A note on the Kaplan-Yorke dimension

Taoufik Bakri[1] and Ferdinand Verhulst *[2]

[1]TNO Sustainable Urban Mobility & safety, PO Box 96800, 2509 JE The Hague, The Netherlands.
[1]taoufik.bakri@tno.nl
[2] Mathematisch Instituut, PO Box 80.010, 508TA Utrecht, Netherlands
[2]f.verhulst@uu.nl

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Abstract

The computation of the Kaplan-Yorke dimension is a convenient tool for studying strange attractors. We will consider the uncertainties of using Lyapunov-exponents and will suggest an improvement. Illustrations are computed for the systems NE9 and Lorenz attractor.

Key words: Kaplan-Yorke dimension, fractal, chaos, NE9, Lorenz attractor

1 Introduction

In nonlinear dynamics multifrequency oscillations arise in many applications of various disciplines. The mathematical equations describing these oscillations contain parameters that upon changing can display many different types of bifurcations; see for instance [Kuznetsov (2023)].
A remarkable phenomenon is the emergence of chaotic motion that is often characterised by geometric structures. There are differences between conservative and dissipative systems, we will focus here on the latter. One has identified various scenarios leading to chaos with as a prominent one period-doubling. In this case one starts with a periodic solution for certain parameter values. Changing the parameters leads to a periodic solution with double period, changing again redoubles the period and so on ad infinitum. In the limit of parameter changes one has a bifurcation sequence of parameters producing chaotic motion.

*Corresponding author

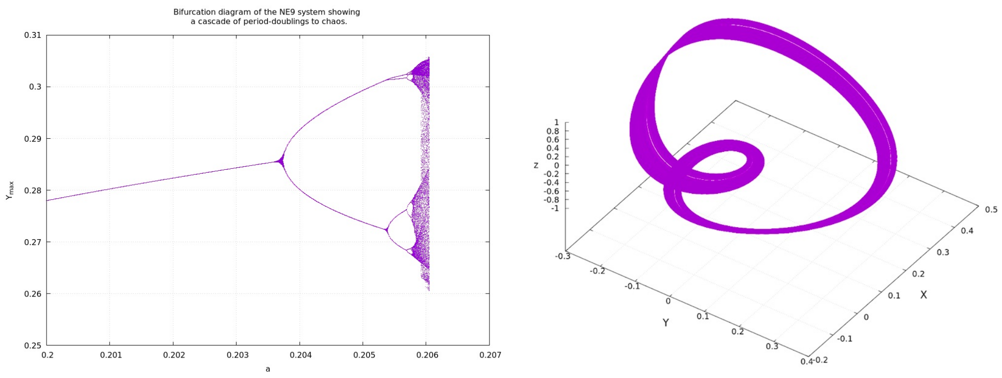

Figure 1: Left bifurcation sequence showing period doublings in system NE9. Right the 3-dimensional chaotic attractor. Courtesy CHAOS 32, [Bakri. & Verhulst[2022)].

We use an illiustration of system NE9 from [Bakri. & Verhulst[2022)]. In fig. 1 we show left the period doubling sequence and right the resulting chaotic attractor.
This geometric object has usually a fractal structure with a dimension that is not a natural number. For an introduction to fractals with both geometric and dynamical aspects see [Verhulst (2000)] sections 14.7-9. Analysis of a number of cases shows that it helps if one can identify a map that shows more details of the dynamics. In the case of system NE9 in fig. 1 this would be a Poincaré map of a transversal of the flow into itself. Such a map shows a Cantor set of returning points in the 2-dimensional transversal. An introduction to qualitative and quantitative aspects of chaos in dynamical systems is [Broer & Takens (2011)], containing many examples and a description of various types of dimensions and entropy.
One can study the geometric aspects of the chaotic attractor by the concepts of limit capacity and Hausdorff dimension; they represent a purely geometric concept of dimension. A dimension characterising both the geometry of the attractor and the dynamics is the correlation dimension. For applications another tool is to compute Lyapunov-exponents and the Kaplan-Yorke dimension. This quantity, indicated by D_{KY}, was conjectured as a dynamical dimension in [Kaplan & Yorke (1979)]. The Lyapunov-exponent generalises the idea of eigenvalue to show the expansion and contraction near a manifold. To explain the procedure we consider a 1-dimensional map $f : X \to X$ generated by a dynamical system. The Lyapunov-exponent $\mu(x_0)$ of a point $x_0 \in X$ is

$$\mu(x_0) = \lim_{n\to\infty} \frac{\ln |df^n(x_0)|}{n},$$

with $df^n(x_0)$ denotes the derivative of the n times iterated map at x_0 and the condition that the limit exists.
We can generalise this to n-dimensional maps to obtain n Lyapunov exponents. For introductions see [Guckenheimer & Holmes (1996)] or [Verhulst (2000)]. However, at this point we immediately have a problem. For a smooth manifold we can obtain tangent maps, we expect the limit to exist. For a manifold with a fractal dimension the limit will not

exist or if it does, it will critically depend on the choice of the initial points. What we can expect in general and at best is a range of Lyapunov exponents with certain uncertainties dependent on the initial values.

In [Kuznetsov *et al.* (2019)], [Stankevich *et al* (2020)] and [Bakri & Verhulst (2025)] the Kaplan-Yorke dimension is computed for the construction of charts of Lyapunov-exponents for interacting systems depending on parameters. Important analytic tools for such calculations are mathematically sound approximation techniques, see for instance [Verhulst (2023)] (ch. 9), and bifurcation theory, see for instance [Kuznetsov (2023)]. Such analysis is supplied by numerical bifurcation tools, see [MATCONT (2019)].
In applications the dynamics of strange attractors can be characterised by the presence of negative Lyapunov-exponents and one or more positive ones. A seminal paper on the analysis of strange attractors and the characterisation of geometric and dynamical dimensions by Lyapunov-exponents is [Grassberger & Procaccia (1983)].
Discussions on the meaning and use of Lyapunov-exponents started somewhat later, see for instance [Takens (2010)], [Boyd (2020)], and [Simó(1989)]. In a discussion of attractors of certain maps in [Gräger. & Jäger (2013)] it is noted that taking suitable limits for $t \to \infty$ to obtain exponents and dimensions it might be useful to obtain upper and lower limits for the large time behaviour. As stated before the problem with the existence of these limits is that we have no suitable tangentspace.
Interesting cases arise if we have autonomous equations of motion. A manifold like a torus in such a system has at least one Lyapunov-exponent zero. An additional periodic solution on such a manifold will add another Lyapunov-exponent zero. J.C. Sprott noted in [Sprott (2013)] that in the case of a torus uncertainties may arise in the numerical computation of Lyapunov- exponents that are zero or very close to zero. A number of examples are given in [Bakri & Verhulst (2025)] with special interesting case a torus flattened by canards; this case is both qualitatively and numerically remarkable.

2 Discussion of examples

If we have a smooth 2-torus we can linearise locally and computation of the Lyapunov-exponents will present no problem as we have locally the dichotomy in linear spaces. If the manifold is fractal we cannot simply characterise local spaces. To obtain the Kaplan-Yorke dimension we can make an interpolation using the Lyapunov-exponents, but especially with exponents near zero this may add to the earlier mentioned numerical uncertainties.

System NE9

We illustrate these problems by system (1) that contains interactions between the chaotic NE9 system from [Jafari *et al.* (2013)], the first 3 equations, and the well-known Van der

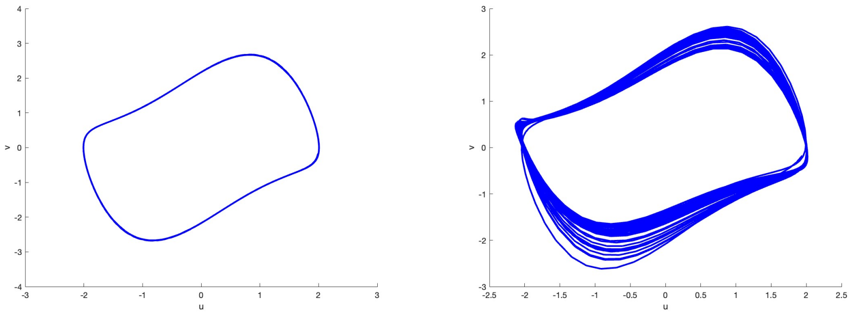

Figure 2: Left the unperturbed phase-plane of the Van der Pol- equation with $b = c = 0$ in system (1). Right the perturbed case with $a = 0.55, b = -1, c = 1$. The initial conditions for $NE9$ are $x(0) = 0.5, y(0) = 0, z(0) = 1$, for the perturbed Van der Pol-equation $u(0) = 2, v(0) = 0$.

Pol-equation (containing a unique periodic solution). The system is:

$$\begin{cases} x' &= y, \\ y' &= -x - yz, \\ z' &= -xz + 7x^2 - a, \\ u' &= v + bx, \\ v' &= -u + (1 - u^2)v + cy. \end{cases} \tag{1}$$

If $b = c = 0$ the systems NE9 and Van der Pol-equation are decoupled. From [Jafari *et al.* (2013)] we have if $a = 0.55$ in the NE9 system for the Lyapunov-exponents $0.0504, 0, -0.3264$ and Kaplan-Yorke dimension $D_{ky} = 2.154$.
The 2 decoupled systems produce each at least 1 Lyapunov-exponent zero so we have 2 Lyapunov-exponents zero in the unperturbed case $b = c = 0$. In fig. 2 right we show a solution of system (1) with $b = -1, c = 1$. What is the Kaplan-Yorke dimension D_{KY} of the limiting solution? The fluctuations of the dimension are shown in fig. 3 for two different time intervals but with the same initial conditions. The fluctuating dimension values D_{KY} are within the same bounds and illustrate the fractal and irregular character of the attractor.

The Lorenz attractor

For the Lorenz system there is a wealth of literature. The system is a highly simplified description of the dynamics of Rayleigh-Bénard convection that describes the instability of rising air on a relative warm Earth surface with an atmosphere where air moves to cooler

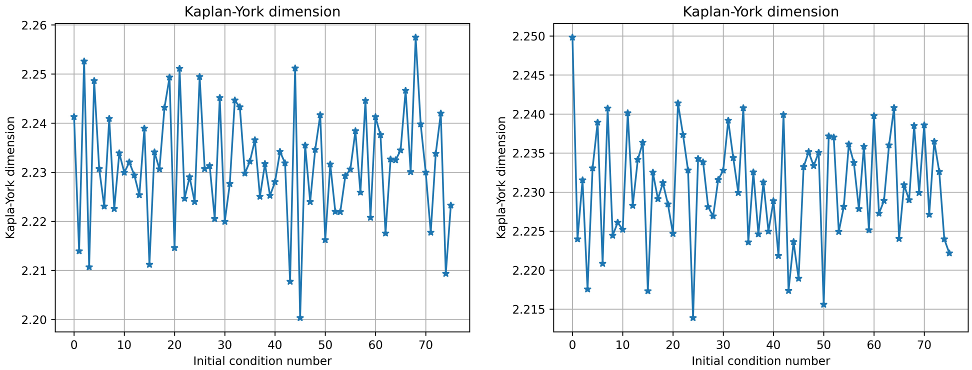

Figure 3: Left the Kaplan-Yorke dimension with $b = -1, = c = 1$ for system (1) starting with 75 different initial conditions, $x(0)$ runs from 0.5 to 1, $z(0)$ from 1 to 1.5. The time interval is 10^5. Right the dimension D_{KY} for the same system with the same 75 initial conditions but on a longer time interval 10^6. In both cases the fluctuations satisfy $2.215 < D_{KY} < 2.250$ (apart from some outliers). [Jafari *et al.* (2013)] Produces $D_{KY} = 2.154$.

regions. The equations for the system with the Lorenz attractor are:

$$\begin{cases} \dot{x} &= 10(-x + y), \\ \dot{y} &= -xz + 28x - y, \\ \dot{z} &= xy - \frac{8}{3}z. \end{cases} \tag{2}$$

In [Viswanath (2004)] a Hausdorff dimension of 2.0627160 is computed based on analysing Cantor sets on a suitable Poincaré map. According to [Sprott (2003)] we have by quadratic interpolation $D_{KY} = 2.112$. We find the value $D_{KY} = 2.062$ (see fig. 4) that is very close to the Hausdorff dimension.

3 Conclusions and discussion

1. Our analysis of the Kaplan-Yorke dimension was stimulated by considering interactions of quasi-periodic oscillations in a dissipative setting, see [Bakri & Verhulst (2025)]. In these problems clustering of Lyapunov-exponents near zero, cascading period doublings, chaos and hyperchaos were observed.

2. The Hausdorff dimension is geometric and represents an upper bound for the correlation dimension. In [Grassberger & Procaccia (1983)] it is stated that the Kaplan-Yorke dimension D_{KY} can be considered as an improved upper bound for the correlation dimension and so yields a more precise description of the dynamics near the strange attractor. For such estimates we have to take into account a certain spreading of D_{KY} values for fractal manifolds. This claim is made explicit for our calculation of the Kaplan-Yorke dimension of the chaotic Lorenz system.

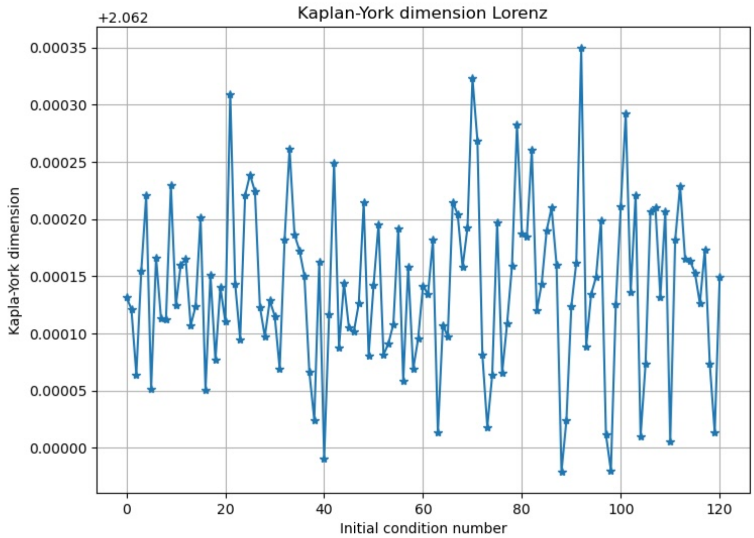

Figure 4: Kaplan-Yorke dimension for the Lorenz system (2) starting with 100 different initial conditions. The dimension D_{KY} shows very small fluctuations near 2.062.

3. We conjecture that in the case of a Hausdorff dimension close to a natural number the Kaplan-Yorke dimension will be close to the geometric dimension.

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High precision numerics was obtained using MATCONT ode 78 under MATLAB.

Conflict of Interest

The authors have no conflicts of interest, no funding was used.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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