# Evolution to mirror-symmetric galaxies

Ferdinand Verhulst
Mathematisch Instituut, University of Utrecht
PO Box 80.010, 3508 TA Utrecht, The Netherlands
email: f.verhulst@uu.nl

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**Abstract**

The evolution of a rotating axisymmetric galaxy from an asymmetric state to a state of mirror symmetry with respect to the galactic plane has as basic result that in the asymmetric initial state the perpendicular $z$ normal mode is unstable for the $1:1$ and $1:2$ resonances. Dynamically this results in a transfer of mass and momenta towards the galactic plane. The timescale of evolution to symmetric equilibrium will determine in these cases the final distribution function describing position and velocities. In the case of the $1:1$ resonance we have in the final stage apart from the angular momentum integral 2 adiabatic invariants describing nonlinear dynamics. For the $1:2$ resonance the dynamics in the final stage is simpler, apart from the angular momentum integral the dynamics is governed by the 2 actions. In the first sections the results of mathematical analysis have been summarised, examples of evolution are given in section 3.

Key words: mirror-symmetry; evolution; rotating systems; resonance.
MSC classes: 34C14, 34C29, 34E10, 37J25, 37J65, 70H11, 85A05

# 1 Modeling complex evolution

The dynamics of galaxies cannot be understood without considering their evolution. There are many examples in astrophysics where evolution has produced systems with certain symmetries, think of globular clusters, disk galaxies and elliptical galaxies. In the solar system planets usually take a spherical shape with rotation producing a certain flattening at the poles. Orbits of satellites around planets with tidal friction often tend to evolve to circular orbits in a plane while locking into $1:1$ resonance; this resonance is present for the Earth-Moon system where one rotation of the Moon around its axis corresponds with one revolution around Earth.
We consider the evolution of galaxies with, because of observations, special interest in the evolution of the position and velocity distributions of stars from a relatively

asymmetric state to a symmetric one.
The evolution of galaxies is a very complex affair. *Our basic assumption is that the systems are already in an axisymmetric state of mass distribution and evolve slowly to mirror-symmetry with respect to the plane of the galaxy.* One can drop the assumption of axi-symmetry but this adds one degree of freedom and one timescale to the system with more computational possibilities. Note that different timescales are involved. For instance in the orbital evolution of a satellite around a heavy mass with tidal friction one can reach the state of 1 : 1 resonance with the motion still far from planar.
Apart from the basic assumption our formulation of the Hamiltonian is completely general except for the choice of explicit time-dependence. Changing the shape of time-dependence there is no difficulty in repeating the calculations.

The collisionless Boltzmann equation describes the evolution of the distribution of particles $f(t, \mathbf{x}, \mathbf{v})$ where $\mathbf{x}$ indicates the position, $\mathbf{v}$ the velocity; the collective gravitational potential ruling the dynamics of the system is $\Phi(t, \mathbf{x})$, see [2] ch. 4. With Lagrangian derivative $d/dt$ the equation is $df/dt = 0$. This is a first order partial differential equation with characteristics given by the Hamiltonian equations of motion. The axisymmetry implies when introducing cylindrical coordinates $R, \phi, z$:

$$\frac{\partial \Phi}{\partial \phi} = 0, \tag{1}$$

producing the angular momentum integral $J$ (in [2] called $L_z$) enabling us to reduce the spatial 3-dimensional motion to 2 degrees of freedom (dof). The angular momentum integral is:

$$R^2 \dot{\phi} = J. \tag{2}$$

We want to describe the dynamical consequences of evolution to a mirror-symmetric state by expanding around the circular orbits in the galactic plane at position $(R, z) = (R_0, 0)$ putting $R = R_0 + x$. To study evolution to symmetry we consider as a model the time-dependent two dof Hamiltonian:

$$H = \frac{1}{2}(\dot{x}^2 + x^2) + \frac{1}{2}(\dot{z}^2 + \omega^2 z^2) - (\frac{1}{3}a_1 x^3 + a_2 x z^2) - e^{-\varepsilon^n t}(\frac{1}{3}a_3 z^3 + a_4 x^2 z). \tag{3}$$

The parameter $\varepsilon$ is small and positive, with $n$ it determines the timescale of evolution. We will choose $n = 1$ or 2. The epicyclic frequency in the galactic plane has been scaled to 1, the vertical frequency is $\omega$, so $x$ corresponds with deviations of $R_0$, the radius of circular orbits. The two velocity dispersions differ considerably; for velocity dispersions in the plane of the galaxy see [3], for dispersions in the halo [1]. Velocity dispersions in galaxies is an ongoing research topic. To make the local analysis more transparent we rescale the coordinates $x = \varepsilon \bar{x}$ etc. Dividing by $\varepsilon^2$ and leaving out the bars we obtain the equations of motion:

$$\begin{cases} \ddot{x} + x & = \varepsilon(a_1 x^2 + a_2 z^2) + \varepsilon e^{-\varepsilon^n t} 2 a_4 x z, \\ \ddot{z} + \omega^2 z & = \varepsilon 2 a_2 x z + \varepsilon e^{-\varepsilon^n t}(a_3 z^2 + a_4 x^2). \end{cases} \tag{4}$$

A more general formulation is given in [9] also including computational details.. For the mathematical analysis one uses first and second order averaging, see [6] or [10].

. If we choose $a_1 = 1, a_2 = -1, a_3 = a_4 = 0$ we have the famous Hénon-Heiles problem [5]. The possible values of $\omega$ depend on the galactic potential constructed. An example describing an axisymmetric rotating oblate galaxy can be found in [2] eq. (3-50).

A time-independent example with in the centre of the galaxy a very massive nucleus produces with mass $M$ large the potential:

$$\Phi = \Phi_0(\sqrt{R^2 + z^2}) + \Phi_1(R, z), \Phi_0 = -\frac{M}{\sqrt{R^2 + z^2}}, \tag{5}$$

where, at least near the centre, $\Phi_1$ is small with respect to $\Phi_0$. It is well-known that, assuming rotation and expanding in a neighbourhood of the centre and near the circular orbits we find for the epecyclic orbits and the orbits perpendicular to the galactic plane the $1 : 1$ resonance. Outside the disk of the galaxy extending the galactic plane this resonance may again be prominent.

# 2 Evolution of the dynamics to mirror-symmetry

Assuming $\omega \geq 1$ the resonances with most dynamical consequences are generally $1 : 1$, $1 : 2$ and $1 : 3$, for the background of nonlinear resonance see [6], ch. 10.

It will be useful to introduce polar coordinates

$$x = r_1 \cos(t + \psi_1),\ z = r_2 \cos(\omega t + \psi_2), \tag{6}$$

and the actions $E_x, E_z$ by:

$$E_x = \frac{1}{2}(\dot{x}^2 + x^2) = \frac{1}{2}r_1^2,\ E_z = \frac{1}{2}(\dot{z}^2 + \omega^2 z^2) = \frac{\omega^2}{2}r_2^2. \tag{7}$$

The dynamics of prominent resonances produces periodic solutions, adiabatic invariants and interesting stability changes during evolution.

## 2.1 The $1 : 1$ resonance

If the epicyclic frequency and the vertical frequency are equal or close, the $1 : 1$ resonance becomes important. The dynamics of the $1 : 1$ resonance turns out to be the most complicated case. Depending on the system parameters we find normal modes (families of periodic solutions in the galactic plane and perpendicular to it) and stability changes during evolution. Averaging the equations of motion we find that to $O(\varepsilon)$ all averaged terms vanish. Significant dynamics takes place on a longer timescale so we choose $n \geq 2$ to consider longer timescales, we have to use second order averaging. The combination angle $\chi = \psi_1 - \psi_2$ plays a part at second order. Both time-dependent and time-independent terms are strongly active on intervals of time $O(1/\varepsilon^2)$. We summarise from [8] the results for the end-stage of mirror-symmetry $a_3 = a_4 = 0$.

- The equations of motion have in the mirror-symmetric case 2 adiabatic invariants:
$$E_0 = \frac{1}{2}(r_1^2 + r_2^2) = \frac{1}{2}(\dot{x}^2 + x^2 + \dot{z}^2 + z^2) \tag{8}$$
and:
$$I_3 = r_1^2 r_2^2 \cos 2\chi + \alpha r_1^4 + \beta r_1^2, \tag{9}$$
with $\alpha, \beta$ rational functions of $a_1, a_2$.
- The epicyclic $x, \dot{x}$ normal mode is in the mirror-symmetric stage an exact solution, it is unstable for $-1/3 < a_1/(3a_2) < 2/15$ and $1/3 < a_1/(3a_2) < 2/3$.
- The vertical $z, \dot{z}$ normal mode is obtained as $O(\varepsilon)$ close to perpendicular motion. It is unstable for $-1/3 < a_1/(3a_2) < 1/3$.
- There are in-phase periodic solutions $\chi = 0, \pi$ existing and out-phase periodic solutions $\chi = \pi/2, 3\pi/2$.

We expect the dynamics of the mirror-symmetric case to describe the orbits of system (4) on intervals of time larger than $1/\varepsilon^2$. During evolution the in-phase and out-phase solutions are present with constant amplitude and are approaching periodicity. The dynamics will after a long time be governed by the angular momentum integral (2) and the 2 adiabatic invariants (8)-(9) producing a complicated velocity distribution and varying actions as demonstrated in the examples.

## 2.2 The $1:2$ resonance

A prominent case for general 2 dof systems is the $1:2$ resonance ($\omega = 2$). We find the active combination angles $\chi_1 = 2\psi_1 - \psi_2$ and in the mirror-symmetric state $\chi_2 = 4\psi_1 - 2\psi_2$. First order averaging produces instability of the vertical $z$ normal mode ($x = \dot{x} = 0$); the epicyclic $x$ normal mode does not exist on the interval of time $O(1/\varepsilon)$ but emerges on a longer timescale. There are 2 adiabatic invariants valid on intervals $O(1/\varepsilon)$:
$$\frac{1}{2}r_1^2 + 2r_2^2 = E_0, \quad a_4 r_1^2 r_2 \cos\chi_1 = I_3, \tag{10}$$
with constants $E_0, I_3$ and $r_1, r_2$ radii in polar coordinates, the asymmetry ($a_4$) is prominent. In the original coordinates we have:
$$\frac{1}{2}(\dot{x}^2 + x^2) + \frac{1}{2}(\dot{z}^2 + 4z^2) = E_0, \; a_4(x^2 z - \dot{x}^2 z + 2x\dot{x}\dot{z}) = I_3.$$
On longer intervals of time the terms with coefficients $a_3, a_4$ will vanish and the symmetric terms become dominant. The first order averaged system admits families of solutions with constant amplitude and decreasing phases on intervals $O(1/\varepsilon)$ if:
$$\chi_1 = 0, \pi, r_1^2 = \frac{4}{3}E_0, r_2^2 = \frac{1}{6}E_0. \tag{11}$$
The solutions with $\chi_1 = 0$ are again called in-phase, the solutions with $\chi_1 = \pi$ out-phase. Second order averaging is needed to consider longer intervals of time and new phenomena. We summarise some results of [9].

1. During an interval of time of order $1/\varepsilon$ the integrals (adiabatic invariants) (10) are active, the system is dominated by the asymmetric $a_4$ term. On this interval of time it will govern the orbital dynamics and accordingly the corresponding distribution function in phase-space.
2. On time intervals of order $1/\varepsilon^2$ the time-independent system involving the coefficients $a_1, a_2$ dominates the dynamics. In [7] it is shown that for this system, depending on $a_1, a_2$, 2 resonance manifolds can exist on the energy manifold In this case the resonance manifold with $4\psi_1 - 2\psi_2 = 0$ has stable $2:4$ resonant periodic orbits surrounded by tori, for $4\psi_1 - 2\psi_2 = \pi$ the $2:4$ resonant periodic orbits also exist but are unstable. The resonance manifolds have size $O(\varepsilon)$, the dynamics takes place on intervals of time of order $1/\varepsilon^3$. Outside the resonance manifolds the dynamics is characterised by the adiabatic invariants $E_x, E_z$.
3. The instability of the $z$ normal mode persists at second order but leads to stability in the final stage of mirror symmetry.. The epicyclic $x, \dot{x}$ normal mode does not exist in the first stage (time order $1/\varepsilon$) but the normal mode emerges in the final stage of mirror symmetry.

# 3 Examples of evolution

Dynamically interesting are the cases where we start in the beginning of evolution near an unstable orbit, say a normal mode, and move to a different dynamics when reaching a state close to mirror-symmetry. This changes the velocity distribution drastically. Such evolution can happen in the case of $1:1$ and of $1:2$ resonance.
As described above these cases are dynamically different. The $1:1$ resonance when reaching a near mirror-symmetric state will still show nonlinear interaction between the modes; the normal modes persist but depending on the parameters can be stable or unstable. In the case of the $1:2$ resonance near mirror-symmetry implies that the dynamics has at the end of evolution the character of a $2:4$ resonance with little interaction of the modes; the normal modes persist and are stable. During evolution of the $1:2$ resonance the velocity distribution may change drastically.
The figures in the next subsections shows clearly the different types of dynamics of the $1:1$ and $1:2$ resonances in the final stage of mirror-symmetry.

### 3.1 Evolution of the $1:1$ resonance

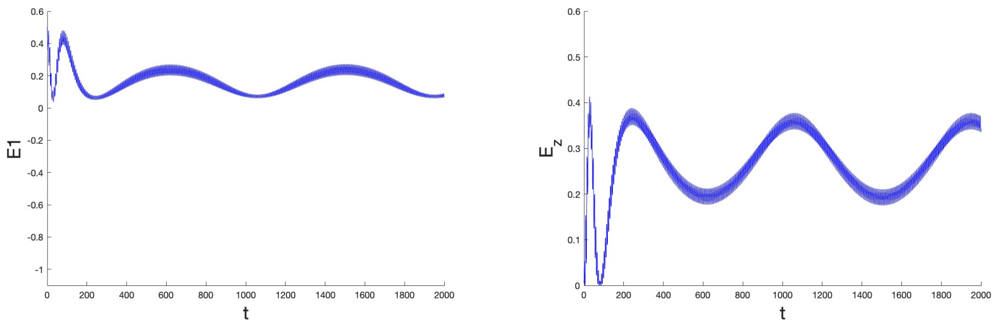


Figure 1: Left the behaviour of the action $E_x(t)$ of system (4) in $1:1$ resonance for the case $n = 2$ near the epicyclic normal mode; initial conditions $x(0) = 1, z(0) = 0.1, E_x(0) = 0.5, E_z(0) = 0.005$. Parameter values $a_1 = -1.5, a_2 = 1, a_3 = 0, a_4 = 4, \varepsilon = 0.1$. Right we display the corresponding $z$ behaviour by plotting $E_z(t)$. It takes around 200 timesteps to settle in the stationary state which is drastically different from the initial state. In the final stage the normal modes are stable.

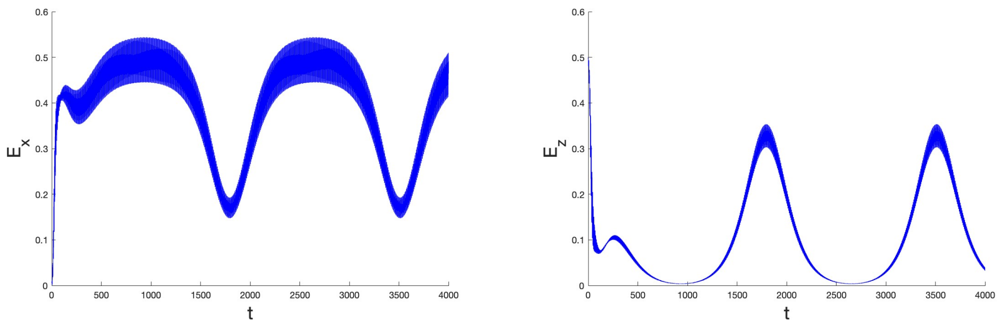


Figure 2: Left the behaviour of the action $E_x(t)$ of system (4) in $1:1$ resonance for the case $n = 2$ starting near the vertical (z) normal mode; initial conditions $x(0) = 0.1, z(0) = 1, E_x(0) = 0.005, E_z(0) = 0.5$. Parameter values $a_1 = -1.5, a_2 = 1, a_3 = 0, a_4 = 4, \varepsilon = 0.1$. In the final stage the normal modes are stable.

In figs 1-2 the choice of $a_1, a_2$ implies the system has stable epicyclic and stable vertical normal modes for the final mirror-symmetric case ($a_3 = a_4 = 0$). Considering time evolution starting in an asymmetric state, our choice of $a_3, a_4$ keeps the normal modes but destabilises them in the initial stage; we have $a_1 = -1.5, a_2 = 1, a_3 = 0, a_4 = 4$. When the time-dependent perturbation has become negligible the orbits have moved into general position. The time-independent case will approximate the dynamics after a long initial interval of time, the in-phase periodic solutions exist in this mirror-symmetric case and are stable.

## Evolution to the Hénon-Heiles dynamics

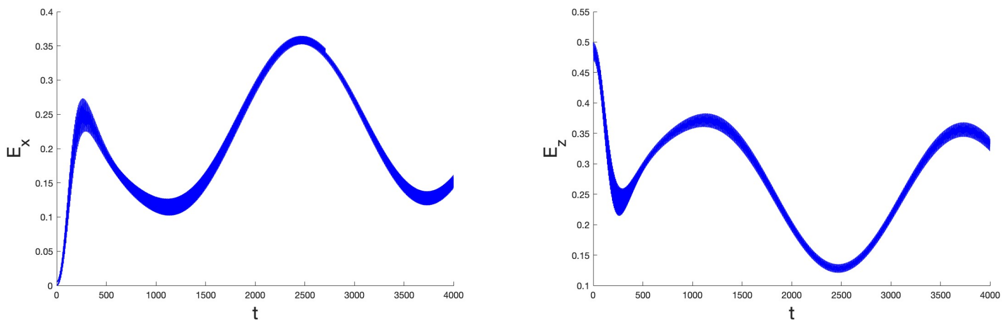


Figure 3: Evolution to the Hénon-Heiles system based on system (4) for the case $n = 2, a_1 = 1, a_2 = -1, a_3 = 1, a_4 = 4$ with $\varepsilon = 0.05$ intially close to motion perpendicular to the galactic plane with $x(0) = 0, z(0) = 1$ and zero velocities. Left action $E_x$, right $E_z$. Because of the instability of the $z$ normal mode in the asymmetric initial stage the dynamics evolves to motion on a torus around a periodic solution in general position with in the final stage $E_x + E_z$ approximately constant.

The Hénon-Heiles case is included as there are many details available in the literature on this system [5]. If $a_3 = a_4 = 0$ and $\varepsilon = 1$ the system is chaotic; large-scale chaos sets in at $E_0 \geq 1/12 = 0.083$, the energy manifold is bounded if $0 \leq E_0 \leq 1/6$. We consider the case of regular dynamics. In fig. 3 (left) we present for $\varepsilon = 0.05$ action $E_x(t)$ starting close to the unstable $z$ normal mode. As we see in fig. 3 (right) the motion into the halo becomes reduced whereas the component of motion in the galactic plane $E_x$ has become much larger.

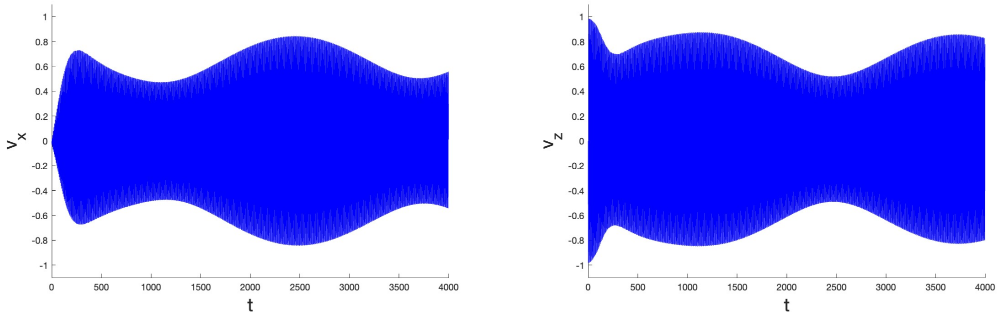


Figure 4: Evolution to the Hénon-Heiles system of the velocity distribution based on system (4) with the same parameters as in fig. 3. The motion is initially close to motion perpendicular to the galactic plane with $x(0) = 0, z(0) = 1$ and zero velocities. The velocity field corresponds with energy transfer to the galactic plane $x$ direction.

The instability of the $z$ normal mode produces a drastic change of the velocity distribution. This is illustrated in fig. 4 where the evolution is shown of the velocities $v_x, v_z$ from the asymmetric case to the case of mirror-symmetry.

Instead of the evolution to the Hénon-Heiles family one can formulate more general conclusions. In section 2 we have listed open sets of parameters where the mirror-symmetric potentials ($a_3 = a_4 = 0$) have existing normal $x$ and $z$ modes that are unstable. The examples given here are typical for the dynamics.

### 3.2 Evolution of the $1:2$ resonance

. On a timescale $O(1/\varepsilon)$ the $z$ normal mode exists and is unstable; the epicyclic $x$ normal mode is not present but emerges during evolution. In fig. 5 we start near the $z$ normal mode position. The solutions move into general position on tori around periodic solutions. The asymmetric past of the system has changed the overall dynamics drastically, but this depends strongly on the initial state where the potential is still asymmetric and the timescale of evolution.
In fig. 6 we put $\varepsilon = 0.05$ and 0.03, the other parameters and initial conditions are as in fig. 5. The evolution to mirror-symmetry takes longer as $\varepsilon$ is smaller than in fig. 5 with considerable influence on the position and velocity distribution. It is remarkable how sensitive the final dynamics is to the transition timescale of asymmetry to mirror-symmetry.

### 3.3 Evolution of the $1:3$ resonance

The $1 : 3$ resonance is dynamically different and less interesting. Most of the analysis can be deduced from [8], Putting $\omega = 3$ in system (4) we find after 2nd order averaging:

$$\dot{r}_1 = O(\varepsilon^3),\ \dot{r}_2 = O(\varepsilon^3). \tag{12}$$

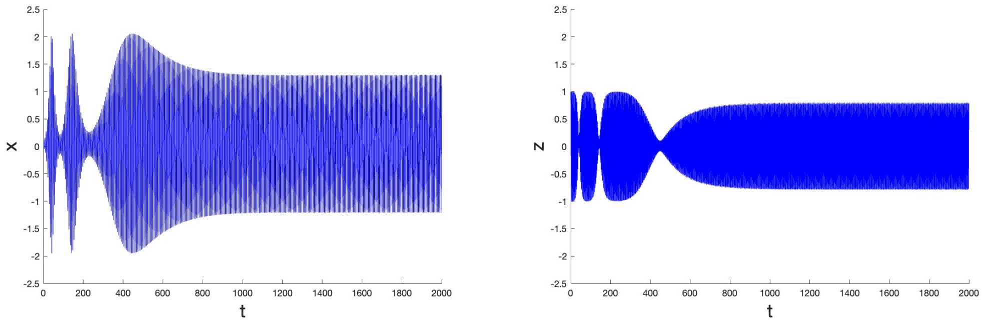


Figure 5: The 1 : 2 resonance. The behaviour of system (4) starting near the $z$-normal mode. We have $x(0) = 0.1, z(0) = 1$ with initial velocities zero, $\varepsilon = 0.08, \omega = 2, a_1 = 0.5, a_2 = 1, a_3 = -1, a_4 = 3$. The dynamics produces a position and velocity distribution that has changed considerably by the evolution to mirror symmetry.

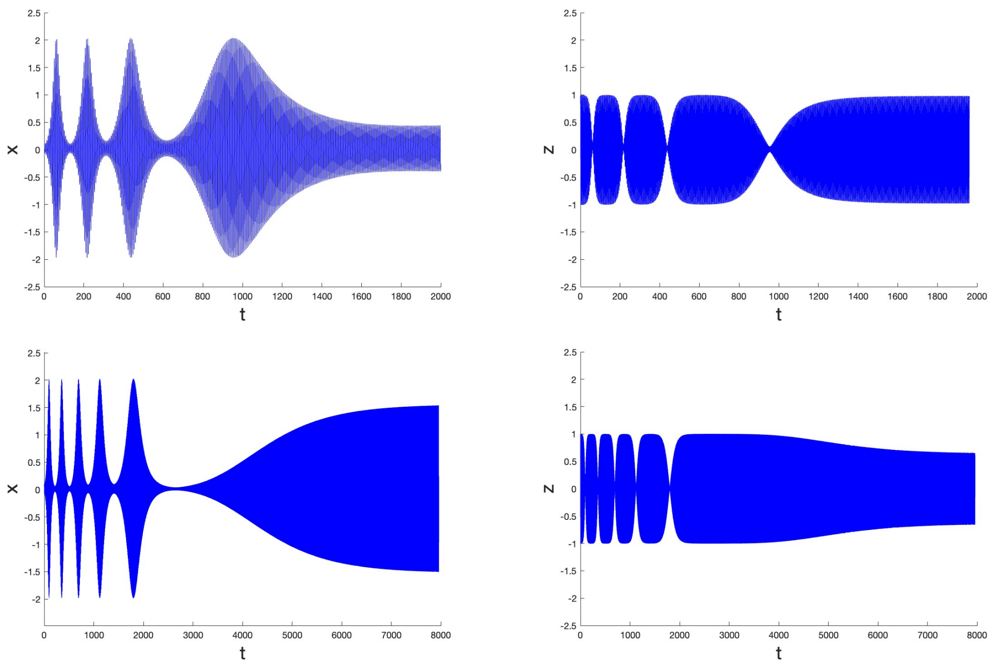


Figure 6: The 1 : 2 resonance. Decreasing $\varepsilon$ lengthens the transition time from asymmetry to symmetry and changes the transition dynamics. In fig. 5 we used $\varepsilon = 0.08$. Keeping the other parameters and the initial conditions equal we show the $x(t), z(t)$ evolution for the cases $\varepsilon = 0.05$ (top), 0.03 (below).

This remarkable result shows that for the 1 : 3 resonance the slowly vanishing asymmetry of the potential *plays no part* for the amplitudes to order $\varepsilon^3$. The theory of higher order resonance, see [6] and [7], shows that the combination angle $\chi_3 = 6\psi_1 - 2\psi_2$ plays a crucial role. With high precision the distribution function at the 1 : 3 resonance will depend on $J$ and the 2 actions.

## 3.4 Consequences for the distribution function at the main resonances

We conclude and summarise. Suppose we start with a collection of particles (stars) characterised by a distribution function satisfying the Boltzmann equation that is collisionless assuming negligible dynamical friction. The system is already in an axi-symmetric state but the evolution to mirror symmetry to the galactic plane is still going on. We have, apart from angular momentum $J$, two active integrals of motion strongly depending on the local resonance between epicyclic and vertical oscillations. The distribution function will be a function of the 3 integrals. Near resonance the integrals will change during evolution.

A basic aspect of the evolution to mirror-symmetry is the instability near the $z$ normal mode (or in general, motion perpendicular to the galactic plane) in the original asymmetric state when close to 1 : 1 or 1 : 2 resonance. In this stage matter will be moved to the galactic disk and $x$-components of the velocities starting near perpendicular motion will grow.

The timescale of evolution as given by the choice of $\varepsilon$ will also be very important for the state of the final dynamics, see for examples figs 5-6.

Both the 1 : 1 or 1 : 2 resonance show this instability but there is also a difference. The 1 : 1 resonance is characterised by nonlinear interaction of the 2 modes both in the asymmetric and in the final symmetric state. In the final state we have for the 3 integrals of motion $J$, $E_0$ from (8) and $I_3$ from (9).

For the 1 : 2 resonance we have in the initial asymmetric stage, (on a timescale of order $1/\varepsilon$) the 3 integrals of motion $J, E_0, I_3$ with $E_0, I_3$ given by (10). In the final stage, say on timescales of order $1/\varepsilon^3$, the integrals of motion will be $J, E_x, E_z$. So angular momentum and the 2 actions will dominate the dynamics of the 1 : 2 resonance in the mirror-symmetric stage.

This also holds throughout for the evolution of the 1 : 3 and other resonances from asymmetry to mirror-symmetry. We leave out many interesting details of the higher order resonances, see [9] for the theoretical background. The $O(\varepsilon)$ constancy of the actions in the 1 : 3 case and in the mirror-symmetric 1 : 2 case can be illustrated numerically.

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# References


[1] W.R. Brown, M.J. Geller, S.J. Kenyon and A. Diaferio, *Velocity dispersion profile of the Milky Way halo*, Astron. J. 139, pp. 59-67 (2010).

[2] J. Binney and S. Tremaine, *Galactic Dynamics*, Princeton Series in Astrophysics, Princeton UP 3rd printing (1994).

[3] W. Dehnen, D.E. MacLaughlin and J. Sachania, *The velocity dispersion and mass profile of the Milky Way*, MN Roy Astr Soc 369, pp. 1688-1692 (2006).

[4] F.G.. Gustavson, *On constructing formal integrals of a Hamiltonian system near an equilibrium point*, Astron. J. 71, pp. 670-686 (1969).

[5] M. Henon and C. Heiles, *The applicability of the third integral oi motion: some numerical experiments*, Astron. J. 69, pp. 73-79 (1964).

[6] J.A. Sanders, F. Verhulst and J. Murdock, *Averaging methods in nonlinear dynamical systems* 2nd ed., Appl. Math. Sciences 59, Springer, New York etc., (2007).

[7] J.M. Tuwankotta and F. Verhulst , *Symmetry and resonance in Hamiltonian systems*, SIAM J. Appl. Math. 61 pp. 1369-1385, (2000).

[8] F. Verhulst, *Discrete symmetric dynamical systems at the main resonances with applications to axi-symmetric galaxies*, Phil. Trans. roy. Soc. London 290, pp. 435-465 (1979).

[9] Ferdinand Verhulst, *Evolution to Mirror-Symmetry in Rotating* Systems, Symmetry 13, 1189, doi.org/10.3390/sym13071189 (2021).

[10] Ferdinand Verhulst, *A Toolbox of Averaging Theorems, ordinary and partial differential equations*, Surveys and Tutorials in the Applied Mathematical Sciences vol. 12, Springer, (2023).