in t. This seems to settle the question that on this timescale *to any order of approximation*, the time variable τ suffices. Multiple timing and averaging are asymptotically equivalent when considering approximations on the timescale $1/\varepsilon$.
The suggestion that this statement carries over to longer timescales is not correct. Multiple timing may actually obscure behaviour on longer timescales because of the linearisation procedure which is involved. This is shown by the Mathieu equation in example 12.27. We present another, simpler example.

Example 2.2
Suppose that averaging to second order has produced the system

$$\begin{aligned}\dot{x} &= \varepsilon^2 y,\\ \dot{y} &= -\varepsilon x.\end{aligned}$$

The solutions read, when adding initial values,

$$\begin{aligned}x(t) &= x(0)\cos(\varepsilon^{\frac{3}{2}}t) + \varepsilon^{\frac{1}{2}}y(0)\sin(\varepsilon^{\frac{3}{2}}t),\\ y(t) &= -\varepsilon^{-\frac{1}{2}}x(0)\sin(\varepsilon^{\frac{3}{2}}t) + y(0)\cos(\varepsilon^{\frac{3}{2}}t).\end{aligned}$$

Expanding the cos and sin functions in an infinite Taylor series, the solutions can clearly be written as functions of $\tau = \varepsilon t$ in the form (3), but the *intrinsic* time variable is $\varepsilon^{\frac{3}{2}}t$. This time variable characterises the oscillation frequency on a timescale longer than $1/\varepsilon$.
Extracting ε, introducing τ and applying Poincaré expansion, as used in multiple timing, produces expansions of the form

$$\begin{aligned}x(\tau) &= x(0) + \varepsilon(y(0)\tau - \frac{1}{2}x(0)\tau^2) + \varepsilon^2\cdots,\\ y(\tau) &= y(0) - x(0)\tau + \varepsilon(\frac{1}{2}y(0)\tau^2 - \frac{1}{6}x(0)\tau^3) + \varepsilon^2\cdots.\end{aligned}$$

To obtain the averaging result and the insight that the time variable $\varepsilon^{\frac{3}{2}}t$ plays a part, we have to include all (infinite) terms of the Poincaré expansion. In practice one stops at a low level of approximation, thus missing this insight.
A disadvantage of this simple example is that the solution for $y(t)$ grows to $O(\varepsilon^{-\frac{1}{2}})$ on timescales beyond $1/\varepsilon$. We can remedy this by choosing $x(0) = 0$.