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Nonintegrability of 3 DoF Hamiltonian Resonant Systems

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We review a number of methods to prove nonintegrability of Hamiltonian systems and focus on 3 Degrees-of-Freedom (DoF) systems listing the known results for the prominent resonances. Associated with the Hamiltonian systems are the averaged-normal forms that provide us with geometric insight, approximations of orbits and measures of chaos. Symmetries do change the qualitative and quantitative pictures; we illustrate this for the 1:2:1 resonance with discrete symmetry in the 1st and 3rd DoF. In this case, the averaged-normal form is still nonintegrable, but it becomes integrable when adding discrete symmetry in all DoF. Apart from the short-periodic solutions obtained by averaging, we find many periodic solutions. There is numerical evidence of the presence of Šilnikov bifurcation which clarifies the presence of nonintegrability phenomena qualitatively and quantitatively.

Keywords: Hamiltonian resonance; nonintegrability; symmetry; averaging-normalization; Šilnikov bifurcation; algebraic method.

1. Introduction

Time-independent Hamiltonian systems with n Degrees-of-Freedom (DoF) are generally nonintegrable for nondense sets of the parameters (coefficients of the Hamiltonian function) with positive measure; see [Broer & Sevryuk, 2010]. The standard formulation for Hamiltonian systems is after introducing an analytic function $H(p,q)$ of $2n$ variables on a suitable domain:

$$H(p,q), \quad p = (p_1, \ldots, p_n), \quad q = (q_1, \ldots, q_n) \tag{1}$$

inducing equations of motion:

$$\dot{p} = \frac{\partial H}{\partial q}, \quad \dot{q} = -\frac{\partial H}{\partial p}. \tag{2}$$

System (2) has at least one integral of motion, $H(p,q)$. Orbits starting at $p(0), q(0)$ will remain on the energy manifold $H(p(0), q(0)) = E_0$ with E_0 a constant. We call n the number of DoF. We suppose that near stable equilibria, we can expand $H = H_2 + H_3 + H_4 + \cdots$ with $H_j, j = 2, 3, 4, \ldots$ homogeneous polynomials of degree j. $H_2(p,q)$ is

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supposed to be a Morse function, if H_2 is positive definite in a neighborhood of the origin, the energy manifold will be compact near the origin of phase-space.

Normalizing the Hamiltonian near stable equilibrium gives us explicit estimates of the dynamics determined by the equations of motion. Equivalent methods are Birkhoff–Gustavson normalization, see [Gustavson, 1966] and averaging methods keeping the equations of motion conservative, see [Sanders *et al.*, 2007]. An important question then is whether the normalized system is integrable or not. In addition, we have the basic question what nonintegrability of the normalized Hamiltonian means dynamically.

There are many aspects of the formulation of Hamiltonian normal forms that can be found in [Sanders *et al.*, 2007, Chapter 10]; we will presuppose the formulations of this material.

In the case of 2 DoF, the normal form near a stable equilibrium will always be integrable. A consequence is that intersections of stable and unstable manifolds in this case will always have very small angles, sometimes vanishing at any order of normalization. Interesting dynamics in 2 DoF happens at somewhat larger values of the energy with respect to stable equilibrium as can be shown by the analysis using Melnikov integrals.

This motivates us to consider Hamiltonian systems with 3 DoF near stable equilibrium. We will summarize the results for first-order resonances and discuss examples with symmetry later. The numerical value of H_2 (for given initial conditions) is indicated by E_0; near stable equilibrium, we can rescale $p \to \varepsilon p$, $q \to \varepsilon q$ with ε a small positive parameter. Dividing by ε^2, we obtain from Hamiltonian (1) the Hamiltonian $H_2 + \varepsilon\alpha H_3 + \varepsilon^2\beta H_4$ with H_3, a cubic polynomial in (p, q), H_4 quartic in (p, q), etc. We clearly have the energy $H(p(0), q(0)) = E_0 + O(\varepsilon)$ for all times.

The normal form $\overline{H}$ to cubic and quartic terms is indicated by

$$\begin{aligned} \overline{H}^3 &= H_2 + \overline{H}_3, \\ \overline{H}^4 &= H_2 + \overline{H}_3 + \overline{H}_4. \end{aligned} \tag{3}$$

Note that according to [Weinstein, 1973], an n DoF Hamiltonian system near a stable equilibrium contains at least n families of periodic solutions parameterized by the energy. We will keep this in mind when looking for periodic solutions in particular systems.

1.1. *Coordinate transformations for approximations*

Suppose we consider an n DoF Hamiltonian system with Hamiltonian $H = H_2 + H_3$ as in (3). As we shall see, one can devise transformations to simplify the quadratic nonlinear terms in the equations of motion. As indicated above, we obtain a simplified system of equations of the form $\overline{H} = H_2 + \overline{H}_3 + R$. The term R will consist of quartic and higher-order degree elements, the corresponding equations of motion will in general be as difficult to analyze as the original system. The next step is well known and often applied: we truncate the system to $\overline{H}^3 = H_2 + \overline{H}_3$ and try to analyze the truncated system. Very rarely the relations between the solutions of the truncated system and the original one are investigated. We have here very basic questions. Do periodic solutions found in the truncated system persist in the original system? Does chaos persist? On what time intervals do we have mathematical approximations of the original system? The questions on periodic solutions have been answered in [Poincaré, 1892, 1893, 1899, Vol. 1] using the implicit function theorem; for Hamiltonian systems, we have to consider the periodic solutions on a fixed energy manifold. Explicit approximation estimates can be found in [Sanders *et al.*, 2007; Verhulst, 2023]; the estimation theory is technically complicated. Chaos in the normal form describes phenomena that cannot be destroyed by higher-order ε-terms.

It can be confusing that different coordinate systems play a part in Hamiltonian systems, for introductions, see [Arnold, 1978, 1983]. Apart from the canonical variables (p, q), one uses action-angle variables τ_j, Φ_j defined by

$$\begin{aligned} &q_j = \sqrt{2\tau_j}\sin\Phi_j, \quad p_j = \sqrt{2\tau_j}\cos\Phi_j, \\ &\tau_j \geq 0, \quad \Phi_j \in S^1, \ \ j = 1, \ldots, n. \end{aligned} \tag{4}$$

Action-angle coordinates are again canonical in the sense that they can be written in the form (2). We will also use amplitude-phase coordinates, see Sec. 3. They are related to action-angles and are conservative but not canonical; they have the advantage of a more direct physical meaning.

In many special problems, especially in celestial mechanics, many other coordinate systems can be useful.

1.2. *The order of resonance and combination angles*

For the quadratic part of the Hamiltonian H_2, we usually write

$$H_2 = \frac{1}{2}\sum_{j=1}^{n}(p_j^2 + \omega_j^2 q_j^2), \quad \text{or}$$

$$H_2 = \frac{1}{2}\sum_{j=1}^{n}\omega_j(p_j^2 + q_j^2), \quad \omega_j > 0.$$

The parameters ω_j are called the (linear) frequencies. We have a resonance if we can find natural numbers $a_1, \ldots, a_m \neq 0$ such that

$$\sum_j^m a_j\omega_j = 0$$

for a number of frequencies from the set $\omega_1, \ldots, \omega_n$. As we shall see in Sec. 2, the presence of resonances causes the importance of effective combination angles and interesting nonlinear dynamics.

1.3. *Methods signaling nonintegrability*

An important step in the analysis of a Hamiltonian system is to establish whether the system is integrable or not. What nonintegrability means for the dynamical system is a second question. In the important case of near-integrability, we have the KAM theorem that gives general conditions for perturbed n DoF integrable Hamiltonian systems to contain an infinite number of n-tori, the measure of which tends to 1 as the perturbation tends to zero. For precise formulations and an extensive discussion of the theory, see [Broer & Sevryuk, 2010]. Basic results for 2 DoF Hamiltonian systems are discussed in [Levi, 2010].

Two DoF systems

As a typical example, we consider 2 DoF Hamiltonian systems close to the basic 2:1 resonance. To restrict the number of parameters, we consider potential problems $H = H_2 + \varepsilon H_3$ with

$$\begin{cases} H_2 = \dfrac{1}{2}(\dot{x}^2 + \omega^2 x^2 + \dot{y}^2 + y^2), \quad \omega^2 = 4 + \varepsilon d, \\ H_3 = -(a_1x^3 + a_2y^3 + a_3x^2y + a_4xy^2). \end{cases} \tag{5}$$

The parameter d allows for detuning of the exact resonance. If $a_3 = a_4 = 0$, the Hamiltonian (5) is integrable; we assume that the parameters $a_1, \ldots, a_4$ are nonzero. After normalization, we find for the normalized Hamiltonian $\overline{H}$:

$$\begin{aligned} \overline{H} &= H_2 + \varepsilon\overline{H}_3, \\ \overline{H}_3 &= \frac{1}{4}a_4(2xy^2 - x(\dot{y}^2 + y^2) + \dot{x}\dot{y}y). \end{aligned} \tag{6}$$

The normalized Hamiltonian (6) is integrable with integrals $H_2, \overline{H}_3$. The solutions of the equations of motion induced by $\overline{H}$ approximate the solutions of the equations of motion induced by the original Hamiltonian (5) with error $O(\varepsilon)$ on the timescale $1/\varepsilon$. Hamiltonian (5) will not be integrable in general, its parameter space is four-dimensional, it can be seen as a perturbation of Hamiltonian (6). In this case, integrability in parameter space has measure zero ($a_3a_4 \neq 0$). However, although we have nonintegrability of Hamiltonian (5), the measure of invariant 2-tori near stable equilibrium tends to 1 if $\varepsilon \to 0$.

Poincaré introduced a map characterizing the phase-flow of a Hamiltonian system near stable equilibrium. The orbits move on energy manifolds that are topologically equivalent to spherical surfaces. In the case of 2 DoF, an energy manifold is S^3. As the flow is recurrent, for an introduction and references, see [Verhulst, 2023], we can construct a two-dimensional transversal to the flow and, because of the recurrence, the flow induces a map of the transversal into itself. A periodic solution will produce a fixed point of the map, orbits near a stable periodic solution will generally be organized on 2-tori surrounding the periodic solution producing closed curves on the transversal. The 2-tori separate the flow on S^3. Evidence of this example and other cases, see [Holmes *et al.*, 1988], suggest that near stable equilibrium of 2 DoF Hamiltonian systems, the distance between the tori and the asymptotic estimates from the normalization is exponentially small (like $\exp(-1/\varepsilon)$).

The analysis will be qualitatively and quantitatively different for 3 DoF Hamiltonian systems.

Three DoF systems

Near stable equilibrium, the energy manifolds will be S^5. A transversal of the flow on a fixed energy manifold will still be four-dimensional, the KAM theorem guarantees the presence of invariant 3-tori containing quasi-periodic solutions. However, the 3-tori will not separate the orbits of S^5, there is more space for chaotic solutions.

The consequence of this geometric argument is that the integrability of the normalized Hamiltonian system is an important question as it directly affects the presence of chaotic solutions. This means that we have to look for a third integral of the normalized Hamiltonian, independent of H_2 and $\overline{H}_3$.

We will review a number of methods signaling nonintegrability. The first three techniques lead also to insight in the dynamics. The last two leave interesting questions open on the relation between nonintegrability and the corresponding dynamics; this question has been settled for specific Hamiltonian systems like the Hénon–Heiles system, see [Kozlov, 1996; Morales-Ruiz, 2000]. More details will be given in the following section for the first-order resonances in 3 DoF.

(1) Intersection of manifolds
The original idea was conceived by Poincaré and can be found in [Poincaré, 1892, 1893, 1899, Vol. 3]. Considering a saddle point of a Poincaré map, one can identify stable and unstable manifolds emerging from the saddle. The complexity of transversal crossings of the manifolds produces the wild behavior that nowadays is identified as chaos and horseshoe dynamics. Most applications of the idea are a mixture of analytic and numerical dynamics, see also [Holmes *et al.*, 1988]. The use of the so-called Melnikov integral is based on the same ideas.

(2) Devaney's adaptation of Ŝilnikov bifurcation
This extends both the ideas of Poincaré and Ŝilnikov, see [Devaney, 1976]. Suppose we have a Hamiltonian system containing an isolated periodic solution that is complex unstable. A Poincaré map of the system shows for this solution a fixed point with two adjoint complex eigenvalues. If one can find an isolated homoclinic of this periodic solution, this will contain a horseshoe map and we have chaos near this periodic solution. Again, most applications of the idea are a mixture of analytic and numerical dynamics.

(3) Laskar's frequency method
Consider a Hamiltonian system that is close to an integrable one. Assume that the system satisfies the conditions of the KAM theorem, producing an infinite set of tori around a stable periodic solution. If the system is actually integrable there will be periodic and quasi-periodic solutions on the tori. If the system is nonintegrable, a frequency analysis on the tori will show jumps in the frequencies between the tori, see [Laskar, 1993]. They will correspond to gaps between the tori containing chaotic solutions. Using the frequency method is technically difficult, it was used in 1990 by Laskar to demonstrate chaos in a Solar System model. It can also be used in dissipative systems with symmetries that allow infinite sets of tori, see [Bakri & Verhulst, 2022].

(4) Analysis of singularities. Duistermaat [1984] discusses the 1:2:1 resonance for 3 DoF Hamiltonian systems. It turns out that $\overline{H}_3 = 0$ contains a family of periodic solutions and it is possible to analyze the period function P of this family. Complex continuation of P shows infinite branching of the period function which excludes integrability. The approach is different from Ziglin–Morales–Ramis theory to be discussed next. The author discusses the dynamics by looking at possible intersections of stable and unstable manifolds of the family of periodic solutions; he needs H_4 terms to obtain transversal intersection.

(5) An algebraic method (Ziglin–Morales–Ramis theory). We will use an observation by Lyapunov in 1894, see [Kozlov, 1996, Chapter 5] for extensive discussions of the background; see also [Morales-Ruiz, 2000]. Lyapunov's result is that if we find multivalued solutions from a variational equation of a periodic solution, this result extends to the original Hamiltonian system; a modern treatment involves monodromy groups. According to Lyapunov, if the general solution of the variational equations is not single-valued, then the solutions of the original nonlinear equations are also not single-valued; this excludes integrability. In a more recent formulation, one uses the Galois group; for details on this use of differential Galois theory, see [Morales-Ruiz, 2000]. Non-Liouvillian

or closed-form solutions imply a noncommutative differential Galois group, which violates the necessary condition of Morales and Ramis theory for integrability of Hamiltonian systems. In this case, the solution is multivalued, so, there is no additional analytic first integral and in most cases, not even a meromorphic integral. An application is given in Appendix A.

Integrability of Hamiltonian systems is exceptional. Important questions concern the description of the dynamics in the case of nonintegrability and the measure of chaos in nonintegrable systems. For both questions, the theory of normalization and averaging plays a crucial role.

2. The First-Order Resonances in 3 DoF

From [Sanders *et al.*, 2007, Chapter 10], we list the first-order resonances in 3 DoF. We add the general normal forms and the results on integrability. Note that for special values of the coefficients, systems that are in general nonintegrable may be integrable. This may also happen by assuming symmetries.

2.1. *Integrability results of normal forms*

We leave out the ε in the expressions. The Hamiltonian $H_2 + H_3$ has 56 free parameters, the reduction to normal forms is impressive.

- **The 1:2:2 resonance.**

The normal form is in action-angle coordinates τ, ψ to third order:

$$\begin{aligned}\overline{H}^3 &= \tau_1 + 2\tau_2 + 2\tau_3 \\ &\quad + 2[a_1\sqrt{2\tau_2}\cos(2\psi_1 - \psi_2 - a_2) \\ &\quad + a_3\sqrt{2\tau_3}\cos(2\psi_1 - \psi_3 - a_4)], \qquad (7)\end{aligned}$$

with constants $a_1, \ldots, a_4$. For potential problems, we have $a_2 = a_4 = 0$. The actions and the two combination angles are slowly varying.

The normal form $\overline{H}^3$ in (7) is integrable, see, for potential problems, [Martinet *et al.*, 1981] and, for the general Hamiltonian case, [van der Aa & Verhulst, 1984]. The proofs are by inspection of the equations of motion.

It was shown in [Christov, 2020] by algebraic methods that for an open set of parameters, the normal form $\overline{H}^4$ is not integrable. The implication is that generically, the 1:2:2 resonance behaves to $O(\varepsilon)$ integrable and on smaller subsets to $O(\varepsilon^2)$ dynamically nonintegrable. High precision numerics in [Christov, 2020] shows chaotic behavior in Poincaré sections.

- **The 1:2:1 resonance.**

The normal form is in action-angle coordinates τ, ψ to third order:

$$\begin{aligned}\overline{H}^3 &= \tau_1 + 2\tau_2 + 2\tau_3 \\ &\quad + 2\sqrt{2\tau_2}[a_1\cos(2\psi_1 - \psi_2 - a_2) \\ &\quad + a_3\sqrt{\tau_1\tau_3}\cos(\psi_1 - \psi_2 + \psi_3 - a_4) \\ &\quad + a_5\tau_3\cos(2\psi_3 - \psi_2 - a_6)]. \qquad (8)\end{aligned}$$

We have three combination angles and 56 parameters in $H_2 + H_3$, reduced to six parameters in the normal form. In the case of potential problems, we have $a_2 = a_4 = a_6 = 0$.

It was shown in [Duistermaat, 1984] that for an open set of parameters in the Hamiltonian, the normal form $H_2 + \overline{H}^3$ is nonintegrable. The technique used in [Duistermaat, 1984] was that in certain subsets, one can identify solutions and continue time analytically into the complex domain. This leads to the presence of an essential singularity in the sense of complex analysis. Duistermaat [1984] remarks on the nonintegrable dynamics of this resonance, these observations are interesting but far from complete.

Using Ziglin–Morales–Ramis theory based on differential Galois theory, the nonintegrability of the normal form (8) was also shown in [Christov, 2012]. No explicit dynamics was indicated for this resonance but the algebraic method is also useful for other resonance problems.

- **The 1:2:3 resonance.**

The normal form is in action-angle coordinates τ, ψ to third order:

$$\begin{aligned}\overline{H}^3 &= \tau_1 + 2\tau_2 + 3\tau_3 \\ &\quad + 2\sqrt{2\tau_1\tau_2}[a_1\sqrt{\tau_1}\cos(2\psi_1 - \psi_2 - a_2) \\ &\quad + a_3\sqrt{\tau_3}\cos(\psi_1 + \psi_2 - \psi_3 - a_4)]. \qquad (9)\end{aligned}$$

We have two combination angles and four parameters in the normal form. In the case of potential problems, we have $a_2 = a_4 = 0$.

It was shown in [Hoveijn & Verhulst, 1990] that the normal form (9) is nonintegrable for open sets of the parameters. The analysis is based on [Devaney, 1976] where one has to identify a complex unstable periodic solution. One has to identify a transversal homoclinic orbit with inward and outward spiraling orbits. The dynamics near the homoclinic fits into the theory of Šilnikov showing that in such a case, the dynamics contains locally a horseshoe map. The system induced by Hamiltonian (9) contains a complex unstable periodic solution associated with the τ_2 normal mode, two heteroclinic orbits and a continuous set of homoclinics. This complicates the application of Devaney's theory in [Devaney, 1976]. In [Hoveijn & Verhulst, 1990], this is solved by extending to the normal form $\overline{H}^4$; it is shown by precise numerics that the infinite set of homoclinics breaks up to produce a transversal homoclinic orbit. This implies the presence of Šilnikov dynamics corresponding to chaotic behavior. A Poincaré section in [Hoveijn & Verhulst, 1990] illustrates the dynamics.

It should be noted that a horseshoe map is a structurally stable phenomenon in the context of symplectic maps, so, the nonintegrability also extends to the original Hamiltonian.

Interestingly, when using Ziglin–Morales–Ramis theory and special solutions of the equations of motion, the nonintegrability of the normal form (9) for open sets of the parameters was shown in [Christov, 2012] by studying the dynamics of the normal form (9) without the need of H_4. On the other hand, the use of Šilnikov bifurcation provides knowledge of the dynamics.

- **The 1:2:4 resonance.**

The normal form is in action-angle coordinates τ, ψ to third order:

$$\begin{aligned}\overline{H}^3 &= \tau_1 + 2\tau_2 + 4\tau_3 \\ &\quad + 2[a_1\tau_1\sqrt{2\tau_2}\cos(2\psi_1 - \psi_2 - a_2) \\ &\quad + a_3\tau_2\sqrt{2\tau_3}\cos(2\psi_1 - \psi_3 - a_4)]. \end{aligned} \tag{10}$$

We have two combination angles and four parameters in the normal form. In the case of potential problems, we have $a_2 = a_4 = 0$.

Identifying particular solutions and their characteristics using Ziglin–Morales–Ramis theory, nonintegrability of normal form (10) was shown for open sets of the parameters in [Christov, 2012].

2.2. *Models and symmetries*

In mathematics, one usually aims at general results but in applications, one meets special cases. A special case that arises very often is the presence of symmetries. One can think of the symmetry of a swinging pendulum in the angle of deflection with respect to a symmetry-axis. Another case is the spherical symmetry induced by gravitational forces. Interesting examples are also found in systems of coupled oscillators as in the Fermi–Pasta–Ulam problems, see for instance [Fermi *et al.*, 1955; Verhulst, 1979; Rink & Verhulst, 2000]. So, examples that are nongeneric from a mathematical point of view are sometimes of utmost importance in applications.

We will consider the consequences of symmetry assumptions for 3 DoF Hamiltonian systems in 1:2:1 resonance leading to a possible degeneration of the normal forms. The implication is that in applications, we have to check the analysis of Sec. 2.1 for special cases.

3. A Symmetric 1:2:1 Resonance

Consider the Hamiltonian:

$$H(p,q) = H_2 + \varepsilon H_3 + \varepsilon^2 H_4, \tag{11}$$

for $b_2 \neq 0, b_9 \neq 0$:

$$H_2 = \frac{1}{2}(p_1^2 + q_1^2) + \frac{1}{2}(p_2^2 + 4q_2^2) + \frac{1}{2}(p_3^2 + q_3^2),$$

$$-H_3 = b_2 q_1^2 q_2 + b_9 q_3^2 q_2.$$

The Hamiltonian shows mirror-symmetry (discrete symmetry) in two of the DoF, q_1, q_3. The equations of motion are for $H_2 + \varepsilon H_3$:

$$\begin{cases} \ddot{q}_1 + q_1 = \varepsilon 2 b_2 q_1 q_2, \\ \ddot{q}_2 + 4q_2 = \varepsilon(b_2 q_1^2 + b_9 q_3^2), \\ \ddot{q}_3 + q_3 = \varepsilon 2 b_9 q_3 q_2. \end{cases} \tag{12}$$

It has been shown by Christov, see Appendix A, that system (12) is not integrable.

3.1. *The case $b_2 = b_9$*

If we have $b_2 = b_9$, then apart from the Hamiltonian, we have a second integral of motion:

$$q_3\dot{q}_1 - q_1\dot{q}_3 = C, \tag{13}$$

with C, a constant. The integral implies conservation of angular momentum between the first and third mode. In this case, we can reduce the system to 2 DoF, the normal form will be integrable. In the sequel, we will assume $b_2 \neq b_9$.

3.2. *Periodic solutions of a symmetric 1:2:1 resonance*

System (12) has a q_2 normal mode P_1, it is harmonic:

$$q_2(t) = e^{i2t}, \quad p_1 = q_1 = p_3 = q_3 = 0. \tag{14}$$

The normal mode can also be written as $q_2(t) = r_0 \cos(2t + \phi_0)$ with amplitude r_0 and phase ϕ_0 constant. We can apply averaging-normalization to system (12) by canonical transformations or by amplitude-phase averaging. The second approximation approach keeps the approximating system conservative and is mathematically equivalent to standard canonical normalization; see [Sanders *et al.*, 2007, Chapter 10] or [Verhulst, 2023]. We transform

$$\begin{cases} q_1(t) = r_1 \cos(t + \phi_1(t)), \\ \dot{q}_1(t) = -r(t) \sin(t + \phi_1(t)), \\ q_2(t) = r_2(t) \cos(2t + \phi_1(t)), \\ \dot{q}_2(t) = -2r_2(t) \sin(2t + \phi_1(t)), \\ q_3(t) = r_3(t) \cos(t + \phi_3(t)), \\ \dot{q}_3(t) = -r_3(t) \sin(t + \phi_3(t)). \end{cases} \tag{15}$$

Substituting in system (12) and assuming $q_2(t) \neq 0$, we find the following after averaging:

$$\begin{cases} \dot{r}_1 = -\dfrac{\varepsilon}{2} b_2 r_1 r_2 \sin(2\phi_1 - \phi_2), \\ \dot{\phi}_1 = -\dfrac{\varepsilon}{2} b_2 r_2 \cos(2\phi_1 - \phi_2), \\ \dot{r}_2 = \dfrac{\varepsilon}{8}(b_2 r_1^2 \sin(2\phi_1 - \phi_2) \\ \qquad + b_9 r_3^2 \sin(2\phi_3 - \phi_2)), \\ \dot{\phi}_2 = -\dfrac{\varepsilon}{8r_2}(b_2 r_1^2 \cos(2\phi_1 - \phi_2) \\ \qquad + b_9 r_3^2 \cos(2\phi_3 - \phi_2)), \\ \dot{r}_3 = -\dfrac{\varepsilon}{2} b_9 r_3 r_2 \sin(2\phi_3 - \phi_2), \\ \dot{\phi}_3 = -\dfrac{\varepsilon}{2} b_9 r_2 \cos(2\phi_3 - \phi_2). \end{cases} \tag{16}$$

Amplitude-phase averaging has to be handled with care at the zeros of the amplitudes. For r_1, r_3, the singularity is removed in system (16), we have to exclude $r_2 = 0$.

System (16) has the first integral:

$$\frac{1}{2}(r_1^2 + 4r_2^2 + r_3^2) = E_0, \tag{17}$$

with E_0, a constant. The second independent integral of the normalized system (16) is $\overline{H}^3$:

$$\overline{H}^3 = b_2 r_1^2 r_2 \cos(2\phi_1 - \phi_2) + b_9 r_3^2 r_2 \cos(2\phi_3 - \phi_2). \tag{18}$$

With reference to [Duistermaat, 1984], we present the second integral of system (16) also in standard canonical coordinates:

$$\begin{aligned} \overline{H}^3 = q_2[b_2(q_1^2 - p_1^2) + b_9(q_3^2 - p_3^2)] \\ + p_2(b_2 q_1 p_1 + b_9 q_3 p_3). \end{aligned} \tag{19}$$

Note that in [Duistermaat, 1984], the roles of q_2 and q_3 have been exchanged. It is remarked in [Duistermaat, 1984] that the submanifold $\overline{H}^3 = 0$ contains a set of periodic solutions. However, on the reduced Hamiltonian, they have nodal eigenvalues, so they are not useful to apply the [Devaney, 1976] scenario for nonintegrability. We will return to the integrability problem.

The normal mode $q_2(t) = r_0 \cos(2t + \phi_0)$ is unstable. This can be proved by perturbing $q_2(t) = r_0 \cos(2t + \phi_0) + u$, inserting in system (12) and linearizing at $q_1 = q_3 = 0$. The linearized system is

$$\begin{aligned} &\ddot{q}_1 + q_1 = \varepsilon 2 b_2 r_0 \cos(2t + \phi_0) q_1, \quad \ddot{u} + 4u = 0, \\ &\ddot{q}_3 + q_3 = \varepsilon 2 b_9 r_0 \cos(2t + \phi_0) q_3, \end{aligned}$$

with for q_1, q_3 the well-known Mathieu equation with prominent instability Floquet-tongue. We find near the normal mode nodal instability (two positive and two negative real eigenvalues of the reduced Hamiltonian at the normal mode).

System (16) contains at least two invariant manifolds: M_1 given by $q_3(t) = 0, t \geq 0$ and M_2 given by $q_1(t) = 0, t \geq 0$; M_1 and M_2 show 2 DoF dynamics that can be analyzed by averaging. We find in M_1 the unstable normal q_2-mode from $q_1 = 0$. Putting

$$\chi_1 = 2\phi_1 - \phi_2$$

we note that the amplitude is constant if $\chi_1 = 0, \pi$. The equation for χ_1 is

$$\frac{d\chi_1}{dt} = \varepsilon b_2 \left(-r_2 + \frac{r_1^2}{8r_2}\right) \cos \chi_1. \tag{20}$$

$\chi_1(t)$ is stationary if $r_1^2 = 8r_2^2$. From the integral (17), we find for the periodic solutions indicated by P_2, P_3:

$$r_1^2 = \frac{4}{3}E_0, \quad r_2^2 = \frac{1}{6}E_0, \quad r_3 = 0. \tag{21}$$

Similar results are found for manifold M_2 with periodic solutions P_4, P_5:

$$r_3^2 = \frac{4}{3}E_0, \quad r_2^2 = \frac{1}{6}E_0, \quad r_1 = 0. \tag{22}$$

In M_1 and M_2, we find apart from the normal mode two periodic solutions that are stable within the manifolds, altogether five periodic solutions. The 2 DoF stability follows from the well-known analysis of the 1:2 resonance, see, for instance [Sanders *et al.*, 2007, Chapter 10]. The stability analysis in M_1 (and in M_2) takes place by considering the normalized (averaged) system on the energy manifold and identifying periodic solution with the second integrals of motion as critical points. The critical points have for 2 DoF systems generically two real or two purely imaginary eigenvalues, in this case purely imaginary.

Are the periodic solutions given by (21) and (22) stable or unstable in the full 3 DoF system?

Consider the periodic solutions in M_1. It is no restriction of generality to put $\phi_1(0) = 0$, so we have for the two periodic solutions from system (12):

$$\begin{aligned} q_1(t) &= 2\sqrt{\frac{E_0}{3}} \cos t, \\ q_2(t) &= \pm\sqrt{\frac{E_0}{6}} \cos 2t. \end{aligned} \tag{23}$$

Putting

$$\begin{aligned} q_1 &= 2\sqrt{\frac{E_0}{3}} \cos t + u_1, \\ q_2 &= \pm\sqrt{\frac{E_0}{6}} \cos 2t + u_2, \\ q_3 &= u_3, \end{aligned}$$

substituting in system (12) and linearizing, we find for u_3:

$$\ddot{u}_3 + u_3 = \pm\varepsilon 2 b_9 u_3 \sqrt{\frac{E_0}{6}} \cos 2t.$$

So, we have for u_3 after linearization the Mathieu equation with parametric forcing in the main Floquet instability tongue. The periodic solutions (23) are unstable in 3 DoF. With an analogous reasoning, we have instability for the two periodic solutions with $q_2 > 0$ in M_2. Solutions that are close to P_2, P_3 with $q_3(0)$ small will leave a neighborhood of M_1 rotating around an unstable invariant manifold that serves as guiding center. A similar observation holds for P_4, P_5.

3.3. *Phase-locked invariant manifolds*

Special solutions of system (16) are obtained by choosing

$$\cos(2\phi_1 - \phi_2) = 0, \quad \cos(2\phi_3 - \phi_2) = 0, \tag{24}$$

resulting in four possible choices: $2\phi_1 - \phi_2 = \pi/2$, $3\pi/2$, $2\phi_3 - \phi_2 = \pi/2$, $3\pi/2$.

For the amplitudes, we have the equations

$$\begin{aligned} \dot{r}_1 &= \mp\frac{\varepsilon}{2} b_2 r_1 r_2, \\ \dot{r}_2 &= \pm\frac{\varepsilon}{8} b_2 r_1^2 \pm \frac{\varepsilon}{8} b_9 r_3^2, \\ \dot{r}_3 &= \mp\frac{\varepsilon}{2} b_9 r_3 r_2, \end{aligned} \tag{25}$$

that can be integrated.

3.4. *The complexity of nonintegrability*

We will use the recurrence theorem for system (12) to demonstrate complexity, see [Verhulst, 2016; Verhulst, 2023, Chapter 8.3]. We start near to the normal mode $q_2(t)$ and will use the Euclidean distance $d(t)$ to indicate the distance from the initial point in phase-space. For a Hamiltonian system with energy $O(1)$, we have by requiring a distance d_0 an upper limit L for the recurrence time T_r:

$$T_r \leq O\left(\frac{1}{d_0^{2n-1}}\right), \tag{26}$$

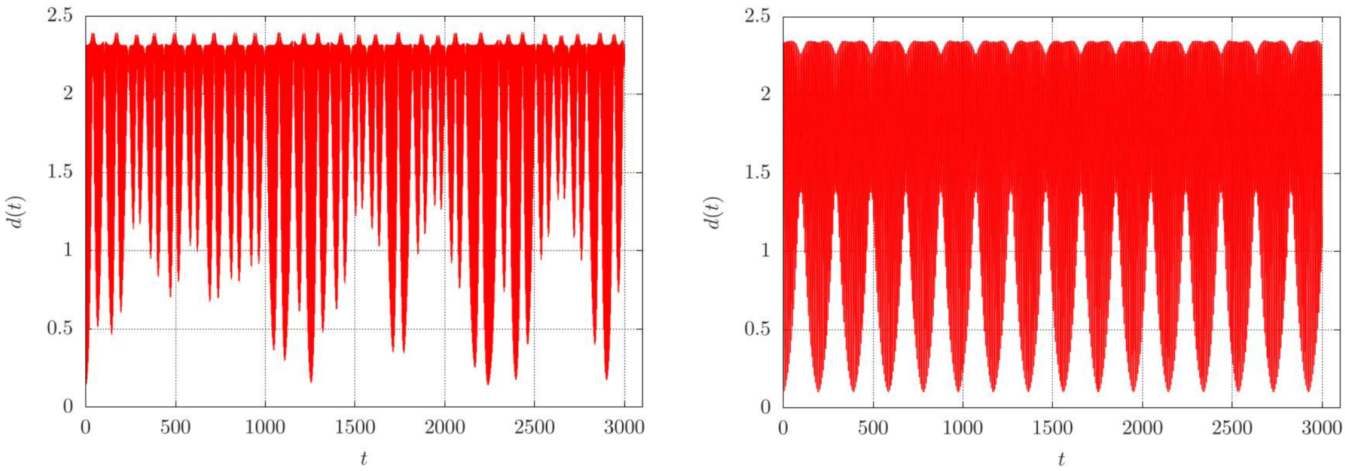

Fig. 1. Recurrence indicated by Euclidean distance $d(t)$ for system (12) starting near the normal mode $q_2(t)$ (left figure) with $q_1(0) = 0.1$, $q_2(0) = 1$, $q_3(0) = -0.1$, $b_2 = 1$, $b_9 = -2$ with initial velocities zero. We choose $\varepsilon = 0.1$ and $d_0 = 0.1$. We need 2390 time-steps for recurrence. On the right is the case near the periodic solution P_2 in M_1 starting at $q_1(0) = 2/\sqrt{3}$, $q_2(0) = 1/\sqrt{6}$, $q_3(0) = 0.1$ and initial velocities zero.

with n the number of DoF, see [Verhulst, 2016]. To the left of Fig. 1, we have $d(t)$ for system (12) starting near the q_2 normal mode. Choosing $d_0 = 0.1$, we have an upper limit for 3 DoF of 10^5 time-steps. The recurrence time depends on the presence of periodic solutions, tori and the repeated passage through resonance zones. For this case, we have recurrence at 2390 time-steps.

To the right of Fig. 2, we start near a periodic solution P_2 in M_1 with $r_1(0) = 2/\sqrt{3}, r_2(0) = 1/\sqrt{6}, r_3(0) = 0.1$. The solution is stable in M_1 but unstable by coupling to the q_3-mode. The recurrence is much stronger as we have stability with respect to the q_1, q_2 modes.

Numerical explorations show that there are many other periodic solutions in the region near P_2, see also Appendix B. We give an example in Fig. 2, where we show Poincaré sections near a periodic orbit with nearby chaotic behavior.

We will turn now to a periodic solution in general position using again Poincaré maps, see Fig. 3. The eigenvalues of the fixed point of the Poincaré map are $[0.8923391 + 0.451365622i,\ 0.8923391 - 0.451365622i,\ 1.03766315,\ 1., 0.96370388]$, so, two complex and one positive. We computed the 1D unstable manifold of the Poincaré map corresponding with an unstable manifold. This orbit follows oscillatingly a homoclinic solution as the guiding

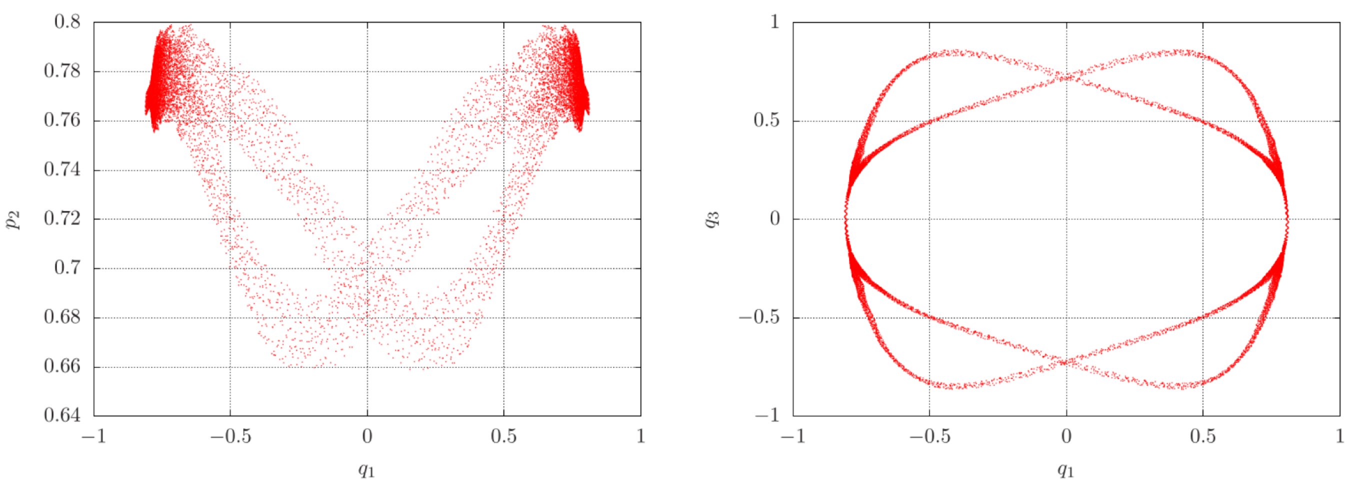

Fig. 2. Poincaré section for fixed energy at $q_2 = 0$ for system (12) near periodic solution P_2 in M_1. The section is four-dimensional, we show two projections with dots indicating chaos. We have chosen $b_2 = 1, b_9 = -1.1756926$, $\varepsilon = 0.1$ with $q_1(0) = 2/\sqrt{3}$, $q_2(0) = 1/\sqrt{6}$, $q_3(0) = 0.011$ and initial velocities zero.

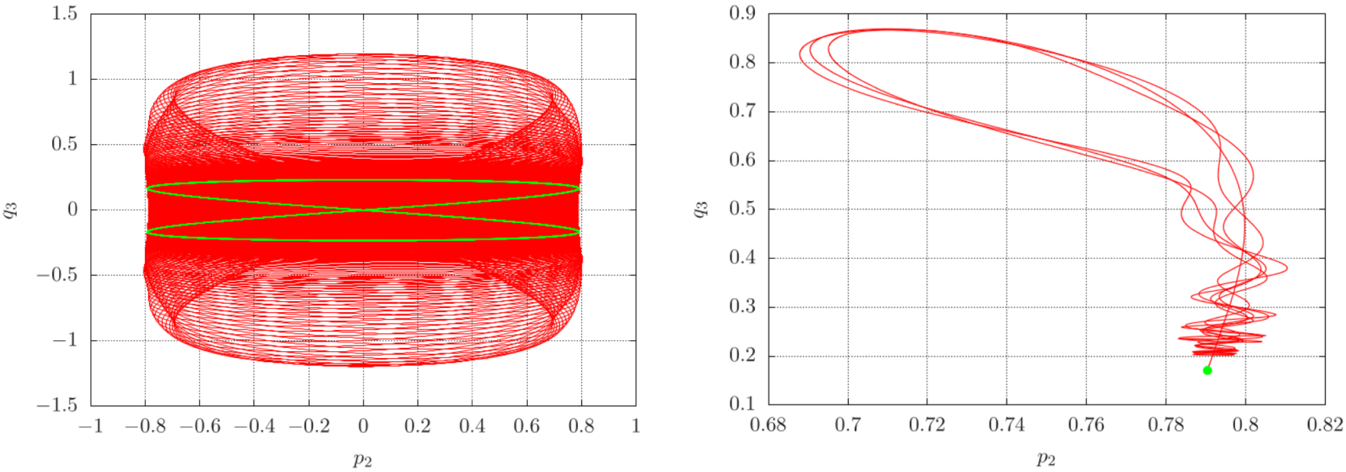

Fig. 3. Poincaré section for fixed energy at $q_2 = 0$ for system (12) near a numerically detected saddle periodic solution in general position with a one-dimensional unstable manifold. We have chosen $b_2 = 1$, $b_9 = -1.1756926$, $\varepsilon = 0.1$ with $q_1[0] = -0.81151$; $p_1[0] = -0.8091$; $q_2[0] = 0$; $p_2[0] = 0.79038$; $q_3[0] = 0.17045$; $p_3[0] = -0.15606$. On the left, we show a projection of the periodic orbit on the p_2, q_3 plane (closed green curve). Starting near the saddle in the direction of the unstable eigenvector $v = [-0.11750252, -0.18952268, 0, -0.03178473, -0.7913451, 0.56836342]$ yields the red curve as orbit. On the right, we numerically computed the 1D unstable manifold of the saddle fixed point of the Poincaré map corresponding with the unstable cycle and show Ŝilnikov-like behavior. The solution on the unstable manifold follows oscillatingly a homoclinic solution as the guiding center.

center in Fig. 3 with an interval of nearly 250 timesteps. This is the right scenario for Ŝilnikov bifurcation leading to chaos; it remains a conjecture as one has still to show that the orbit is a transversal homoclinic.

3.5. *The manifold* $\overline{H}^3 = 0$

We return to the observation that periodic solutions exist on the manifold $\overline{H}^3 = 0$. From the expression (18), we have that $\overline{H}^3 = 0$ if r_2 (or q_2) vanishes and if $r_1 = r_3 = 0$, the case of the q_2 normal mode (14) that we discussed earlier. Until now, we have not discussed the dynamics for $q_2 = 0$ and solutions that are ε-close to $q_2 = 0$. To avoid singularities in the variational system, see (16), we use for q_2 the co-moving transformation:

$$\begin{cases} q_1(t) = r_1 \cos(t + \phi_1(t)), \\ \dot{q}_1(t) = -r(t)\sin(t + \phi_1(t)), \\ q_2(t) = y_1 \cos 2t + \dfrac{1}{2} y_2 \sin 2t, \\ \dot{q}_2(t) = -y_1 2 \sin 2t + y_2 \cos 2t, \\ q_3(t) = r_3(t)\cos(t + \phi_3(t)), \\ \dot{q}_3(t) = -r_3(t)\sin(t + \phi_3(t)). \end{cases} \tag{27}$$

Substitution in system (12) and averaging over t produce

$$\begin{cases} \dot{r}_1 = -\varepsilon \dfrac{b_2}{2} r_1 \left(y_1 \sin 2\phi_1 + \dfrac{1}{2} y_2 \cos 2\phi_1 \right), \\ \dot{\phi}_1 = -\varepsilon \dfrac{b_2}{2} \left(y_1 \cos 2\phi_1 - \dfrac{1}{2} y_2 \sin 2\phi_1 \right), \\ \dot{y}_1 = \dfrac{\varepsilon}{8} (b_2 r_1^2 \sin 2\phi_1 + b_9 r_3^2 \sin 2\phi_3), \\ \dot{y}_2 = \dfrac{\varepsilon}{4} (b_2 r_1^2 \cos 2\phi_1 + b_9 r_3^2 \cos 2\phi_3), \\ \dot{r}_3 = -\varepsilon \dfrac{b_9}{2} r_3 \left(y_1 \sin 2\phi_3 + \dfrac{1}{2} y_2 \cos 2\phi_3 \right), \\ \dot{\phi}_3 = -\varepsilon \dfrac{b_9}{2} \left(y_1 \cos 2\phi_3 - \dfrac{1}{2} y_2 \sin 2\phi_3 \right). \end{cases} \tag{28}$$

To find periodic and other special solutions, we will look for zeros on the right-hand sides of system (28). Solutions of system (28) with $y_1(t) = y_2(t) = 0$ can have stationary amplitudes r_1, r_3 and phases ϕ_1, ϕ_3. Nontrivial solutions for r_1, r_3 arise from the equations for y_1, y_2 if

$$\sin 2(\phi_1 - \phi_3) = 0, \tag{29}$$

or $\phi_1 - \phi_3 = 0, \pi/2$. If $\phi_1 - \phi_3 = 0$, we require $b_2/b_9 < 0$, if $\phi_1 - \phi_3 = \pi/2$, we should have

$b_2/b_9 > 0$, so, we have families of solutions satisfying $r_1^2+r_3^2 = 2E_0$, $q_2(t) = 0$, $t \geq 0$ and E_0 a positive fixed constant.

We find two families of special solutions, in a different formulation, this result was noted already in [Duistermaat, 1984].

We linearize near the critical points of system (28); we find a 6×6 matrix with two eigenvalues as zero, as the periodic solutions are families parametrized by the energy. Consider the case $\phi_1 = \phi_3 = 0$, $b_2 = 1$, $b_9 = -1$, $r_1 = r_3 = 1$. The eigenvalues are $\pm 0.5\,\varepsilon i, \pm 0.5\,\varepsilon i$, 0, 0. On choosing $\phi_1 = \pi/2$, $\phi_2 = 0$, $b_2 = b_9 = 1$, $r_1 = r_2 = 1$, we find the same eigenvalues, a Hamiltonian Hopf or Krein bifurcation.

These first-order results point at neutral stability of the special solutions. We have to obtain second-order averaging-normalization to discuss the possibility of a break-up of the families. For second order, we find for the ε^2 terms:

$$\begin{cases} \dot{r}_1 : -\dfrac{1}{64} b_2 b_9 \varepsilon^2 r_1 r_3^2 \sin[2(\phi_1 - \phi_3)], \\ \dot{\phi}_1 : -\dfrac{1}{64} b_2 \varepsilon^2 (8 b_9 r_3^2 + 6 b_1 (4 y_1^2 + y_2^2) + b_2 (9 r_1^2 + 4 y_1^2 + y_2^2) + b_9 r_3^2 \cos[2(\phi_1 - \phi_3)]), \\ \dot{y}_1 : -\dfrac{1}{256} \varepsilon^2 y_2 (24 b_1 (b_2 r_1^2 + b_9 r_3^2) + 4 (b_2^2 r_1^2 + b_9^2 r_3^2) + 15 b_1^2 (4 y_1^2 + y_2^2)), \\ \dot{y}_2 : \dfrac{1}{64} \varepsilon^2 y_1 (24 b_1 (b_2 r_1^2 + b_9 r_3^2) + 4 (b_2^2 r_1^2 + b_9^2 r_3^2) + 15 b_1^2 (4 y_1^2 + y_2^2)), \\ \dot{r}_3 : \dfrac{1}{64} b_2 b_9 \varepsilon^2 r_1^2 r_3 \sin[2(\phi_1 - \phi_3)], \\ \dot{\phi}_3 : -\dfrac{1}{64} b_9 \varepsilon^2 (8 b_2 r_1^2 + 9 b_9 r_3^2 + 24 b_1 y_1^2 + 4 b_9 y_1^2 + (6 b_1 + b_9) y_2^2 + b_2 r_1^2 \cos[2(\phi_1 - \phi_3)]). \end{cases} \tag{30}$$

We find

$$\begin{aligned} \frac{d}{dt}(\phi_1(t) - \phi_2(t)) &= -\frac{1}{64}\varepsilon^2 [8 b_2 b_9 (r_3^2 - r_1^2) + 9 (b_2^2 r_1^2 - b_9^2 r_3^2) \\ &\quad + b_2 b_9 (r_3^2 - r_1^2) \cos 2(\phi_1 - \phi_3)]. \end{aligned} \tag{31}$$

Using that the right-hand side of Eq. (31) vanishes, $\cos 2(\phi_1 - \phi_3) = \pm 1$ and from the integral (17) $r_1^2 + r_3^2 = 2E_0$ we find $r_1 = r_3$.

We have no evidence for change of stability.

4. The 3 DoF Symmetric 1:2:1 Resonance

In a number of applications, we have discrete symmetry in 3 DoF. In this case, we have $b_1 = b_2 = b_9 = 0$ or H_3 vanishes, which, in fact is in an expansion of the Hamiltonian, and all H_m terms with m odd vanish. We put

$$-H_4 = \frac{1}{4} c_1 q_1^4 + \frac{1}{4} c_2 q_2^4 + \frac{1}{4} c_3 q_3^4 + \frac{1}{2} c_5 q_1^2 q_3^2. \tag{32}$$

We have for H_4 (32) only the quartic terms ($O(\varepsilon^2)$ terms), so, the singularities of amplitude-phase transformation (15) vanish. Averaging produces:

$$\begin{cases} \dot{r}_1 = -\varepsilon^2 \dfrac{1}{8} c_5 r_1 r_3^2 \sin 2(\phi_1 - \phi_3), \\ \dot{\phi}_1 = -\varepsilon^2 \left[\dfrac{3}{8} c_1 r_1^2 + \dfrac{1}{4} c_5 r_3^2 \times \left(1 + \dfrac{1}{2}\cos 2(\phi_1 - \phi_3)\right)\right], \\ \dot{r}_2 = 0, \\ \dot{\phi}_2 = -\varepsilon^2 \dfrac{3}{16} c_2 r_2^2, \\ \dot{r}_3 = -\varepsilon^2 \dfrac{1}{8} c_5 r_1^2 r_3 \sin 2(\phi_1 - \phi_3), \\ \dot{\phi}_3 = -\varepsilon^2 \left[\dfrac{3}{8} c_3 r_3^2 + \dfrac{1}{4} c_5 r_1^2 \times \left(1 + \dfrac{1}{2}\cos 2(\phi_1 - \phi_3)\right)\right]. \end{cases} \tag{33}$$

The 1:1 resonance between the q_1 and q_3 modes plays a part with periodic solutions found from the zeros of the right-hand sides of system (33). The periodic solutions are located in so-called resonance manifolds. We have the condition $\sin 2(\phi_1 - \phi_3) = 0$ and require for the combination angle $\chi_1 = \phi_1 - \phi_3$:

$$\frac{d\chi_1}{dt} = -\frac{\varepsilon^2}{4}\left(\frac{3}{2}(c_1 r_1^2 - c_3 r_3^2) + c_5(r_3^2 - r_1^2)\right.$$
$$\left. \times \left(1 + \frac{1}{2}\cos 2\chi_1\right)\right), \tag{34}$$

with $\cos 2\chi_1 = \pm 1$. We conclude that $H_2 + \overline{H}_4$ is integrable with integrals:

$$H_2 = E_0, \quad \overline{H}_4 = \overline{H}_4(p_0, q_0), \quad r_2(t) = r_2(0). \tag{35}$$

An extended analysis would include the H_6 terms, or, in the sense of approximation theory, $O(\varepsilon^4)$ terms. This kind of analysis involves interaction between low- and higher-order resonances, an approach has been formulated in [Verhulst, 2023, Chapter 8]. In general, we expect from H_6 new phenomena induced by terms $q_1^4 q_2^2$, $q_3^4 q_2^2$ in the resonance zones arising from the 1:1 resonances induced by H_4. We demonstrate this for the simplified but typical case:

$$-H_4 = \frac{1}{2}c_5 q_1^2 q_3^2, \quad -H_6 = \frac{1}{4}d_1 q_1^4 q_2^2 + \frac{i}{4}d_2 q_2^2 q_3^4. \tag{36}$$

According to averaging-normalization with $c_1 = c_2 = c_3 = 0$, system (33) produces the presence of the 1:1 resonance. We have for the combination angle $\chi_1 = \phi_1 - \phi_3$:

$$\dot{\chi_1} = -\varepsilon^2\left[\frac{3}{8}(c_1 r_1^2 - c_3 r_2^2) + \frac{1}{4}c_5(r_3^2 - r_1^2)\right.$$
$$\left. \times \left(1 + \frac{1}{2}\cos 2(\phi_1 - \phi_3)\right)\right]. \tag{37}$$

The amplitudes r_1, r_3 satisfying Eq. (37) will be constant with $t \geq 0$, $\sin 2(\phi_1 - \phi_3) = 0$. The corresponding solutions of system (33) will be periodic. For a fixed value of the energy E_0, there are two periodic solutions located in two resonance zones. The zones are small neighborhoods of the resonance manifolds determined by Eq. (37) and $\phi_1 = \phi_3$ or $\phi_1 - \phi_3 = \pi/2$. The solutions of system (33) approximate the solutions of the Hamiltonian system [Eq. (32)] with precision $O(\varepsilon^2)$ on the timescale $1/\varepsilon^2$.

In the resonance zones where the right-hand sides of system (33) vanish, the $O(\varepsilon^4)$ terms may be of importance. The equations of motion induced by Hamiltonians (36) are

$$\begin{cases} \ddot{q}_1 + q_1 = \varepsilon^2 c_5 q_1 q_3^2 + \varepsilon^4 d_1 q_1^3 q_2^2, \\ \ddot{q}_2 + 4q_2 = \varepsilon^4 \frac{1}{2} d_1 q_1^4 q_2 + \varepsilon^4 \frac{1}{2} d_2 q_2 q_3^4, \\ \ddot{q}_3 + q_3 = \varepsilon^2 c_5 q_1^2 q_3 + \varepsilon^4 d_2 q_2^2 q_3^3. \end{cases} \tag{38}$$

System (38) has three manifolds governed by 2 DoF, $q_1(t) = 0$, $q_2(t) = 0$, $q_3(t) = 0$, $t \geq 0$. The three normal modes exist as exact solutions, in addition, we have the 2:4 resonances in the q_1, q_2 and the q_2, q_3 manifolds. If $c_5 = 0$, we have after averaging for the H_6 terms

$$\begin{cases} \dot{r}_1 : -\varepsilon^4 \frac{1}{32} d_1 r_1^3 r_2^2 \sin(4\phi_1 - 2\phi_2), \\ \dot{\phi}_1 : -\varepsilon^4 \frac{1}{32} d_1 r_1^2 r_2^2 \cos(4\phi_1 - 2\phi_2), \\ \dot{r}_2 : \varepsilon^4 \frac{1}{128}(d_1 r_1^4 r_2 \sin(4\phi_1 - 2\phi_2) \\ \qquad + d_2 r_2 r_3^4 \sin(4\phi_1 - 2\phi_2)), \\ \dot{\phi}_2 : -\varepsilon^4 \frac{1}{64}\left[d_1 r_1^4 \left(3 + \frac{1}{2}\cos(4\phi_1 - 2\phi_2)\right)\right. \\ \qquad \left. + d_2 r_3^4 \left(3 + \frac{1}{2}\cos(4\phi_3 - 2\phi_2)\right)\right], \\ \dot{r}_3 : -\varepsilon^4 \frac{1}{32} d_2 r_3^3 r_2^2 \sin(4\phi_3 - 2\phi_2), \\ \dot{\phi}_3 : -\varepsilon^4 \frac{1}{32} d_2 r_3^2 r_2^2 \cos(4\phi_3 - 2\phi_2). \end{cases} \tag{39}$$

The 2:4 resonance produces periodic solutions with different frequencies. The solutions are obtained slightly differently as $q_2(t) = 0, t \geq 0$ satisfies the averaged H_6 terms with arbitrary initial conditions for the q_1, q_3 modes.

5. Discussion and Conclusions

In general, Hamiltonian systems with two or more DoF are nonintegrable. This is an important statement but the problem is the condition "in general". Models in the natural sciences are never "general" but have specific assumptions, in particular the assumptions of certain symmetries. There is a second important issue. If a Hamiltonian is nonintegrable, this can be a major dynamics affair or a small-scale localized phenomenon. The theory of averaging and normal forms clarifies this issue quantitatively. A third important issue arises when we ask ourselves what the nature of nonintegrability is, and what the dynamical consequences are.

In this paper, we reviewed the methods of demonstrating nonintegrability and focus on 3 DoF systems at the first-order resonances near stable equilibrium. The 1:2:1 resonance is one of the complicated systems, which got attention in the literature. We show that if the system has discrete symmetry in the first and third DoF, we have nonintegrability characterized by Šilnikov bifurcation. Apart from the short-periodic families, we find many other periodic solutions. If the 1:2:1 resonance has discrete symmetry in the 3 DoF, the normal form is integrable. In this case, nonintegrability of the original system, if present, will be dynamically a small-scale affair.

Conflicts of Interest

The authors have no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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The numerical experiments were carried out with Matcont under Matlab with ode 78 and independently with ode 78 under C with a different computer. Dr. Ognyan Christov provided the proof in Appendix A and commented on the paper.

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Appendices

Appendix A

The 1:2:1 Nonintegrability

Consider Hamiltonian (11) with H_4 omitted. The nonintegrability of the symmetric 1:2:1 resonance with equations of motion (12) was proved by [Christov, 2023] assuming $b_2 \neq 0, b_9 \neq 0$. The proof of nonintegrability of system (12) runs as follows.

As shown in Sec. 3, a solution of the equations of motion (12) is the q_2 normal mode:

$$\begin{aligned} &q_2(t) = e^{i2t}, \quad p_2 = \dot{q}_2, \\ &p_1(t) = q_1(t) = p_3(t) = q_3(t) = 0, \quad t \geq 0. \end{aligned} \tag{A.1}$$

Putting $q_1 = \zeta, q_2 = \zeta_2 + e^{i2t}, q_3 = \zeta$, the variational equations along the normal mode (A.1) are

$$\begin{cases} \ddot{\zeta}_1 + (1 - 2b_2 e^{i2t})\zeta_1 = 0, \\ \ddot{\zeta}_2 + 4\zeta_2 = 0, \\ \ddot{\zeta}_3 + (1 - 2b_9 e^{i2t})\zeta_3 = 0. \end{cases} \tag{A.2}$$

The equations for ζ_1, ζ_3 are of the form

$$\ddot{\zeta} + (1 - 2be^{i2t})\zeta = 0, \quad b \neq 0. \tag{A.3}$$

Changing the independent variable by

$$\tau = e^{i2t}, \tag{A.4}$$

and indicating differentiation with respect to τ by accent $'$ we find

$$\zeta'' + \frac{1}{\tau}\zeta' + \frac{2b\tau - 1}{4\tau^2}\zeta = 0. \tag{A.5}$$

Equation (A.5) is a complex valued equation with complex time. After transforming $\zeta = z/\tau^{1/2}$, we have the standard form:

$$\ddot{z} + \frac{b}{2\tau}z = 0. \tag{A.6}$$

The double confluent Heun equation is of the form

$$\begin{aligned} &\ddot{Z} + r(\tau)Z = 0, \\ &r(\tau) = \frac{\alpha^2}{4} - \frac{\gamma}{\tau} - \frac{\delta}{\tau^2} - \frac{\beta}{\tau^3} + \frac{\alpha^2}{4\tau^4}, \end{aligned} \tag{A.7}$$

see [Duval & Loday-Richaud, 1992, pp. 236–237]. In the case of Eq. (A.6), we have

$$\alpha = \beta = \delta = 0, \quad \gamma = -\frac{b}{2}.$$

The analysis of Eq. (A.7) is based on the Kovacic algorithm in [Duval & Loday-Richaud, 1992]. It says that for the double confluent Heun equation, there are two cases for the differential Galois group to be solvable:

(1) $\alpha = \beta = \gamma = 0$ which implies $b = 0$.
(2) $\alpha = \beta\gamma = 0$ and there exists $m \in \mathbb{Z}$ such that $\delta = \frac{(3+2m)(1-2m)}{16}$.

In our case, $b \neq 0$ and there is no $m \in \mathbb{Z}$ to match $\delta = 0$, hence Eq. (A.6) does not admit Liouvillian solutions. In fact, the solutions of Eq. (A.6) can be expressed as Bessel functions. The implication is that the Galois group is $SL(2, \mathbb{C})$, obviously noncommunitative. Extending this to Eq. (A.5) and using the Morales–Ramis theorem, see [Morales-Ruiz & Ramis, 2010] for a survey, we conclude that the Hamiltonian system (12) is nonintegrable.

Appendix B

Numerically Detected Periodic Orbits of System (12) in the Case $b_2 \neq b_9$

One of the most straightforward ways to numerically detect periodic orbits of a continuous dynamical system is to compute the fixed points of its corresponding Poincaré return map $P(x)$ and its powers $P^n(x), n \in \mathbb{N}_{>0}$. In this setting, we used the following parameters: $\varepsilon = 0.1$, $b_1 = 0$, $b_9 = -1.1756926$ and $b_2 = 1$. The Poincaré section in this experiment is the five-dimensional vector space $q_2 = 0$. Starting with initial values of q_1, p_1, p_2 varying between 0 and 1 with stepsize equal to 0.01 and using the discrete symmetry of system (12), many periodic orbits were detected that are spatially distributed in a nonrandom way. The stability of the computed cycles was determined as well. See Fig. 4.

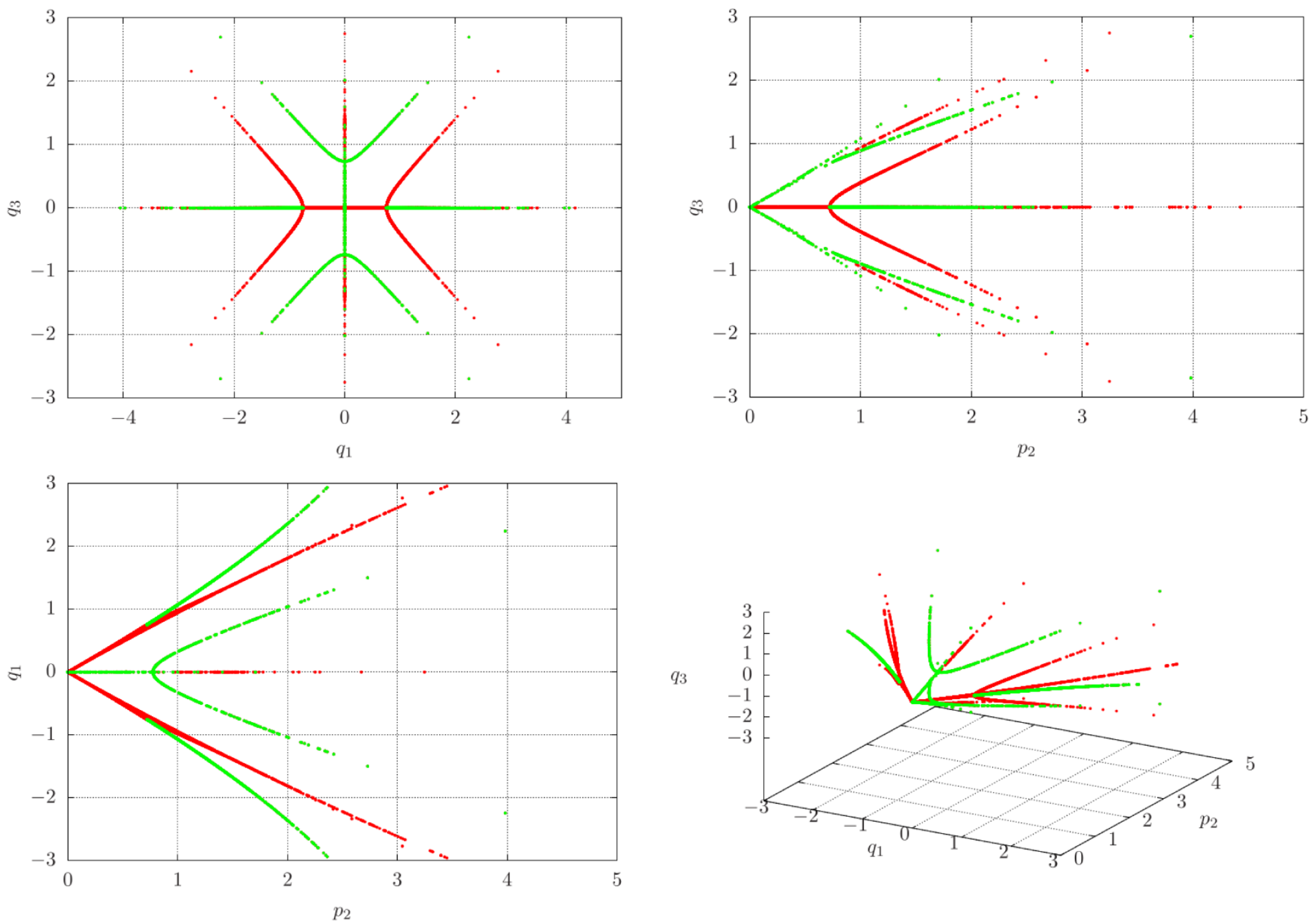

Fig. 4. Numerically computed fixed points obtained by continuation of the Poincaré return map (with cross-section $\Sigma : q_2 = 0$) of system (12) with different projections. Each dot represents a periodic solution. Green dots correspond with (Lyapunov) stable cycles. The red dots represent unstable periodic orbits.