Averaging in damping by parametric stiffness excitation

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Abstract

Stability investigations of vibration quenching employing the concept of actuators with a variable stiffness are presented. Systems with an arbitrary number of degrees of freedom with linear spring- and damping-elements are considered, that are subject to self-excitation as well as parametric stiffness excitation. General conditions for full vibration suppression and conditions of instability are derived analytically by applying a singular perturbation of first and second order. The analytical predictions are compared for exemplary systems by numerical time integration and show a great improvement of former results. These basic results obtained can be used for accurate design of a control strategy for actuators.

Key words: parametric excitation, self-excitation, vibration suppression, dynamic stability, perturbation technique

1. Introduction

Vibrations that occur in a dynamical system can be classified with respect to their causes as forced, self-excited or parametrically excited vibrations, e.g. see [20], [27] or [3]. The interaction between two types of vibration is considered: self-excited and parametrically excited vibrations. Self-excited vibrations can occur if a system has access to an external reservoir of energy. Other than with forced vibrations where the frequency of the excitation is prescribed, here the system itself determines the frequency at which the energy is transferred into the system. This type of vibration may occur due to interaction between a structure and a fluid, as in the case of an airplane wing which tends to unstable vibrations when reaching a critical speed. Such vibrations are also called flow-induced vibrations. The destruction of the Tacoma-Bridge in 1940 is an impressive example how dangerous self-excited vibrations by fluid-structure interaction can be, see [18]. Other examples are unstable vibrations of turbo machinery coming from flow-induced vibrations, friction-induced vibrations in brakes (squeal noise) or stick-slip phenomena in tool machines leading to marks of the cutting tool at the surface of the workpiece. Also unstable bogie motions of a rail vehicle at very high speeds belongs to the group of self-excited vibrations. Models for a large class of self-excited vibrations are the Van der Pol- as well as the Rayleigh-oscillator, see [36]. They describe self-excitation by a non-linear force f_{se},

$$f_{se}^{v} = c\dot{x}\left(1+\gamma x^2\right) \qquad \text{and} \qquad f_{se}^{r} = c\dot{x}\left(1+\gamma \dot{x}^2\right), \tag{1}$$

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respectively, where x represents a deflection and c is a negative damping coefficient. The present paper mainly analyses the stable motion of a system, for which the equations of motion may be linearised around the equilibrium position $x, \dot{x} = 0$. The non-linear forces (1) yield $f_{se} = c\dot{x}$ as the linearised expression. Hence, this type of self-excited vibrations can be described by a negative damping coefficient. It should be noted that linearisation works well for a description of the stability regions. In the case of instability, the solution will grow and nonlinear terms will become active after some time.

On the other hand, parametrically excited vibrations occur if one or more coefficients of the differential equations are not constant but periodically time-varying. The frequency of the parameter change is prescribed explicitly as a function of time and is independent of the motion of the system, e.g. by the rotational speed of a shaft. Examples are the pendulum with periodically varying length or periodically moving pivot point, a rotating shaft with nonsymmetric cross-section, a periodically varying stiffness of gear-wheels, or flywheels with variable inertia. Parametrically excited systems and structures have been studied extensively in the past because of the interesting phenomena which occur in such systems. A parametrically excited system may exhibit parametric resonances if the parametric excitation frequency is close to

$$\eta_n = \frac{|\Omega_k \mp \Omega_l|}{n}, \qquad k, l = 1, 2, \dots n. \tag{2}$$

For $k \neq l$ the frequency η_n denotes parametric combination frequency of difference or summation type depending on the sign in eq. (2) and for upper sign and $k = l$ parametric principal frequency, see [43]. Here Ω_k and Ω_l denote the jth and kth fundamental frequency of the undamped system. There are several publications dealing with single or coupled differential equations having a time-periodic coefficient, see i.e. [5], [43], [6], [15] or [31], and the literature cited therein. The main focus there is to investigate the destabilising effect of parametric excitation, i.e. the instability boundary curves in the system parameter domain. The non-resonant cases did not seem to be interesting for applications. The mechanism of damping by parametric excitation as proposed here is based on the effect of coupling modes by parametric excitation and leads to artificial additional damping in the system by the parametric excitation at this frequency. A specific parametric excitation that stabilises an otherwise unstable system is called to be at parametric *anti*-resonance. The main contributions with respect to parametric anti-resonances can be found in [35–37,39,13,16,1,10].

The present study is motivated by the investigations [39] and [13]. Firstly, the stability of a general two degrees of freedom system is investigated that is under the influence of self-excited and parametric excitation, respectively. Analytical stability boundaries are derived employing the concept of first and second order averaging. Arbitrary shape functions that can be represented by Fourier series are considered for parametric excitation. Secondly, the general results are applied to two mechanical example systems. The conditions resulting from the first order averaging coincide with the conditions obtained in the literature before. The necessity of a second order approximation is emphasised. On the one hand, the conditions resulting from second order approximation overcome the error of the first order approximation near the parametric resonance and anti-resonance frequencies $|\Omega_1 \mp \Omega_2|$. On the other hand, the second order approximation enables a prediction of the stability boundary near the parametric frequencies of second order, $|\Omega_1 \mp \Omega_2|/2$.

2. Equations of motion

Parametric excitation may appear by periodic variations of one or more system parameters. The equations of motion of a linear mechanical system with m degrees of freedom and without external forcing are

$$\mathbf{M}(\eta\tau)\ddot{\mathbf{x}} + \mathbf{C}(\eta\tau)\dot{\mathbf{x}} + \mathbf{K}(\eta\tau)\mathbf{x} = \mathbf{0}, \tag{3}$$

with the time τ, the parametric excitation frequency η, the position vector $\mathbf{x}$, see [27] or [45]. The matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ correspond to the mass/inertia, damping and stiffness coefficients. These system matrices are time-dependent, periodic matrices of size $m \times m$. The equations (3) describe the dynamics of m modes and represent a system of m coupled linear differential equations with periodic coefficients.

For the case of a pure stiffness variation as considered here, the inertia and damping matrices are kept constant while the stiffness matrix is varied periodically with frequency η

$$\mathbf{M}(\eta\tau) = \mathbf{M}_0, \quad \mathbf{C}(\eta\tau) = \mathbf{C}_0, \quad \mathbf{K}(\eta\tau) = \mathbf{K}_0 + \varepsilon\mathbf{K}_c c(\eta\tau),$$

where the index 0 denotes the constant part and the index c the time-dependent part of the corresponding matrix. The periodic shape function c is, without loss of generality, restricted to a Fourier cosine series of the form

$$c(\eta\tau) = \sum_{m=1}^{\infty} a_m \cos(m\eta\tau), \qquad \text{with} \qquad a_m = \frac{2}{T}\int_0^T c(\eta\tau)\cos(m\eta\tau)d\tau \tag{4}$$

Rescaling the damping matrix by a positive parameter ε, $\mathbf{C}_0 \mapsto \varepsilon\mathbf{C}_0$, the general linear equations of motion in eq. (3) simplify to

$$\mathbf{M}_0\mathbf{z}'' + \varepsilon\mathbf{C}_0\mathbf{z}' + \mathbf{K}_0\mathbf{z} = -\varepsilon\mathbf{K}_c\mathbf{z}c(\eta\tau). \tag{5}$$

Note that the matrices $\mathbf{K}_c$ allow the description of a time-periodic variation for one or more stiffness coefficients. The only restriction here is, that these variations all occur with the same frequency η and phase. Restricting our study to systems with distinct eigenvalues, the equations of motion can be transformed to the quasi-normal form

$$\mathbf{z}'' + \Omega^2\mathbf{z} = -\varepsilon\left(\Theta\mathbf{z}' + \mathbf{Q}\mathbf{z}c(\eta\tau)\right), \tag{6}$$

where

$$\Omega^2 = \mathbf{T}^{-1}\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{T}, \quad \Theta = \mathbf{T}^{-1}\mathbf{M}_0^{-1}\mathbf{C}_0\mathbf{T}, \quad \mathbf{Q} = \mathbf{T}^{-1}\mathbf{M}_0^{-1}\mathbf{K}_c\mathbf{T}. \tag{7}$$

The transformation matrix $\mathbf{T}$ is defined by the diagonal matrix of squared natural frequencies Ω_i^2. Using Einstein summation, a convention that repeated indices are implicitly summed over, eq. (6) can be rewritten in the comprehensive form

$$z_i'' + \Omega_i^2 z_i = -\varepsilon\left\{\Theta_{ij}z_j'(\tau) + Q_{ij}z_j a_m\delta_m\cos(m\eta\tau)\right\}. \tag{8}$$

Herein i is the free index. For a system with two degrees of freedom $i, j = 1, 2$.

The main purpose of this work is explore the ability of damping vibrations by parametric excitation of a self-excited, weakly damped system. From now on the factor ε is assumed to be small, leading to a weakly coupled system of equations in eq. (8) and small damping.

3. Analytical stability analysis

3.1. *Time transformation and quasi-periodic averaging*

Applying a time transformation to eq. (8) in order to normalise the frequency η to one in eq. (2)

$$\eta\tau \mapsto t, \qquad z_i(\tau) = z_i(t/\eta) = \bar{z}_i(t). \tag{9}$$

Substituting in eq. (8), dividing by η^2 and omitting the bar yields

$$\ddot{z}_i + \frac{\Omega_i^2}{\eta^2}z_i = -\frac{\varepsilon}{\eta}\Theta_{ij}\dot{z}_j - \frac{\varepsilon}{\eta^2}Q_{ij}z_j a_m\delta_m\cos mt. \tag{10}$$

Allowing a small detuning of second order near η_n of the form

$$\eta = \eta_n + \varepsilon\sigma + \varepsilon^2\beta + \mathcal{O}\left(\varepsilon^3\right) \tag{11}$$

and expanding the coefficients $1/\eta$ and $1/\eta^2$ to Taylor series for small values of parameter ε gives

$$\ddot{z}_i + \varpi_i^2 z_i = -\frac{\varepsilon}{\eta_n^2}f_i\left(t, \mathbf{z}\right) + \frac{\varepsilon^2}{\eta^{(n),3}}g_i\left(t, \mathbf{z}\right) + \mathcal{O}\left(\varepsilon^3\right), \tag{12}$$

with the abbreviations $\varpi_i = \Omega_i/\eta_n$, $\mathbf{z} = [z_1, \dot{z}_1, z_2, \dot{z}_2]^T$ and

$$f_i\left(t, \mathbf{z}\right) = \eta_n\Theta_{ij}\dot{z}_j + Q_{ij}z_j a_m\delta_m\cos mt - 2\eta_n\varpi_i^2\sigma z_i, \tag{13a}$$

$$g_i\left(t, \mathbf{z}\right) = \sigma\eta_n\Theta_{ij}\dot{z}_j + 2\sigma Q_{ij}z_j a_m\delta_m\cos mt - \eta_n\varpi_i^2\left(3\sigma^2 - 2\eta_n\beta\right)z_i. \tag{13b}$$

For a *second* order approximation all terms in eq. (12) of higher order than ε^2 are neglected. Similar to the classical method of estimating the particular solution from the homogenous solution by variation of parameters, the coordinate transformations $z_i, \dot{z}_i \mapsto u_i, v_i$ defined as

$$z_i = u_i \cos \varpi_i t + v_i \sin \varpi_i t, \qquad \dot{z}_i = -u_i \varpi_i \sin \varpi_i t + v_i \varpi_i \cos \varpi_i t \tag{14}$$

are performed. By introducing the abbreviations $s_i = \sin \varpi_i t$ and $c_i = \cos \varpi_i t$, equation (12) is transformed to

$$\eta^{(n),3} \varpi_i \dot{u}_i = h_i^s(t, \mathbf{u}) = -\left(\varepsilon \eta_n f_i(t, \mathbf{u}) + \varepsilon^2 g_i(t, \mathbf{u})\right) s_i, \tag{15a}$$

$$\eta^{(n),3} \varpi_i \dot{v}_i = h_i^c(t, \mathbf{u}) = \left(\varepsilon \eta_n f_i(t, \mathbf{u}) + \varepsilon^2 g_i(t, \mathbf{u})\right) c_i, \tag{15b}$$

with the state vector $\mathbf{u} = [u_1, v_1, u_2, v_2]^T$ and

$$f_i(t, \mathbf{u}) = -\eta_n \Theta_{ij} \left(-u_j \varpi_j s_j + v_j \varpi_j c_j\right) + \left(2\Omega_i \varpi_i \sigma \delta_{ij} - Q_{ij} a_m \delta_m \cos mt\right) \left(u_j c_j + v_j s_j\right), \tag{16a}$$

$$g_i(t, \mathbf{u}) = \sigma \eta_n \Theta_{ij} \left(-u_j \varpi_j s_j + v_j \varpi_j c_j\right) - (\eta_n \varpi_i^2 (3\sigma^2 - 2\eta_n \beta) \delta_{ij} - 2\sigma Q_{ij} a_m \delta_m \cos mt)(u_j c_j + v_j s_j). \tag{16b}$$

The functions $\tilde{F}_i^{s/c}$ on the right-hand side of this system of equations are quasi-periodic – they are not periodic but they can be split into a finite sum of different periodic terms of the following form

$$h_i^{s,c}(t, \mathbf{u}) = \sum_{k=1}^{N} h_{i,k}^{s,c}(t, \mathbf{u}),$$

with N fixed and $f_{i,k}^{s,c}(\mathbf{u}, t)$ and $g_{i,k}^{s,c}$ are T_k-periodic in t. For this case *averaging in the general case* in [40] can be applied and the time-averages $\hat{\mathbf{f}}(\mathbf{u})$ and $\hat{\mathbf{g}}(\mathbf{u})$ of the nominal vector fields $\mathbf{f}(t, \mathbf{u}) = [f_1 s_1, f_1 c_1, f_2 s_2, f_2 c_2]^T$ and $\mathbf{g}(t, \mathbf{u}) = [g_1 s_1, g_1 c_1, g_2 s_2, g_2 c_2]^T$ result in

$$\hat{\mathbf{f}}(\mathbf{u}) = \langle \mathbf{f}(t, \mathbf{u}) \rangle = \sum_{k=1}^{N} \frac{1}{T_k} \int_0^{T_k} \mathbf{f}(t, \mathbf{u})\, dt, \qquad \hat{\mathbf{g}}(\mathbf{u}) = \langle \mathbf{g}(t, \mathbf{u}) \rangle.$$

The integration over T_k is carried out for fixed values of $\mathbf{u}$. Hence, for averaging first the periods of the right hand sides of eq. (15) have to be determined. With the help of decomposition theorems the arising products of the trigonometric terms can be rearranged as a sum of basic trigonometric terms. For this simple system with two modes 12 different periods arise for each value of n. Averaging over a basic trigonometric term yields always zero, except for the case where a term becomes resonant, i.e. the argument of a cosine function vanishes.

Subsequently, the second-order averaging as outlined in [41] is performed. By introducing the new vector fields

$$\mathbf{y}(t, \mathbf{u}) = \int_0^t \left(\mathbf{f}(s, \mathbf{u}) - \hat{\mathbf{f}}(\mathbf{u})\right) ds - \mathbf{a}(\mathbf{u}),$$

where $\mathbf{a}$ is chosen such that the average of $\mathbf{y}$ vanishes, and

$$\dot{\mathbf{v}}(t) = \varepsilon \hat{\mathbf{f}}(\mathbf{v}) + \varepsilon^2 \left\langle \frac{\partial \mathbf{f}(t, \mathbf{v})}{\partial \mathbf{x}} \mathbf{y}(t, \mathbf{v}) \right\rangle + \varepsilon^2 \langle \mathbf{g}(t, \mathbf{v}) \rangle, \qquad \mathbf{v}(0) = \mathbf{u}(0)$$

the averaged solution of eq. (15) obeys

$$\mathbf{u}(t) = \mathbf{v}(t) + \varepsilon \mathbf{y}(t, \mathbf{v}(t)) + \mathcal{O}(\varepsilon^2)$$

and consequently

$$\hat{\mathbf{u}}(t) = \mathbf{v}(t) + \varepsilon \langle \mathbf{y}(t, \mathbf{v}(t)) \rangle + \mathcal{O}(\varepsilon^2) = \mathbf{v}(t) + \mathcal{O}(\varepsilon^2).$$

The last three equations can be summarised as

$$\dot{\hat{\mathbf{u}}}(t) = \varepsilon \hat{\mathbf{f}}(\hat{\mathbf{u}}) + \varepsilon^2 \left\langle \frac{\partial \mathbf{f}(t, \hat{\mathbf{u}})}{\partial \mathbf{x}} \mathbf{y}(t, \hat{\mathbf{u}}) \right\rangle + \varepsilon^2 \langle \mathbf{g}(t, \hat{\mathbf{u}}) \rangle. \tag{17}$$

Herein the difference between the solutions $\mathbf{u}$ of the original and $\hat{\mathbf{u}}$ of the averaged system is of order ε^2, $\hat{u}_i(t) - u_i(t) = \mathcal{O}(\varepsilon^2)$, on the timescale $1/\varepsilon$. Finally, the *first order approximation* is simply the solution of

$$\dot{\hat{\mathbf{u}}}(t) = \varepsilon \hat{\mathbf{f}}(\hat{\mathbf{u}}) + \mathcal{O}(\varepsilon^2). \tag{18}$$

where $\hat{u}_i(t) - u_i(t) = \mathcal{O}(\varepsilon)$, again on the timescale $1/\varepsilon$.

3.2. *Averaging near combination frequencies $n = m$*

In this section we discuss the stability near a combination frequency of order n as defined in eq. (2) due to the effect of the n-th Fourier coefficient in eq. (4),

$$n = m. \tag{19}$$

3.2.1. *First order averaging*

Averaging eq. (15) according to eq. (18) in the vicinity of a parametric combination frequency of order n as defined in eq. (2) results in

$$\dot{\hat{u}}_i = \frac{\varepsilon}{\eta_n^2 \varpi_i} \left\{ -\frac{\eta_n}{2}\Theta_{ii}\varpi_i \hat{u}_i \pm \frac{Q_{ij}}{4} a_n \hat{v}_j - \Omega_i \varpi_i \sigma \hat{v}_i \right\}, \tag{20a}$$

$$\dot{\hat{v}}_i = \frac{\varepsilon}{\eta_n^2 \varpi_i} \left\{ -\frac{\eta_n}{2}\Theta_{ii}\varpi_i \hat{v}_i - \frac{Q_{ij}}{4} a_n \hat{u}_j + \Omega_i \varpi_i \sigma \hat{u}_i \right\}, \tag{20b}$$

where the upper signs correspond to $\eta_n = |\Omega_1 - \Omega_2|/n$ and the lower signs to $\eta_n = (\Omega_1 + \Omega_2)/n$. Rewriting in matrix notation yields

$$\begin{bmatrix} \dot{\hat{u}}_1 \\ \dot{\hat{v}}_1 \\ \dot{\hat{u}}_2 \\ \dot{\hat{v}}_2 \end{bmatrix} = \frac{\varepsilon}{\eta_n^2} \begin{bmatrix} -\frac{\eta_n}{2}\Theta_{11} & -\Omega_1\sigma & 0 & \pm\frac{\eta_n}{4\Omega_1} a_n Q_{12} \\ \Omega_1\sigma & -\frac{\eta_n}{2}\Theta_{11} & -\frac{\eta_n}{4\Omega_1} a_n Q_{12} & 0 \\ 0 & \pm\frac{\eta_n}{4\Omega_2} a_n Q_{21} & -\frac{\eta_n}{2}\Theta_{22} & -\Omega_2\sigma \\ -\frac{\eta_n}{4\Omega_2} a_n Q_{21} & 0 & \Omega_2\sigma & -\frac{\eta_n}{2}\Theta_{22} \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \hat{u}_2 \\ \hat{v}_2 \end{bmatrix}. \tag{21}$$

Choosing the upper signs and $n = 1$ this coefficient matrix coincides with [1, p.69]. Introducing the complex abbreviations

$$\hat{w}_1 = \hat{u}_1 + j\hat{v}_1, \qquad \hat{w}_2 = \hat{u}_2 \pm j\hat{v}_2, \tag{22}$$

where $j = \sqrt{-1}$ is the complex unit, eq. (20) is equivalent to

$$\dot{\hat{\mathbf{w}}} = \frac{\varepsilon}{\eta_n^2} \begin{bmatrix} -\frac{\eta_n}{2}\Theta_{11} + j\Omega_1\sigma & -j\frac{\eta_n}{4\Omega_1} a_n Q_{12} \\ \mp j\frac{\eta_n}{4\Omega_2} a_n Q_{21} & -\frac{\eta_n}{2}\Theta_{22} \pm j\Omega_2\sigma \end{bmatrix} \hat{\mathbf{w}} = \frac{\varepsilon}{\eta_n^2}\mathbf{B}_1 \hat{\mathbf{w}}, \tag{23}$$

with the complex state vector $\hat{\mathbf{w}} = (\hat{w}_1, \hat{w}_2)^T$. After rescaling time by ε/η_n^2 the characteristic equation of the coefficient matrix is a complex polynomial of order two

$$\det(\lambda \mathbf{I}_2 - \mathbf{B}_1) = 0,$$

$$\lambda^2 + \left(\frac{\eta_n}{2}(\Theta_{11} + \Theta_{22}) - j(\Omega_1 \pm \Omega_2)\sigma\right)\lambda + \left(\frac{\eta_n}{2}\Theta_{11} - j\Omega_1\sigma\right)\left(\frac{\eta_n}{2}\Theta_{22} \mp j\Omega_2\sigma\right) \pm \frac{\eta_n^2 a_n^2 Q_{12} Q_{21}}{16\Omega_1\Omega_2} = 0. \tag{24}$$

Applying the extended Routh-Hurwitz criterion for complex polynomials, see Appendix A, the stability of this polynomial can be determined.

First analysing the case for $\sigma = 0$ in eq. (11), $\eta = \eta_n$, this polynomial is stable according to eqs. (A.2) if and only if

$$\Theta_{11} + \Theta_{22} > 0, \tag{25a}$$

$$4\Omega_1\Omega_2\Theta_{11}\Theta_{22} \pm a_n^2 Q_{12} Q_{21} > 0. \tag{25b}$$

For the general case of $\sigma \neq 0$ in eq. (11), the following two conditions have to be satisfied for the system being stable

$$\Delta_1: \qquad \Theta_{11} + \Theta_{22} > 0, \tag{26a}$$

$$\Delta_2: \qquad a_{22}\sigma^2 + a_{00} > 0. \tag{26b}$$

Note that the conditions in eqs. (26a) and (25a) coincide and are independent of the parametric excitation. The condition in eq. (26b) defines the critical values of the detuning σ in eq. (11)

$$\sigma = \mp\sigma_n, \qquad \sigma_n = \frac{(\Theta_{11} + \Theta_{22})}{2n}\sqrt{-1 \mp \frac{a_n^2 Q_{12} Q_{21}}{4\Omega_1\Omega_2\Theta_{11}\Theta_{22}}}. \tag{27}$$

This expression was derived for $n = 1$ in [37], [39], [16], [1], [9] and for general n in [21] and [10]. However, since [21] assumed a positive definite damping matrix, eqs. (26a) or eq. (25a) could not be derived and damping by parametric excitation was not discovered. The critical value in eq. (27) determines the stability border in the system parameter space. To decide which side of the boundary is stable and which is unstable the additional condition in eq. (25b) is needed. If this condition is fulfilled then a parametric *anti*-resonance near η_n with a frequency width of $2\sigma_1$ in eq. (27) is obtained

$$\eta_n - \varepsilon\sigma_n < \eta < \eta_n + \varepsilon\sigma_n. \tag{28}$$

Otherwise there is no damping by parametric excitation possible and the vibration amplitudes grow without restriction. Note that stability conditions above are affected by the kind of parametric combination frequency chosen in eq. (2). These results hold for a first order perturbation and are valid on the timescale $1/\varepsilon$.

3.2.2. *Second order averaging*

The analysis of first order averaging as performed in the previous section leads to analytical expressions that are easy to interpret and helps to understand the basic principle of damping by parametric excitation. However, in some cases an analytical approximation of first order cannot reproduce accurately the numerical stability boundary, e.g. [13]. Therefore, in this section an analytical approximation of second order is performed in order to gain more precision. In the vicinity of a parametric combination frequency of order n in eq. (2) averaging eq. (15) according to eq. (17) extends the coefficient matrix $\mathbf{B}_1$ in eq. (23) by higher order terms in ε to

$$\dot{\hat{\mathbf{w}}} = \mathbf{B}_2\hat{\mathbf{w}} \tag{29}$$

where

$$\mathbf{B}_2 = \frac{\varepsilon}{\eta_n^2}\mathbf{B}_1 + \frac{\varepsilon^2}{\eta_n^2}\begin{bmatrix} \frac{\sigma}{2}\Theta_{11} + ja_{12} & b_{12} + j\frac{\sigma}{4\Omega_1}a_n Q_{12} \\ b_{21} + j\frac{\sigma}{4\Omega_2}a_n Q_{21} & \frac{\sigma}{2}\Theta_{22} + ja_{21} \end{bmatrix} \tag{30}$$

and

$$\begin{aligned} b_{ij} =& \frac{\eta_n\left(\Theta_{jj}\Omega_i - \Theta_{ii}\Omega_j\right)}{16\Omega_i^2\Omega_j}a_n Q_{ij} + \frac{\left(a_n Q_{jj}\Omega_i - a_n Q_{ii}\Omega_j\right)}{4\Omega_i\left(\Omega_i + \Omega_i\right)n}\Theta_{ij}, \\ a_{ij} =& -a_n^2 Q_{ij}Q_{ji}\frac{\Omega_i^2 - \Omega_i\Omega_j - \Omega_j^2}{32\Omega_i^2\Omega_j^2 n} + \frac{\eta_n a_n^2 Q_{ii}^2}{4\Omega_i\left(\Omega_i + \Omega_j\right)\left(3\Omega_i - \Omega_j\right)} - \frac{\Omega_i\Theta_{ij}\Theta_{ji}}{2\left(\Omega_i + \Omega_j\right)n} + \Theta_{ii}^2\frac{\eta_n}{8\Omega_i} - \frac{\Omega_i}{\eta_n}\sigma^2 + \Omega_i\beta, \end{aligned} \tag{31}$$

Applying the extended Routh-Hurwitz criterion for complex polynomials, see Appendix A, the stability of this coefficient matrix is determined by (omitting positive factors)

$$\Delta_1: \qquad \left(1 - \frac{\varepsilon\sigma}{\eta_n}\right)\left(\Theta_{11} + \Theta_{22}\right) > 0, \tag{32a}$$

$$\Delta_2: \qquad \left(1 - \frac{\varepsilon\sigma}{\eta_n}\right)^2\left(\varepsilon^2 a_{44}\sigma^4 + \varepsilon a_{33}\sigma^3 + \left(\varepsilon^2 a_{22} + a_{02}\right)\sigma^2 + \varepsilon a_{11}\sigma + \varepsilon^2 a_{20} + a_{00}\right) > 0. \tag{32b}$$

Note, that the coefficients a_{00} and a_{22} are equivalent to the ones in eq. (26b). The solution of eq. (32b) for ε^0 was derived during the stability analysis of first order in eq. (27). Applying $\sigma = \mp\sigma_n$ enables a rescaling by the positive factor ε^1, and eq. (32b) can be rewritten as a quadratic polynomial in β as

$$c_2(\varepsilon^1, \varepsilon^2, \varepsilon^3)\beta^2 + \left(c_1(\varepsilon^1, \varepsilon^2, \varepsilon^3) + 2a_{02}\right)\beta + c_0(\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3) > 0. \tag{33}$$

This quadratic polynomial may be solved directly but would lead to cumbersome terms. As long as the implicit function theorem, see [41], is satisfied,

$$\left.\frac{\partial\Delta_2}{\partial\beta}\right|_{\varepsilon=0} = 2a_{02} = \frac{n^2\eta_n^4}{2}\Theta_{11}\Theta_{22}\sigma_n \neq 0, \tag{34}$$

a convergent series exists in the vicinity of $\varepsilon = 0$ such that

$$\beta = \beta_n^{(0)} + \varepsilon\beta_n^{(1)} + \mathcal{O}(\varepsilon^2). \tag{35}$$

The condition in eq. (34) is violated at the trivial stability boundaries where a modal damping parameter vanishes, $\Theta_{ii} = 0$, as well as at the tip of a stability boundary, $\sigma = 0$.

Applying eq. (35) to eq. (33) and collecting coefficients of equal power in ε and subsequently solving gives:

$$\varepsilon^0: \quad \sigma = \mp\sigma_n \text{ from eq. (27)}, \tag{36}$$

$$\begin{aligned}\varepsilon^1: \quad n\beta_n^{(0)}\Big|_{\sigma=\mp\sigma_n} &= -\frac{\mp\Omega_2\Theta_{11}^2 + \Omega_1\Theta_{22}^2}{8\Omega_1\Omega_2} \pm \frac{a_n^2 Q_{12}Q_{21}}{16\Omega_1\Omega_2(\Omega_1 \mp \Omega_2)} - \frac{\Theta_{12}\Theta_{21}}{2(\Omega_1 \mp \Omega_2)} \\ &+ \frac{a_n^2 Q_{11}^2}{4\Omega_1\left(\Omega_1 \pm \Omega_2\right)\left(3\Omega_1 \mp \Omega_2\right)} + \frac{a_n^2 Q_{22}^2}{4\Omega_2\left(\Omega_1 \pm \Omega_2\right)\left(\Omega_1 \mp 3\Omega_2\right)} \\ &+ \frac{a_n^2\left(\Theta_{11} - \Theta_{22}\right)\left(\Theta_{12}Q_{21} + Q_{12}\Theta_{21}\right)\left(\Omega_2 Q_{11} \mp \Omega_1 Q_{22}\right)}{16\Omega_1\Omega_2\left(\Omega_1 + \Omega_2\right)\left(\Omega_2 - \Omega_1\right)\Theta_{11}\Theta_{22}},\end{aligned} \tag{37}$$

With the abbreviations

$$a_{ij} = Q_{ij}\Theta_{22}\Omega_1^3 \mp (Q_{ij}\Theta_{11} - 4\Theta_{ij}Q_{22})\Omega_1^2\Omega_2 - (Q_{ij}\Theta_{22} + 4\Theta_{ij}Q_{11})\Omega_1\Omega_2^2 \pm Q_{ij}\Theta_{11}\Omega_2^3, \tag{38a}$$

$$\gamma = \frac{32(\Omega_1 - \Omega_2)^2(\Omega_1 + \Omega_2)^2\Omega_1^3\Omega_2^3\Theta_{11}^2\Theta_{22}^2}{(\Theta_{11} + \Theta_{22})^2}, \tag{38b}$$

handsome expressions are obtained for

$$\varepsilon^2: \quad a_n^2\gamma n^2\sigma_n\,\beta_n^{(1)}\Big|_{\sigma=-\sigma_n} = -a_n^2\Omega_1\Omega_2(Q_{12}\Theta_{21} + Q_{21}\Theta_{12})^2(\Omega_2 Q_{11} \mp \Omega_1 Q_{22})^2 \pm \Theta_{11}\Theta_{22}a_{12}a_{21}, \tag{39a}$$

and

$$\beta_n^{(1)} = \beta_n^{(1)}\Big|_{\sigma=+\sigma_n} = -\,\beta_n^{(1)}\Big|_{\sigma=-\sigma_n}. \tag{39b}$$

The remaining coefficients of ε^3 and ε^4 in eq. (33) are not considered. From eq. (11) the stable frequency interval in eq. (28) is extended to

$$\eta_n - \varepsilon\sigma_n + \varepsilon^2\beta_n^{(0)} - \varepsilon^3\beta_n^{(1)} + \mathcal{O}\left(\varepsilon^3\right) < \eta < \eta_n + \varepsilon\sigma_n + \varepsilon^2\beta_n^{(0)} + \varepsilon^3\beta_n^{(1)} + \mathcal{O}(\varepsilon^3). \tag{40}$$

3.3. *Averaging near combination frequencies for $n = m + p$*

It was shown in the previous sections that the n-th Fourier coefficient in eq. (4) leads to a stability boundary curve near a combination frequency of order n as defined in eq. (2). In this section we discuss the effect of the $(n-p)$-th Fourier coefficient on the stability near a combination frequency of order n,

$$n = m + p. \tag{41}$$

Compared to the previous section, the order of the Fourier coefficient is shifted from m to $m+p$. Consequently, an averaging of order $p-1$ is necessary in which case the damping coefficients in eq. (8) need to be rescaled from $\varepsilon^1\Theta_{ij}$ to $\varepsilon^{1+p}\Theta_{ij}$. Averaging of higher order is, in general, analytically feasible for the first few orders, therefore, the following study is restricted to $p = 1$.

3.3.1. *First order averaging*

First order averaging of eq. (15) according to eq. (18) for rescaled damping coefficients in the vicinity of a parametric combination frequency of order n as defined in eq. (2) results in

$$\dot{\hat{u}}_i = -\frac{\varepsilon}{\eta_n^2}\Omega_i\sigma\hat{v}_i, \qquad \dot{\hat{v}}_i = \frac{\varepsilon}{\eta_n^2}\Omega_i\sigma\hat{u}_i, \tag{42}$$

instead of eq. (20). Introducing the complex abbreviations in eq. (22) yields the coefficient matrix corresponding to eq. (23)

$$\mathbf{B}_1 = \frac{\varepsilon}{\eta_n^2}\begin{bmatrix} j\Omega_1\sigma & 0 \\ 0 & \pm j\Omega_2\sigma \end{bmatrix}. \tag{43}$$

This matrix possesses purely imaginary eigenvalues only and, consequently, the stability of the original system in eq. (8) cannot be deducted from the averaged one in eq. (42) and a higher order averaging has to be performed.

3.3.2. *Second order averaging*

In Section 3.2.2, second order averaging was applied to obtain higher accuracy of the stability analysis than first order averaging at combination frequencies of order m. In this section, second order averaging is applied to even enable a stability analysis at frequencies of order $m+1$, since the first order averaging in eq. (43) is not decisive. Second order averaging of eq. (15) according to eq. (17) is performed in the vicinity of a parametric combination frequency of order $m+1$. For rescaled damping coefficients, the coefficient matrix $\mathbf{B}_1$ in eq. (43) is extended by higher order terms in ε and the coefficient matrix in eq. (29) is now

$$\mathbf{B}_2 = \mathbf{B}_1 + \frac{\varepsilon^2}{\eta_n^2}\begin{bmatrix} \frac{\eta_n}{4}\Theta_{11} + jc_{12} & jd_{12} \\ jd_{21} & \frac{\eta_n}{4}\Theta_{22} + jc_{21} \end{bmatrix} \tag{44}$$

instead of eq. (30), where

$$c_{ij} = -\frac{a_n^2 Q_{ii}Q_{ij}}{4\Omega_i(3\Omega_i+\Omega_j)} + \frac{a_n^2 Q_{jj}Q_{ji}}{4\Omega_i(\Omega_j+3\Omega_i)}, \tag{45a}$$

$$d_{ij} = a_n^2 Q_{ij}Q_{ji}\frac{(5\Omega_i+3\Omega_j)}{6\Omega_i(\Omega_i+3\Omega_j)(\Omega_j+3\Omega_i)} - \frac{2\Omega_i}{\eta_n}\sigma^2 + a_n^2 Q_{ii}^2\frac{\eta_n}{2\Omega_i(3\Omega_i+\Omega_j)(\Omega_i-5\Omega_j)} + \Omega_i\beta. \tag{45b}$$

Setting $\sigma = 0$ in eq. (11) results in the Routh-Hurwitz determinants according to eq. (A.2)

$$\Delta_1: \quad \frac{\eta_n}{2}(\Theta_{11}+\Theta_{22}) > 0, \tag{46a}$$

$$\Delta_2: \quad \varepsilon^2\left(\frac{\eta_n^4}{16}\Theta_{11}\Theta_{22}\beta^2 + c_1\beta + c_0\right) > 0. \tag{46b}$$

Since the implicit function theorem cannot be applied easily, the quadratic equation is solved directly and gives

$$\beta = \frac{a_n^2 c}{\delta} \mp \frac{\Theta_{11}+\Theta_{22}}{2n}\sqrt{d_n} = \alpha_n^{(1)} \mp \alpha_n^{(2)} \tag{47}$$

with

$$d_n = -1 \mp \frac{a_n^4 Q_{12}Q_{21}}{\Omega_1\Omega_2\Theta_{11}\Theta_{22}}\left(\frac{(Q_{11}-3Q_{22}) + 3\Omega_2(3Q_{11}-Q_{22})}{(3\Omega_2+\Omega_1)(3\Omega_1+\Omega_2)(\Omega_2\pm\Omega_1)}\right)^2, \tag{48a}$$

$$c = ((30Q_{21}Q_{12} - 90Q_{22}^2)\Omega_1^4 + (-6Q_{11}^2 + 78Q_{22}^2 - 56Q_{21}Q_{12})\Omega_2\Omega_1^3 + (-90Q_{11}^2 + 30Q_{21}Q_{12})\Omega_2^4) + (18Q_{22}^2 + 18Q_{11}^2 - 460Q_{21}Q_{12})\Omega_2^2\Omega_1^2 + (-6Q_{22}^2 + 78Q_{11}^2 - 56Q_{21}Q_{12})\Omega_2^3\Omega_1, \tag{48b}$$

$$\delta = 12((-5\Omega_1+\Omega_2)(3\Omega_2+\Omega_1)(3\Omega_1+\Omega_2)(5\Omega_2-\Omega_1)(\Omega_2\pm\Omega_1)\Omega_1\Omega_2). \tag{48c}$$

According to eq. (11), the system is stable if the parametric excitation frequency stays within the following limits

$$\eta_n + \varepsilon^2\alpha_n^{(1)} - \varepsilon^2\alpha_n^{(2)} + \mathcal{O}(\varepsilon^3) < \eta < \eta_n + \varepsilon^2\alpha_n^{(1)} + \varepsilon^2\alpha_n^{(2)} + \mathcal{O}(\varepsilon^3). \tag{49}$$

3.4. *General discussion*

Examining the case where the parametric excitation is not present in the system, $Q_{ij} = 0$, the stability conditions in eqs. (25) or eqs. (46) collapse to

$$\Theta_{11} > 0 \qquad \text{and} \qquad \Theta_{22} > 0. \tag{50}$$

The radicand in eq. (27) and eq. (48a) becomes negative and leads to a purely imaginary critical value σ_n and β in eq. (47), respectively. Hence, the system without parametric excitation is stable if all modal damping coefficients in the main diagonal of the modal damping matrix are positive. If one modal damping parameter vanishes, $\Theta_{11} = 0$ or $\Theta_{22} = 0$, the system is at its trivial stability boundary, in which case (i) the implicit function theorem in eq. (34) is no longer satisfied, and the series in eq. (35) does not converge, (ii) the critical value σ_n in eq. (27), β_1 in eq. (37) and β in eq.(47) becomes infinite, and (iii) due to infinite σ_n stability condition in eq. (32a) is no longer satisfied. However, the derived stability conditions in eqs. (28,40,49) are valid in the region of interest, as will be clear in the next section.

The following cases can be distinguished with respect to eq. (50):

(i) If these conditions hold there is no negative modal damping present in the system and, hence, the system is stable. In this case parametric excitation near a parametric combination frequency η_n in eq. (2) may be used to enhance the system damping and vibration suppression is achieved.

(ii) If one condition in eq. (50) is violated but the less restrictive condition in eq. (26a) is satisfied, the system is unstable but may be stabilised by a proper parametric excitation near the frequency η_n depending on the condition in eq. (26b). This is the case in which damping by parametric excitation can be applied most effectively.

(iii) Finally, if both conditions in eq. (50) are violated the system is unstable and cannot be stabilised by any parametric excitation.

Averaging at combination frequencies in eq. (2) focuses the investigation on the stabilising effect at a single frequency of a certain order n. Sometimes the stability gain near a frequency of order n, η_n, overlaps the stability gain near a frequency of order $n+1$, η_{n+1}. Sometimes the stability gain near a frequency of order n, η_n, that corresponds to the nth Fourier coefficient interacts with the stability gain near a frequency of order $n+1$, η_{n+1}, corresponding the $(n+1)$th Fourier coefficient. In order to account for overlapping and interaction, respectively, between two neighbouring orders the results in eqs. (40) and (49) are, in a first approximation, summed over. For the interaction between the stability gains corresponding to two subsequent Fourier coefficients a_{n-1} and a_n $(n \geq 2)$, the summed stability boundary near the frequency η_n becomes

$$\eta_n - \varepsilon\sigma_n + \varepsilon^2 \left(\beta_n^{(0)} + \alpha_{n-1}^{(1)} - \alpha_{n-1}^{(2)}\right) + \mathcal{O}\left(\varepsilon^3\right) < \eta < \eta_n + \varepsilon\sigma_n + \varepsilon^2 \left(\beta_n^{(0)} + \alpha_{n-1}^{(1)} + \alpha_{n-1}^{(2)}\right) + \mathcal{O}(\varepsilon^3). \tag{51}$$

For overlapping, the stability gain near η_n and the stability gain near η_{n-1} are need to be summed over in the parameter space. The stability boundary in eq. (40) can be written as functions of an arbitrary parameter p as

$$f_{n-}(p) = \eta_n - \varepsilon\sigma_n + \varepsilon^2\beta_n^{(0)} - \varepsilon^3\beta_n^{(1)} \quad \text{for} \quad \eta \leq \eta_n, \quad f_{n+}(p) = \eta_n + \varepsilon\sigma_n + \varepsilon^2\beta_n^{(0)} + \varepsilon^3\beta_n^{(1)} \quad \text{for} \quad \eta \geq \eta_n, \tag{52}$$

and the stability boundary in eq. (49) as

$$g_n(p) = \eta_n + \varepsilon^2\alpha_n^{(1)} - \varepsilon^2\alpha_n^{(2)} \quad \text{for} \quad \eta \leq \eta_n. \tag{53}$$

Assuming that these functions are invertible, the sum functions of the stability curves become

$$h_{n\mp}\left(\frac{\eta}{\eta_1}\right) = \left(f_{n\mp}^{-1}\left(\frac{\eta}{\eta_1}\right) + g_{n-1}^{-1}\left(\frac{\eta}{\eta_1}\right)\right)^{-1} \tag{54}$$

4. Numerical stability analysis

The stability of the system dynamics can be investigated by means of Floquet theory. Applying Floquet theory transforms a linear time-periodic system into a linear time-invariant system by using a Lyapunov transformation. Hence, the stability of the former system can be inferred from that of the latter system. Below is a brief review of Floquet theory and related technical terms based on [15] and [40].

The considered m-dimensional equations of motion (5) with the position vector $\mathbf{x}(t)$ define a system of linear differential equations with periodic coefficients, which can be transformed to a $2m$-dimensional linear time-periodic system of first order differential equations

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\,\mathbf{y}(t), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \qquad \mathbf{A}(t+T) = \mathbf{A}(t) \quad t \geq t_0, \tag{55}$$

with the state vector $\mathbf{y}(t) \in \mathbb{R}^{2m}$ and the piecewise continuous and periodic matrix $\mathbf{A}(t) \in \mathbb{R}^{2m\times 2m}$ with period T. In most cases, the conditions of the Picard-Lindelöf theorem for existence and uniqueness of initial value problems are trivially satisfied by (55), see [15]. Hence, there exist unique solutions of (55) for arbitrarily given initial conditions $\mathbf{y}_0 \in \mathbb{R}^{2m}$. The set of the solutions of (55) form an $2m$-dimensional linear space. Let $\mathbf{y}_1(t), \mathbf{y}_2(t), \ldots, \mathbf{y}_{2m}(t)$ be $2m$ linearly independent solutions, then

$$\mathbf{Y}(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t), \ldots, \mathbf{y}_{2m}(t)] \tag{56}$$

is called the fundamental matrix. If the fundamental matrix is equal to the unity matrix at the initial time $t = t_0$, $\mathbf{Y}(t_0) = \mathbf{I}_{2m}$, then $\mathbf{Y}(t)$ is called the principal fundamental matrix or the state transition matrix for (55). The state transition matrix is denoted by $\Phi(t, t_0)$, where the second argument indicates dependency on initial conditions. Any solution of (55) can be expressed as $\Phi(t, t_0)\,\mathbf{c}$, where $\mathbf{c} \neq \mathbf{0}$, $\mathbf{c} \in \mathbb{R}^{2m}$ is a constant vector. In particular, for $\mathbf{y}_0$ the solution of (55) is given by

$$\mathbf{y}(t) = \Phi(t, t_0)\,\mathbf{y}_0. \tag{57}$$

The state transition matrix evaluated at $t = T$, $\Phi(T, 0)$, is called the monodromy matrix.

Floquet's theorem postulates that each fundamental matrix of (55), and consequently the state transition matrix $\Phi(t, t_0)$, can be written as the product of two $2m \times 2m$-matrices

$$\Phi(t, t_0) = \mathbf{L}(t, t_0)\,e^{(t-t_0)\mathbf{F}}, \qquad \mathbf{L}(t, t_0) = \mathbf{L}(t+T, t_0), \tag{58}$$

where $\mathbf{L}$ is a $2m \times 2m$-matrix-valued function, which is periodic with period T, and $\mathbf{F}$ is a constant $2m \times 2m$-matrix. The eigenvalues of $e^{T\mathbf{F}}$ are called the characteristic multipliers. The Lyapunov transformation

$$\mathbf{z}(t) = \mathbf{L}^{-1}(t, t_0)\ \mathbf{y}(t)$$

transforms the time-periodic system (55) into a linear time-invariant system with constant coefficients

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{F}\,\mathbf{z}(t), \qquad t \geq t_0, \\ \mathbf{z}(t_0) &= \mathbf{y}_0. \end{aligned} \tag{59}$$

By defining

$$t = t_0 : \qquad \Phi(t_0, t_0) = \mathbf{L}(t_0, t_0)$$

we prepare $2m$ sets of initial condition vectors $\mathbf{k}_i$, so that an identity matrix is formed

$$\Phi(t_0, t_0) = \mathbf{L}(t_0, t_0) = [\mathbf{k}_1(t_0), \mathbf{k}_2(t_0), \ldots, \mathbf{k}_{2m}(t_0)] = \mathbf{I}_{2m}, \tag{60}$$

which can be substituted in (58) and leads to

$$\mathbf{F} = \frac{1}{T}\ln\left(\Phi(t_0 + T, t_0)\right). \tag{61}$$

As a result of (61), the stability of the time-periodic system (55) can be determined either from the eigenvalues of the Floquet exponent matrix $\mathbf{F}$ or from the monodromy matrix $\Phi(T, 0)$. Starting from independent sets of initial conditions defined in (60), the monodromy matrix is calculated numerically by repeated integration of the system equations (55) over one period T. By solving these $2m$ initial value problems over one period T the monodromy matrix $\Phi(T, 0)$ is obtained, where the eigenvalues of the monodromy matrix

$$\Lambda = \mathrm{eig}\left\{\Phi(T, 0)\right\}, \tag{62}$$

determine the stability of the system dynamics. The system is asymptotically stable if and only if all of the multipliers are less than one in magnitude

$$\max\left\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_{2m}|\right\} = \begin{cases} < 1 \text{ asymptotically stable system,} \\ > 1 \text{ unstable system.} \end{cases} \tag{63}$$

This approach gives reliable results and the computational cost is very low compared to a direct time integration of eq. (5) over many periods of the parametric excitation. Determining the state vector at the end of a period can be performed by using an integration algorithm for ordinary differential equations like Radau5 [19]. The procedure presented here was implemented by using the software package Matlab [25]. The stability results serve as a reference for the approximate analytical stability analysis.

5. Examples from mechanics

The stability intervals in eq. (28) and eq. (40) are valid for arbitrary system matrices **K**, **C** but positive definite inertia matrix **M**, due to the quasi-normal form transformation in eq. (7). However, only systems with symmetric matrices **K**, **C** are considered here. It can be shown that for such systems a parametric anti-resonance may occur only near $\eta_n = |\Omega_2 - \Omega_1|/n$ but never near $\eta_n = (\Omega_1 + \Omega_2)/n$, see [7] or [8]. Furthermore, only systems with a single time-periodic stiffness coefficient are shown although the analytical stability conditions allow the variation of one or more stiffness coefficients, see eq. (5).

In general, a dynamical process can be modelled by an equivalent mechanical system. In the following the simplest possible systems for which damping by parametric excitation is achievable are presented – a lumped mass system with two degrees of freedom, see Fig. 1. In the following paragraphs we give explicit expressions for the coefficient matrices of the normal form for two simple lumped mass systems, that possess only a single time-periodic stiffness and/or damping coefficient. Such systems in the context of parametric anti-resonance have been initially investigated in [39] and [13].

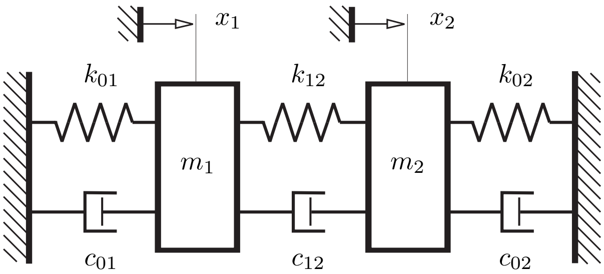


Fig. 1. Lumped mass system with two degrees of freedom

5.1. *System 1*

The first system investigated is a simplified system for which only the stiffness k_{02} attached to the inertial reference frame is time-periodic, $k_{02}(\tau) = k_{02}(1 + \varepsilon\cos(\eta\tau))$, and $k_{01}, c_{12} \equiv 0$, as presented in [39]. The corresponding non-vanishing system matrices of (5) read

$$\mathbf{M}_0 = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad \mathbf{K}_0 = \begin{bmatrix} k_{12} & -k_{12} \\ -k_{12} & k_{12} + k_{02} \end{bmatrix},$$

$$\mathbf{C}_0 = \begin{bmatrix} c_{01} & 0 \\ 0 & c_{02} \end{bmatrix}, \qquad \mathbf{K}_\mathrm{c} = \begin{bmatrix} 0 & 0 \\ 0 & k_{02} \end{bmatrix}. \tag{64}$$

Using the time transformation that is derived from the natural frequency of the subsystem $m_1,\ k_{12}$,

$$\tau = \omega_1 t \qquad \text{with} \qquad \omega_1 = \sqrt{\frac{k_{12}}{m_1}}, \tag{65}$$

and defining the following characteristic parameters

$$\eta = \frac{\omega}{\omega_1}, \quad M = \frac{m_1}{m_2}, \quad \kappa_1 = \frac{c_{01}}{m_1\omega_1}, \quad \kappa_2 = \frac{c_{02}}{m_2\omega_1}, \quad Q^2 = \frac{k_{02}}{m_2\omega_1^2} = \left(\frac{\omega_2}{\omega_1}\right)^2, \tag{66}$$

gives the non-dimensional system matrices

$$\begin{aligned}\bar{\mathbf{M}}_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \bar{\mathbf{K}}_0 = \begin{bmatrix} 1 & -1 \\ -M & M+Q^2 \end{bmatrix}, \\ \bar{\mathbf{C}}_0 &= \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}, \qquad \bar{\mathbf{K}}_{\mathrm{c}} = \begin{bmatrix} 0 & 0 \\ 0 & Q^2 \end{bmatrix}. \end{aligned} \tag{67}$$

The ratios defined in (66) represent relations between the dimensional physical system parameters. For a certain physical system specific values for some of the parameters have to be chosen additionally. The non-dimensional equations of motion are transformed into the quasi-normal form by applying the constant transformation matrix

$$\mathbf{x}(t) = \mathbf{T}\mathbf{z}(t) \qquad \text{with} \qquad \mathbf{T} = \begin{bmatrix} 1 & 1 \\ a_1 & a_2 \end{bmatrix}. \tag{68}$$

The special form of the transformation matrix $\mathbf{T}$ is chosen in order to keep the following expressions simple. The coefficients of the transformation matrix $\mathbf{T}$ read for the system matrices (64)

$$a_1 = \frac{M}{Q^2 + M - \Omega_1^2}, \qquad a_2 = \frac{M}{Q^2 + M - \Omega_2^2} \tag{69}$$

with the eigenvalues

$$\Omega_{1,2}^2 = \frac{1}{2}(1 + M + Q^2) \pm \sqrt{\frac{1}{4}(1 + M + Q^2)^2 - Q^2}. \tag{70}$$

Applying (68) to (64) the equations of motion are transformed into the normal form

$$\bar{\mathbf{K}}_0 \mapsto \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_2^2 \end{bmatrix} = \Omega^2, \qquad \bar{\mathbf{C}}_0 \mapsto [\Theta_{ij}], \qquad \bar{\mathbf{K}}_{\mathrm{c}} \mapsto [Q_{ij}]. \tag{71}$$

The modal damping coefficients yield

$$\begin{aligned} \Theta_{11} &= \frac{-a_2\kappa_1 + a_1\kappa_2}{a_1 - a_2}, \quad \Theta_{12} = \frac{-a_2\kappa_1 + a_2\kappa_2}{a_1 - a_2}, \\ \Theta_{21} &= \frac{a_1\kappa_1 - a_1\kappa_2}{a_1 - a_2}, \quad \Theta_{22} = \frac{a_1\kappa_1 - a_2\kappa_2}{a_1 - a_2}, \end{aligned} \tag{72}$$

and the coefficients of the parametric excitation result in

$$\begin{aligned} Q_{11} &= Q^2 \frac{a_1}{a_1 - a_2}, \quad Q_{12} = Q^2 \frac{a_2}{a_1 - a_2}, \\ Q_{21} &= Q^2 \frac{-a_1}{a_1 - a_2}, \quad Q_{22} = Q^2 \frac{-a_2}{a_1 - a_2}. \end{aligned} \tag{73}$$

The following non-dimensional characteristic system parameters, as defined in eq. (66), are used as default values, see [39],

$$M = 0.5, \qquad \kappa_1 = -0.01, \qquad \kappa_2 = 0.14. \tag{74}$$

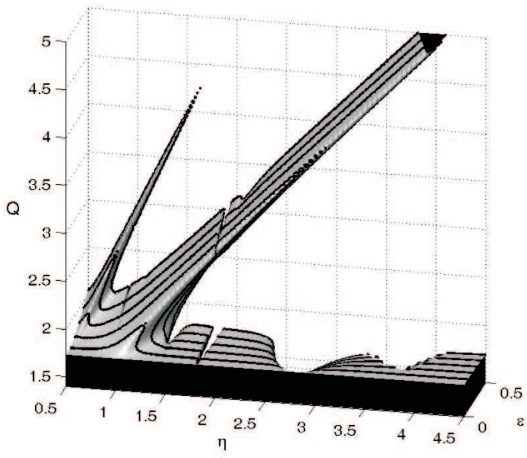
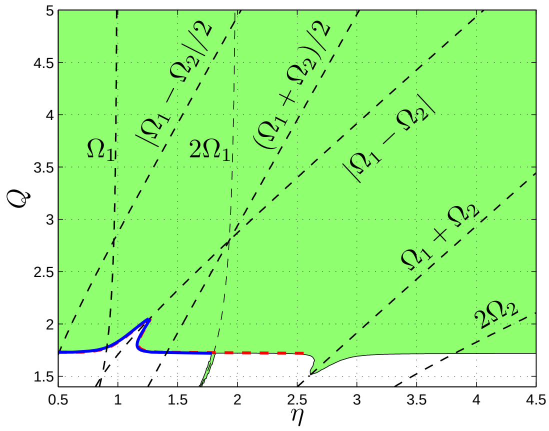
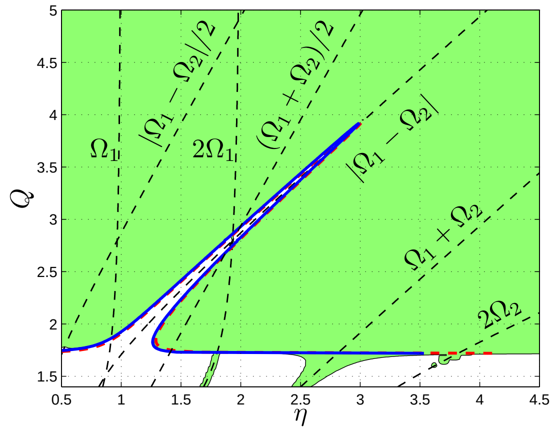


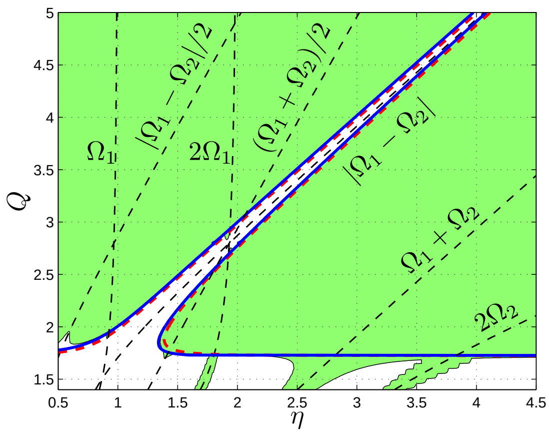
Fig. 2. System 1: Numerical stability domain in dependency of Q, ε and η



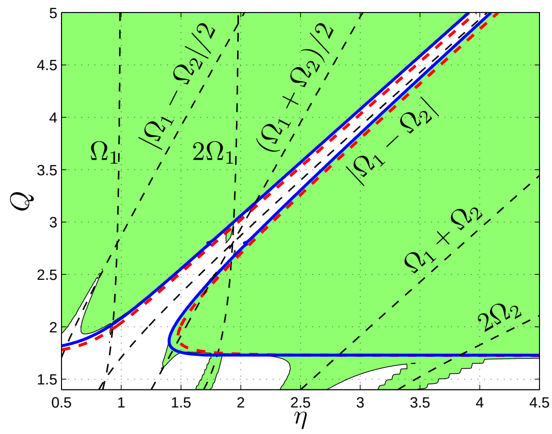
(a) $\varepsilon = 0.1$



(b) $\varepsilon = 0.2$



(c) $\varepsilon = 0.3$



(d) $\varepsilon = 0.4$

Fig. 3. System 1: Stability maps for different scaling factor ε: Comparison between numerical (shaded area is unstable) and analytical first order (dashed line) and second order approximation (solid line). The results for the parametric resonances at $\Omega_1 + \Omega_2$, $2\Omega_1$, etc. are as accurate, but not plotted.

The damping coefficients κ_i satisfy the main stability condition in eq. (26a) and eq. (32a)

$$\Theta_{11} + \Theta_{22} = \kappa_1 + \kappa_2 > 0, \tag{75}$$

according to eq. (72). Note that the damping coefficient κ_1 is chosen to be negative and causes the conventional system without open-loop control ($\varepsilon = 0$) to become unstable if $M < M_{crit} = 2.58$ or $Q > Q_{crit} = 1.73$. The critical system parameters M_{crit} or Q_{crit} follows from

$$\varepsilon = 0: \qquad \Theta_{11} = 0 \quad \text{or} \quad \Theta_{22} = 0 \quad \rightarrow \quad \begin{cases} M_{crit} & \text{for fixed } Q, \\ Q_{crit} & \text{for fixed } M, \end{cases}$$

and represent the stability border for the system without open-loop control. With parametric excitation activated ($\varepsilon \neq 0$), the original stability boundary is deformed and it is possible to stabilise the system beyond these critical system parameters within a certain range of the system and control parameters.

A stability domain for the set of parameters eq. (74) in dependency of the frequency ratio Q and the control parameters ε and η is shown in Fig. 2. Combinations of ε, η and Q, which are enclosed by the shaded surface (indicating the stability boundary) lead to a stable system. A highly complex geometry is obtained. From Fig. 2 it can be seen that the conventional system, $\varepsilon = 0$ and $\kappa_1 < 0$, is stable unless the dimensionless stiffness parameter Q reaches the critical value Q_{crit}. Parametric stiffness excitation, $\varepsilon \neq 0$, generates a major stretch and a narrow spike as additional stability regions.

Slices for constant values of ε of this stability domain are analysed in more detail in Fig. 3. These numerically obtained results are compared to the analytical ones derived in eqs. (28) and (40). A key to the interpretation are the parametric resonance frequencies in eq. (2). As predicted by the analytical analysis, the main area of stability occurs near $\eta_1 = |\Omega_1 - \Omega_2|$ and gets wider for increasing Q. The dashed line denotes the first order approximation in eq. (28) and the solid line the second order approximation in eq. (40). For this system, already the first order approximation is very accurate. The second order approximation introduces an additional shift β_1^0 with which the analytical stability boundary coincides with the numerical one; the expression β^1 is negligible. Analytical and numerical results agree amazingly well if the isolated contribution of the parametric *anti*-resonance frequency $|\Omega_1 - \Omega_2|$ is considered. Even for large values of ε, i.e. Figs. 3c and d, high accuracy is obtained although the underlying analytical procedure assumes ε to be sufficiently small. A significant loss of stability occurs near the parametric resonance frequency $\Omega_1 + \Omega_2$. For clarity of the figure the analytical results at the parametric resonance $\Omega_1 + \Omega_2$ are not plotted but are of same accuracy. Additional regions of instability are caused by the principal parametric resonances at $2\Omega_i/n$. Summarising, an approximation of the equations of motion up to the first order of ε is sufficient to describe the stability of System 1 corresponding to η_1.

In Fig. 3d, an additional small stability area is caused by the second order parametric anti-resonance frequency $(\Omega_2 - \Omega_1)/2$, which is zoomed in Fig. 4 for $\varepsilon = 0.4$. Again, the analytical prediction of first order in eq. (28) and of second order in eq. (40) near η_1 are plotted. Additionally, the analytical prediction of second order in eq. (49) near η_2 is shown. Since the stability regions corresponding to η_1 and η_2 overlap, the sum functions according to eq. (54) are introduced between the second order stability intervals corresponding to eqs. (49) and (40). The analytical sum functions and the numerical stability border are very close and the analytical results even slightly underestimate the size of the stability area in this part of the diagram.

5.2. *System 2*

The second system of interest is a system where only the coupling stiffness k_{12} is time-periodic, $k_{12}(t) = k_{12}\,(1 + \varepsilon \cos \eta\tau)$, and $k_{01}, c_{01} \equiv 0$, as presented in [13]. Contrary to System 1 this system shows the interesting phenomenon of large deformations of its stability domains leading to great deviations between the numerical result and the analytical first-order prediction. The following system matrices alter with respect to eq. (64)

$$\mathbf{C}_0 = \begin{bmatrix} c_{12} & -c_{12} \\ -c_{12} & c_{12} + c_{02} \end{bmatrix}, \qquad \mathbf{K}_c = \begin{bmatrix} k_{12} & -k_{12} \\ -k_{12} & k_{12} \end{bmatrix}. \tag{76}$$

Applying the time transformation that is derived from the natural frequency of the subsystem $m_2,\ k_{02}$,

$$\tau = \omega_2 t \qquad \text{with} \qquad \omega_2 = \sqrt{\frac{k_{02}}{m_2}} \quad \text{and} \quad \omega_1 = \sqrt{\frac{k_{12}}{m_1}} \tag{77}$$

and defining the following characteristic parameters

$$\eta = \frac{\omega}{\omega_2}, \quad M = \frac{m_1}{m_2}, \quad \kappa_1 = \frac{c_{12}}{m_1\omega_2}, \quad \kappa_2 = \frac{c_{02}}{m_2\omega_2}, \quad Q^2 = \frac{k_{12}}{m_1\omega_2^2} = \left(\frac{\omega_1}{\omega_2}\right)^2, \tag{78}$$

the system matrices of the non-dimensional equations of motion yield

$$\begin{aligned} \bar{\mathbf{M}}_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \bar{\mathbf{K}}_0 &= \begin{bmatrix} Q^2 & -Q^2 \\ -MQ^2 & MQ^2+1 \end{bmatrix}, \\ \bar{\mathbf{C}}_0 &= \begin{bmatrix} \kappa_1 & -\kappa_1 \\ -M\kappa_1 & M\kappa_1+\kappa_2 \end{bmatrix}, & \bar{\mathbf{K}}_c &= \begin{bmatrix} Q^2 & -Q^2 \\ -MQ^2 & MQ^2 \end{bmatrix}. \end{aligned} \tag{79}$$

For the system in (76) the coefficients of the transformation matrix as defined in (68) become

$$a_1 = \frac{MQ^2}{1+MQ^2-\Omega_1^2}, \qquad a_2 = \frac{MQ^2}{1+MQ^2-\Omega_2^2} \tag{80}$$

with the eigenvalues

$$\Omega_{1,2}^2 = \frac{1}{2}\left(1+MQ^2+Q^2\right) \pm \sqrt{\frac{1}{4}\left(1+MQ^2+Q^2\right)^2 - Q^2}. \tag{81}$$

The equations of motion are transformed into the quasi-normal form as in eq. (71) with the modal damping coefficients

$$\begin{aligned} \Theta_{11} &= \frac{-\left(a_2+M\right)\left(1-a_1\right)\kappa_1 + a_1\kappa_2}{a_1-a_2}, & \Theta_{12} &= \frac{-\left(a_2+M\right)\left(1-a_2\right)\kappa_1 + a_2\kappa_2}{a_1-a_2}, \\ \Theta_{21} &= \frac{\left(a_1+M\right)\left(1-a_1\right)\kappa_1 - a_2\kappa_2}{a_1-a_2}, & \Theta_{22} &= \frac{\left(a_1+M\right)\left(1-a_2\right)\kappa_1 - a_2\kappa_2}{a_1-a_2}, \end{aligned} \tag{82}$$

and the coefficients of the parametric excitation

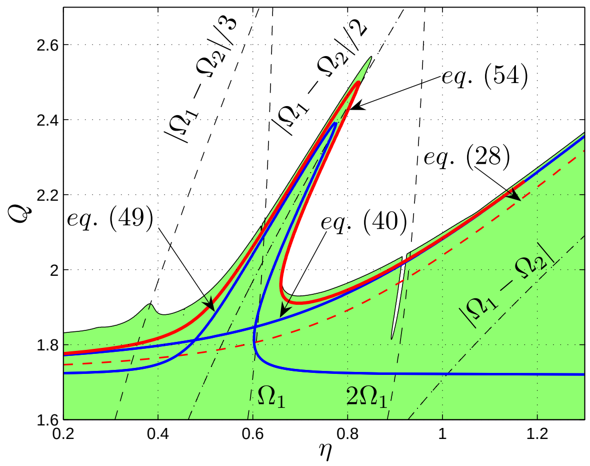


Fig. 4. System 1: Combination frequency of order $n = 2$, $\varepsilon = 0.4$. Eq. (40) indicates the first order approximation, eqs. (40) and (49) the second order approximation near η_1 and η_2, respectively. Eq. (54) combines the effects.

$$
\begin{aligned}
Q_{11} &= Q^2 \frac{-(a_2+M)(1-a_1)}{a_1-a_2}, \quad Q_{12} = Q^2 \frac{-(a_2+M)(1-a_2)}{a_1-a_2}, \\
Q_{21} &= Q^2 \frac{(a_1+M)(1-a_1)}{a_1-a_2}, \quad Q_{22} = Q^2 \frac{(a_1+M)(1-a_2)}{a_1-a_2}.
\end{aligned}
\tag{83}
$$

Using the relations $M = -a_1 a_2$ and $Q^2(1-a_1)(1-a_2) = 1$ simplifies the off-diagonal expressions in (83) to

$$
Q_{21} = \frac{a_1}{a_1-a_2}, \qquad Q_{12} = \frac{-a_2}{a_1-a_2}. \tag{84}
$$

The following non-dimensional characteristic system parameters, as defined in eq. (66), are used as default values, see [13],

$$
M = 0.1, \qquad \kappa_1 = 0.05, \qquad \kappa_2 = -0.05. \tag{85}
$$

The damping coefficients κ_i satisfy the main stability condition in eq. (26a) and eq. (32a)

$$
\Theta_{11} + \Theta_{22} = (1+M)\,\kappa_1 + \kappa_2 > 0, \tag{86}
$$

according to eq. (82). The critical system parameters M_{crit} and Q_{crit} are determined, in analogy to System 1. In contrast to System 1, System 2 possesses two bifurcation points in the parameter Q within the parameter range of interest, which are

$$
Q_{crit,1} = 0.934 \qquad \text{and} \qquad Q_{crit,2} = 0.964. \tag{87}
$$

At these critical values at least one of the modal damping parameters Θ_{ii} vanishes.

A stability domain for the set of parameters eq. (85) in dependency of the frequency ratio Q and the control parameters ε and η is shown in Fig. 5. Here the critical values in eq. (87) are visualised. Slices of this stability domain are displayed in Fig. 6. Again, a parametric anti-resonance frequency near $|\Omega_1 - \Omega_2|$ stabilises System 2. For higher values of ε, an additional region of stability is achieved by a parametric anti-resonance frequency near $|\Omega_1 - \Omega_2|/2$. In contrast to System 1, the first order approximation in eq. (28) is not able to reproduce the numerical stability boundaries. The second order approximation in eq. (40) is necessary even for small values of ε, see Fig. 6a. This approximation introduces additional shifts $\beta^{(0)}$, $\beta^{(1)}$ so that the analytical stability boundary coincides with the numerical one for small values of ε. For the example system considered, the expression $\beta^{(1)}$ is negligible near $\sigma = 0$ and dominant near $\Theta_{11} = 0$. For higher values of ε the benefit of the second order approximation becomes more evident, see Figs. 6b and c. As expected, for very high values of ε, like in Fig. 6d, even a second order approximation is not sufficient to model the stability boundary and a higher order averaging would be necessary. Summarising, while the first order contribution for System 1 in Fig. 3 is very dominant and the second order contribution is negligible, for System 2 in Fig. 6 the first and second order contributions are of similar importance.

6. Conclusions

First and second order averaging of the stability of self-excited and parametrically excited two degrees of freedom systems is investigated. Analytical stability boundaries are derived for a general periodic shape function that can be represented by a Fourier series. The necessity of a second order approximation is emphasised for two mechanical example systems that were studied in the literature before. This higher order approximation improves considerably the prediction of the stability boundary in [13] corresponding to the parametric resonance and anti-resonance frequencies $|\Omega_1 \mp \Omega_2|$. An additional shift of the skeleton line of the first order approximation is found that overcomes the error of the first order approximation. Furthermore, the second order approximation enables a prediction of the stability boundary near the parametric frequencies of second order, $|\Omega_1 \mp \Omega_2|/2$. It is shown that for all known cases and ε small a second order approximation is very accurate.

The presence of the instability domains raises interesting questions regarding the behaviour of the mechanical models for the corresponding values of the parameters. Nonlinear terms have to be included for such an analysis and various bifurcations can be expected, as in [16]. This will be part of future research.

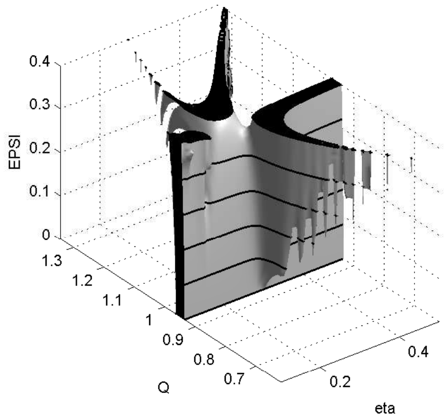
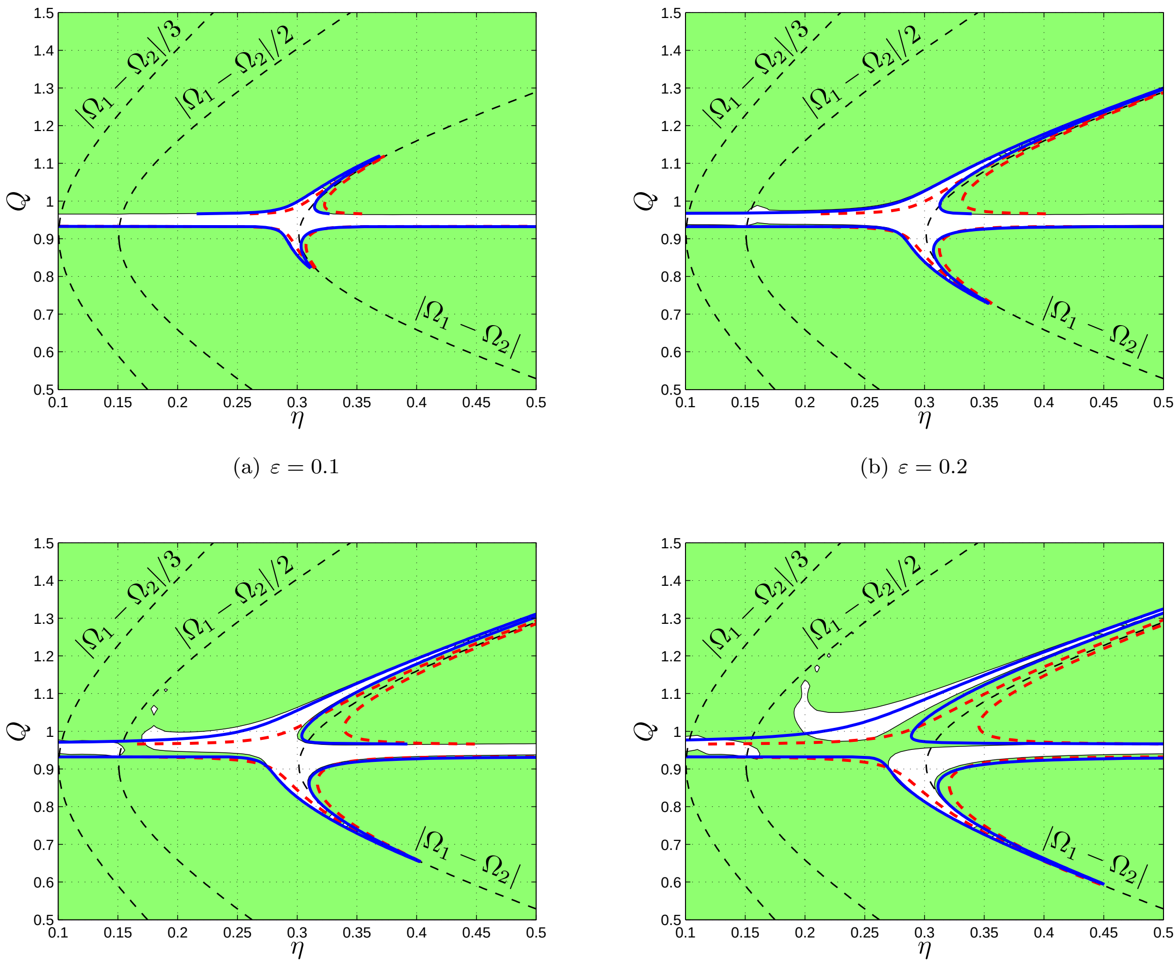


Fig. 5. System 2: Numerical stability domain in dependency of Q, ε and η



(a) $\varepsilon = 0.1$

(b) $\varepsilon = 0.2$

(c) $\varepsilon = 0.3$

(d) $\varepsilon = 0.4$

Fig. 6. System 2: Stability maps for different scaling factor ε: Comparison between numerical (shaded area is unstable) and analytical first order (solid line) and second order approximation (dashed line). $M = 0.1$

References

[1] Abadi, *Nonlinear dynamics of self-excitation in autoparametric systems*, (PhD thesis, Utrecht University, 2003).

[2] H. Bilharz, Bemerkung zu einem Satze von Hurwitz (in German), *Zeitschrift für angewandte Mathematik und Mechanik (ZAMM), Applied Mathematics and Mechanics* **24**(2) (1944) 77–82.

[3] R. Bishop, *Vibration*, Cambridge University Press, 1979.

[4] I. Blekhman, *Selected topics in vibrational mechanics*, (World Scientific, Stability, vibration and control of systems **11**, 2nd ed., 2004).

[5] V.V. Bolotin, *Dynamic stability of elastic systems*, (Holden-Day, San Francisco, 1964).

[6] M. Cartmell, *Introduction to linear, parametric and nonlinear vibrations*, (Chapman and Hall, 1990).

[7] F. Dohnal, *Application of the averaging method on parametrically excited 2dof-systems*, (Technical Report, Institute for Machine Dynamics and Measurement, Vienna University of Technology, Austria, 2003).

[8] F. Dohnal, Suppression of self-excited vibrations by synchronous stiffness and damping variation, in: L. Pesek, ed., *Proceeding Dynamics of Machines* (Prague, 2005) 23 – 30.

[9] F. Dohnal, H. Ecker and A. Tondl, Vibration control of self-excited oscillations by parametric stiffness excitation, in: N.I. Ivanov, M.J. Crocker, eds.,*Proceeding of the Eleventh International Congress of Sound and Vibration (ICSV11)* (St. Petersburg, Russia, 2004) 339 – 346.

[10] F. Dohnal, *Damping of mechanical vibrations by parametric excitation*, (PhD thesis, Vienna University of Technology, 2005).

[11] F. Dohnal, W. Paradeiser and H. Ecker, Experimental Study on Cancelling Self-Excited Vibrations by Parametric Excitation, in: *Proceedings of ASME 2006 InternationalMechanical Engineering Conferences and Exposition* (Chicago, IL, 2006).

[12] F. Dohnal, Damping by parametric stiffness excitation – resonance and anti-resonance, *Journal of Vibration and Control* (2007) accepted.

[13] Ecker, H. and Tondl, A., 2000, "A Suppression of Flow-induced Vibrations by A Dynamic Absorber with Parametric Excitation", in Proceedings of 7th Int. Conf. on Flow-Induced Vibrations, FIV2000, Lucerne, Switzerland, June 19-21, pp.757-763.

[14] N. Eicher, *Einführung in die Berechnung parametererregter Schwingungen* (in German), (Technical University of Berlin, TUB-Documentation, 1981).

[15] M. Farkas, *Periodic motions*, Springer, Berlin, 1994.

[16] S. Fatimah and F. Verhulst, Suppressing flow-induced vibrations by parametric excitation, *Nonlinear Dynamics* **23** (2003) 275–297.

[17] F. Gantmacher, *Matrizenrechnung Teil II, spezielle Fragen und Anwendungen* (in German), (Hochschulbücher für Mathematik, VEB Deutscher Verlag der Wissenschaften, Berlin, 1966).

[18] D. Green and W.G. Unruh, The failure of the Tacoma Bridge: A physical model. *Am. J. Phys.* **74**(8) (2006) 706–716.

[19] E. Hairer and G. Wanner, *Solving ordinary differential equations. Stiff and differential-algebraic problems.* (2nd edition, Springer Series in Computational Mechanics 14, Springer Verlag, 1996).

[20] K. Magnus and K. Popp. *Schwingungen*, Teubner Studienbücher, Mechanik, 1997.

[21] A.A. Mailybayev and A.P. Seyranian, Parametric resonance in systems with small dissipation, *J. Appl. Maths Mechs* **65**(5) (2001) 755–767.

[22] A. Mailybaev, On stability domains of non-conservative systems under small parametric excitation, *Acta Mechanica* **154**(1-4) (2002) 11–33.

[23] K. Makihara, H. Ecker and F. Dohnal, Stability analysis of open-loop stiffness control to suppress self-excited vibrations, *Journal of Vibration and Control* **11** (2005) 643–669.

[24] M. Marden, *Geometry of polynomials*, (Mathematical Surveys **3**, American Mathematical Society, Providence, Rhode Island, 1966).

[25] The Mathworks, 2006, MATLAB Ver.7.2, Natick, USA.

[26] R. Mickens, *An introduction to nonlinear oscillations*, (Cambridge Press, 1981).

[27] P. Müller and W. Schiehlen, *Linear vibrations: a theoretical treatment of multi-degree-of-freedom vibrating systems*, Martinus Nijhoff Publishers, Kluwer Academic, 1985.

[28] A. Nayfeh, *Nonlinear Interactions, Analytical, Computational and Experimental Methods*, (Wiley Interscience, 2000).

[29] A.H. Nayfeh and D.T. Mook, *Nonlinear Oscillations*, (John Wiley and Sons, New York, 3rd edition, 1995).

[30] W. Paradeiser, *Experimenteller Nachweis einer Parameter-Antiresonanz* (in German), (Diploma Thesis, Vienna University of Technology, 2006).

[31] A. Seyranian and A. Mailybaev, *Multiparameter stability theory with mechanical applications*, (World Scientific, *Stability, vibration and control of systems* **13**, 2003).

[32] S.C. Sinha and E. Butcher, Symbolic computation of fundamental solution matrices for linear time-periodic systems, *Journal of Sound and Vibration* **206** (1997) 61–85.

[33] J.J. Thomsen, Some general effects of strong high-frequency excitation: stiffening, biasing and smoothening, *Journal of Sound and Vibration* **253**(4) (2002) 807–831.

[34] J.J. Thomsen, *Vibrations and stability – Advanced theory, analysis and tools*, (Springer Verlag, Berlin Heidelberg New York, 2nd edition, 2003).

[35] A. Tondl, *On the interaction between self-excited and parametric vibrations*, (National Research Institute for Machine Design, Prague, Monographs and Memoranda **25**, 1965).

[36] A. Tondl, *Quenching of self-excited vibrations*, (Academy of sciences, Czech Republic, 1991).

[37] A. Tondl, To the problem of quenching self-excited vibrations, *Acta Technica CSAV* **43** (1998) 109–116.

[38] A. Tondl, Combination resonances and anti-resonances in systems parametrically excited by harmonic variation of linear damping coefficients, *Acta Technica CSAV* **3** (2003) 239–248.

[39] A. Tondl and H. Ecker, Cancelling of self-excited vibrations by means of parametric excitation, in *Proceeding of the 1999 ASME Design Engineering Technical Conference (DETC99), VIB-8071* (Las Vegas, Nevada, 1999) 1–9.

[40] F. Verhulst, *Nonlinear differential equations and dynamical systems*, (Springer-Verlag, Berlin Heidelberg New York, *Texts in applied mathematics* **50**, 2000).

[41] F. Verhulst, *Methods and applications of singular perturbations. Boundary layers and multiple timescale dynamics*, (Springer-Verlag, Berlin Heidelberg New York, *Texts in Applied Mathematics* **50**, 2005).

[42] W.T. Wu, J.H. Griffin and J.A. Wickert, Perturbation method for the Floquet eigenvalues and stability boundary of periodic linear systems, *Journal of Sound and Vibration* **182**(2) (1995) 245–257.

[43] V.A. Yakubovich and V.M. Starzhinskii, *Linear differential equations with periodic coefficients*, **1** and **2**, (Wiley, London, 1975).

[44] J. Warminski and J.M. Balthazar, Vibrations of a parametrically and self-excited system with ideal and non-ideal energy sources, *J. Braz. Soc. Mech. Sci. & Eng.* **25**(4) (2003) 413–420.

[45] J. Wittenburg, *Schwingungslehre: lineare Schwingung, Theorie und Anwendung*, Springer Verlag, 1996.

Appendix A.

According to [2] and [17], for a complex coefficient matrix

$$\mathbf{B} = \begin{bmatrix} B_{11}^r + jB_{11}^i & B_{12}^r + jB_{12}^i \\ B_{21}^r + jB_{21}^i & B_{22}^r + jB_{22}^i \end{bmatrix} \tag{A.1}$$

the Routh-Hurwitz criterion stability conditions become

$$\Delta_1: \quad a_1 > 0, \qquad \Delta_2: \quad a_1^2 a_0 - b_0^2 + a_1 b_0 b_1 > 0, \tag{A.2a}$$

with the abbreviations

$$a_1 = -B_{11}^r - B_{22}^r, \qquad a_0 = B_{11}^r B_{22}^r - B_{11}^i B_{22}^i - B_{12}^r B_{21}^r + B_{12}^i B_{21}^i, \tag{A.3a}$$

$$b_1 = -B_{11}^i - B_{22}^i, \qquad b_0 = B_{11}^r B_{22}^i + B_{11}^i B_{22}^r - B_{12}^r B_{21}^i - B_{12}^i B_{21}^r. \tag{A.3b}$$