Bifurcations and instability phenomena in parametric dynamics

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Abstract

Bifurcations of periodic solutions to tori and bifurcations of tori, producing many different phenomena, are natural in parametric dynamics. We start with a discussion of the role of timescales, followed by some observations for a system of three coupled oscillators. This system is very rich in bifurcations leading in some cases to chaos.

INTRODUCTION

For equations of the form $dx/dt = f(t,x)$ with $f(t,x)$ T-periodic in time, a one dof freedom system requires already knowledge of 2-dimensional maps. So, it is not surprising that one needs in general extended bifurcation theory as codimension one and higher bifurcations, tori and torus-bifurcations, strange attractors, dissipation induced instability etc. But complexity is not exclusive for high-dimensional systems. Detailed studies of low-dimensional, parametrically excited systems are essential for better understanding.

The quantitative tools are various types of normal form analysis, numerical bifurcation theory using packages like AUTO, CONTENT and MATCONT [2], and direct numerical integration . About normal forms there is some confusion which we will discuss first.

NORMAL FORMS AND TIMESCALES

Considering the scientific literature, one observes that the use of asymptotic series to approximate solutions of differential equations takes all kind of different forms: averaging, multiple timing, renormalization etc. In this respect it has been very important to have comparative and unifying studies as [5], [7], to name a few. A basic aspect of the discussion is of course that there is some freedom in using perturbation methods as asymptotic expansions are not unique.

In [5], the equivalence of the averaging method and multiple timing was established for standard equations like $\dot{x} = \varepsilon f(t,x)$ on intervals of time of order $1/\varepsilon$. See also the extensive discussions in [3] and [7].

It is easy to show that for perturbation problems of the form

$$\dot{x} = f(x,t,\varepsilon) = f_0(x,t) + \varepsilon f_1(x,t) + \varepsilon^2 \ldots, \tag{1}$$

the timescales t and εt play a part. A basic problem of the multiple timescales method, however, is that one has to anticipate the timescales that rule the solutions. Such a guess can be correct for simple, or even more complex, but well-understood problems. However, for most research problems the anticipation of timescales is an unnecessary and dangerous restriction. We discuss briefly two classes of problems where multiple timing may be deficient and averaging and other normal form methods give the correct result.

- In bifurcation problems one encounters, after linearization, structural stability problems of matrices, this is characteristic for such problems. A $n \times n$ matrix is called *structurally stable* if it is nonsingular and all eigenvalues have nonzero real part. If we have a zero eigenvalue or purely imaginary eigenvalues, we can expect bifurcations. Apart from this, the presence of multiple eigenvalues affects the form of the expansions and the timescales. In such cases unexpected algebraic timescales can not be avoided. A simple example is the Mathieu-equation, discussed in detail in [8]:

$$\ddot{x} + (1 + \varepsilon a + \varepsilon^2 b + \varepsilon \cos 2t)x = 0, \tag{2}$$

  a and b are free parameters independent of ε. To first order in ε, the instability tongue is found for $a^2 = 1/4$. Choosing the boundary of the tongue for $a = 1/2$, we find to second order $b = 1/32$. The timescales characterising the flow near the Floquet tongue are to second order

$$t,\ \varepsilon t,\ \varepsilon^{\frac{3}{2}} t,\ \varepsilon^2 t.$$

- Problems with resonance manifolds (or zones) may arise in systems of the form

$$\begin{aligned} \dot{x} &= \varepsilon X(x, \phi) + \varepsilon^2 \ldots, \\ \dot{\phi} &= \Omega(x) + \varepsilon \ldots \end{aligned}$$

  with x a Euclidean n-vector, ϕ an angle-vector; the order functions multiplying the right-hand sides are different, but the choice of size here is just an example. Such problems arise in Hamiltonian systems and in dissipative systems; see for instance [7] or [8] and references there. Typical resonance zones are of size $O(\sqrt{\varepsilon})$ with timescale of the dynamics in the resonance zone $\sqrt{\varepsilon}\, t$.

Normal form methods, averaging and renormalization have no need to anticipate the timescales that are relevant for the approximations. Multiple timing, on the other hand, makes restricting choices of timescales but this method is safe to use if we confine the analysis to time intervals of order $1/\varepsilon$ and in general if we understand apriori the nature of the solutions. Extension of validity of approximations beyond order $1/\varepsilon$ is usually not expedient for the multiple timescale method; see the discussions in [7] and [3].

THE MECHANICAL TONDL-MODEL

The model, see fig. 1, considered here, has three degrees of freedom. The equations of motion are studied in [1] and are:

$$\begin{cases} m_1\ddot{y}_1 + \varepsilon b\dot{y}_1 + k_0(1 + \varepsilon \cos \omega t)y_1 - k_1(y_2 - y_1) = 0, \\ m_2\ddot{y}_2 - \varepsilon\beta_0(1 - \gamma_0\dot{y}_2^2)\dot{y}_2 + 2k_1y_2 - k_1(y_1 + y_3) = 0, \\ m_3\ddot{y}_3 + \varepsilon b\dot{y}_3 + k_0(1 + \varepsilon \cos \omega t)y_3 - k_1(y_2 - y_3) = 0. \end{cases} \tag{3}$$

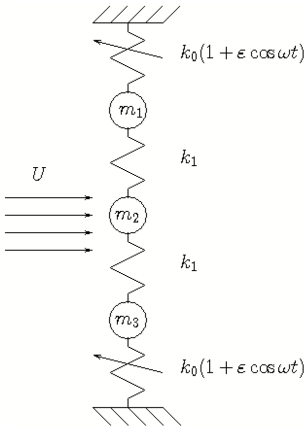

Figure 1. Flow-induced vibrations with linear energy-absorbers

The y_i represent the deflections of the masses m_i, $i = 1, 2, 3$. We have damping or energy-absorbing parameters $k_0, k_1, \beta_0 > 0$. The self-excitation (Rayleigh term) is nonlinear, it models the flow-induced excitation; the springs are linear. The masses are chosen such that we have for the basic frequencies the $1:2:3$-synchronization resonance (the other resonances represent open problems).

Apart from the equilibrium at the origin, we find two nontrivial periodic solutions. In the analysis, the combination of averaging-normalization and numerical bifurcation techniques is very profitable as an equilibrium (critical point) of a normal form corresponds with a periodic solution. A Hopf bifurcation of an equilibrium in a normal form corresponds with a Neimark-Sacker bifurcation. In its turn, a Neimark-Sacker bifurcation of a periodic solution of a normal form corresponds with a 3-torus; see fig. 2.

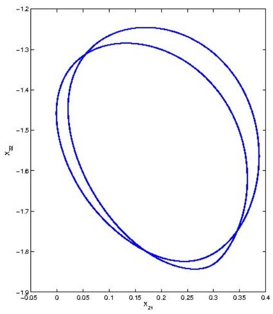

Figure 2. Projection on a plane of a double torus with Lyapunov exponents $\lambda_1 \approx \lambda_2 \approx 0, \lambda_3 = O(10^{-4}), \lambda_4 = O(10^{-3}), \lambda_5, \lambda_6 < 0$. As λ_3 decreases, the torus looses normal hyperbolicity.

This is an enormous advantage as it lifts the analysis to one dimension higher; direct numerical continuation of tori is rather difficult. A bifurcation diagram arising from the periodic solutions shows in general 2- and 3-tori, also torus doubling. One finds Neimark-Sacker, Fold, Chenciner, Fold-Neimark-Sacker, Branching Point, Bogdanov-Takens and Saddle-Node homoclinic bifurcations. The $1:2-$, $1:1-$ and $1:4-$resonances play a part. In a typical scenario, a torus contains stable and unstable periodic solutions. Changing the parameters, transverse intersection of stable and unstable manifolds produces non-smoothness of the torus and transition to chaos; see fig. 3. It is interesting to note, that this scenario agrees with that in the visionary paper by Ruelle and Takens [6].

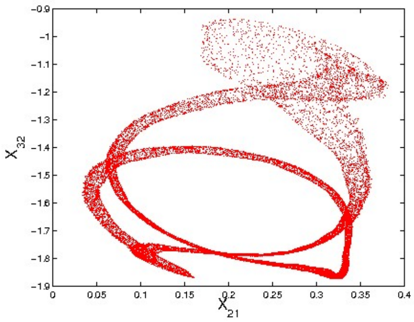

Figure 3. Transition to chaos of a double 3-torus; projection on a plane with Lyapunov exponents $\lambda_1 \approx \lambda_2 \approx\approx \lambda_3 = 0, \lambda_4 = O(10^{-4}), \lambda_5, \lambda_6 < 0$.

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