# Emergence of slow manifolds in nonlinear wave equations

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**Abstract**

Averaging-normalization, applied to weakly nonlinear wave equations provides a tool for identification of slow manifolds in these infinite-dimensional systems. After discussing the general procedure we demonstrate its effectiveness for a Rayleigh wave equation to find low-dimensional invariant manifolds.

Key words: slow manifold, wave equation, Rayleigh, normalization

## 1 Singular perturbations and slow manifolds

Consider the equation

$$\dot{x} = f(x) + \varepsilon g(x)$$

with $\varepsilon$ a small, positive parameter. Suppose that the equation $\dot{y} = f(y)$ that arises if $\varepsilon = 0$, contains an invariant manifold $M_0$. One of the basic questions is, does this invariant manifold persist if $\varepsilon > 0$ in (slightly deformed) shape $M_\varepsilon$? For ODEs, there are many results, see for a survey [13], but for PDEs the literature is still restricted.

The existence and approximation of invariant manifolds of differential equations is strongly related to hyperbolicity properties of $M_0$. In the case of singular perturbations of ODEs with initial values, these hyperbolicity properties are directly related to the attraction properties of the regular (outer) expansion; this also plays an essential part in the actual asymptotic approximations. Under additional assumptions such a regular expansion is associated with the existence of a so-called slow manifold.

Theorems by Tikhonov, O'Malley-Vasil'eva provide the foundations for asymptotic approximations. The theoretical basis for the existence of slow manifolds in ODEs was given by Fenichel; see for references, details and applications [5], [7] and [14].

To be explicit, consider the autonomous system

$$\begin{aligned} \dot{x} &= \varepsilon f(x, y) + \varepsilon^2 \cdots, \quad x \in D \subset \mathbb{R}^n, \\ \dot{y} &= g(x, y) + \varepsilon \cdots, \quad y \in G \subset \mathbb{R}^m, \end{aligned}$$

where the dot denotes differentiation with respect to $t$, $\varepsilon$ is a small, positive parameter. Putting $\varepsilon = 0$ we find from the second equation a family of equilibria given by $g(x, y) = 0$ with $x$ a parameter. The basic assumptions are that all real parts of the eigenvalues of the linearization of the zero set with respect to $y$ are nonzero and that the zero set corresponds with a compact manifold in $\mathbb{R}^{n+m}$. In this case the zero set $y = \phi(x)$ of $g(x, y)$ corresponds with a first-order approximation $M_0$ of the $n$-dimensional (slow) manifold $M_\varepsilon$.

Usually, $y$ is called the fast variable and $x$ the slow variable. In this case of a slow-fast system, because of the presence of the parameter $\varepsilon$, the slow manifold $M_0$ is *normally hyperbolic*.

If $M_0$ is a compact manifold that is normally hyperbolic, it persists for $\varepsilon > 0$, i.e., there exists for sufficiently small, positive $\varepsilon$ a smooth manifold $M_\varepsilon$ close to $M_0$. Corresponding with the signs of the real parts of the eigenvalues, there exist stable and unstable manifolds of $M_\varepsilon$, smooth continuations of the corresponding manifolds of $M_0$, on which the flow is fast.
This idea has been very fruitful for finite dimensional systems; in this paper we will discuss extension to infinite dimensional problems. It turns out that the dimension of the invariant manifolds obtained in this way, is usually much lower than the dimension estimates obtained for inertial manifolds.

## 2 Extension to PDEs

Extension to infinite-dimensional problems is possible but raises special difficulties, depending on the choice of operator and the type of problem formulation. A paper discussing parabolic and hyperbolic problems is [1], see also [2], [3] and [15]. In [6] the emphasis is on the persistence of invariant manifolds in dissipative equations. A prominent technique is contraction which takes often the form of Gronwall's lemma.
We will briefly discuss the results of [2] and [3] where parabolic PDEs have been considered. The equations and their solutions are associated with a Banach space $X$ and a $C^1$ semiflow defined on $X$. First one has to identify a compact, connected invariant manifold $M_0$ of the flow that is normally hyperbolic. One has to prove then that $M_0$ persists under perturbations of the semiflow where *persistence* again means 'small quantitative deformation of $M_0$ without qualitative (topological) changes'. In the case of parabolic equations we have the additional problem that backwards solutions may not exist, so certain maps associated with these equations are not invertible. Also the finite dimensional geometric tools of dynamical systems theory like the smooth continuation of tangent bundles, have to be developed for infinite dynamical systems. Another problem is the lack of compactness that is typical for these systems.
Applications often refer to systems of the form

$$u_t = \Delta u + f(u) + \varepsilon \cdots ,$$

where $u$ is an $n$-vector, $f(u)$ represents the nonlinear terms. The manifold $M_0$ is identified for $\varepsilon = 0$ and conditions are provided in [2] and [3] so that it persists for $\varepsilon > 0$.

## 3 Formulation for wave equations

For wave equations, a suitable linear operator usually generates a group instead of a semigroup. Variation of constants enables us to formulate a slowly varying system that permits normalization. This facilitates the identification of the invariant manifold $M_0$ and the persistence properties. This idea was first explored in [10] with an application to a parametrically excited wave equation:

$$u_{tt} - c^2 u_{xx} + \varepsilon\beta u_t + (\omega_0^2 + \varepsilon\gamma \cos t)u = \varepsilon\alpha u^3,\ t \geq 0, 0 < x < \pi, \tag{1}$$

with Neumann boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and $\beta > 0$ (damping). In [11] the experimental motivation for this model is discussed, for instance a line of coupled pendula with vertical (parametric) forcing or the behavior of water waves in a vertically forced channel. The analysis of slow manifolds of this equation can be found in [10] together with the bifurcational behavior of the invariant manifolds.
More in general, consider semilinear initial value problems of hyperbolic type,

$$u_{tt} + Au = \varepsilon f(u, u_t, t, \varepsilon), \quad u(0) = u_0, u_t(0) = v_0, \tag{2}$$

where $A$ is a positive, self-adjoint linear operator on a separable Hilbert space and $f$ will be specified. Here we will be concerned with the case that we have one space dimension and that for $\varepsilon = 0$ we have a linear, dispersive wave equation by choosing:

$$Au = -u_{xx} + u.$$

To produce a system in vector form, one writes

$$u = u_1,\ \frac{\partial u_1}{\partial t} = u_2,\ \frac{\partial u_2}{\partial t} = -Au_1 + \varepsilon f(u_1, u_2, t, \varepsilon),$$

and uses the operator (with eigenvalues and eigenfunctions), associated with this system.
To focus ideas, consider the case of boundary conditions $u(0,t) = u(\pi,t) = 0$.
In this case, a suitable domain for the eigenfunctions is $\{u \in W^{1,2}(0,\pi) : u(0) = u(\pi) = 0\}$. Here $W^{1,2}(0,\pi)$ is the Sobolev space consisting of functions $u \in L_2(0,\pi)$ that have first-order generalized derivatives in $L_2(0,\pi)$. The eigenvalues are $\lambda_n = \omega_n^2 = \sqrt{n^2+1}, n = 1, 2, \cdots$ the corresponding eigenfunctions $v_n(x) = \sin(nx)$ and the spectrum is nonresonant.
Substitution into Eq. (2) the expansion

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) v_n(x)$$

and taking inner products with $v_m(x)$, $m = 1, 2, \cdots$, produces the infinite set of coupled second-order equations

$$\ddot{u}_n + \omega_n^2 u_n = \varepsilon F(\mathbf{u}), \tag{3}$$

with $\mathbf{u}$ representing the vector with elements $u_n$, $n = 1, 2, 3, \cdots$.
The next step is to transform system (3) into a slowly varying system by the (variation of constants) phase-amplitude transformation

$$u_n(t) = r_n(t)\cos(\omega_n t + \psi_n(t)),\ \dot{u}_n(t) = -\omega_n r_n(t)\sin(\omega_n t + \psi_n(t)). \tag{4}$$

The resulting system is of the form

$$\dot{r}_n = \varepsilon F_1(r_n, \psi_n, t),\ \dot{\psi}_n = \varepsilon F_2(r_n, \psi_n, t),\ n = 1, 2, \cdots \tag{5}$$

with $(F_1(r_n, \psi_n, t), F_2(r_n, \psi_n, t))$ an almost-periodic function in a Banach space, satisfying Bochner's criterion, see for instance [14]. Its average $(F_1^0, F_2^0)$ is defined by:

$$(F_1^0, F_2^0)(r_n, \psi_n) = \lim_{T\to\infty} \frac{1}{T}\int_0^T (F_1(r_n, \psi_n, s), F_2(r_n, \psi_n, s))ds. \tag{6}$$

We can apply normalization by the averaging transformation, see [14] or [12]. An explicit example will be given in the next section. The normalized system will be of the form

$$\dot{z} = \varepsilon G(z) + O(\varepsilon^2) \tag{7}$$

with $z$ representing the infinite dimensional system of transformed phases and amplitudes. After introducing the normalizing transformation, we can still in principle obtain the exact solution by solving the resulting Eq. (7) including the $O(\varepsilon^2)$-terms to find $r_n(t)$ and $\psi_n(t)$. 'In principle', because in nearly all cases the solution of the full system can not be given explicitly. Omitting however the $O(\varepsilon^2)$-terms and solving the resulting equations produces an approximation of the solutions in the following sense:

Assume that the righthandside vector fields of system (5) are continuously differentiable and uniformly bounded on $\bar{D} \times [0,\infty) \times [0,\varepsilon_0]$, where $D$ is an open, bounded set in a suitable Banach space $X$. Solving system (7) to $O(\varepsilon)$ produces and $o(1)$-approximation of $r_n(t), \psi_n(t)$, valid on the timescale $1/\varepsilon$.

For a more precise formulation see [4] or [14].
Suppose that not all the linear normal modes, i.e. the decoupled one degree of freedom solutions in the case $\varepsilon = 0$ (filling two-dimensional manifolds), are solutions of system (3) for $\varepsilon > 0$, but applying averaging-normalization, the normal modes are solutions of the averaged system to $O(\varepsilon)$. The Lyapunov manifolds, smooth continuations for $\varepsilon > 0$ of the two-dimensional normal mode

manifolds, persist for the original system (2) or (3) if we have normal hyperbolicity of the normal mode solutions of the averaged system. In an amplitude-phase representation, this happens for instance if we have attraction in the amplitude equations and parallel flow from the phase equation to $O(\varepsilon)$.
We also have to check that in the spectrum the eigenvalues are sufficiently separated, preferably by a gap size, independent of the mode number.

# 4 A nonlinear Rayleigh wave equation

A benchmark example of a nonlinear wave equation was studied by Keller and Kogelman in [8], who consider a Rayleigh type of excitation described by the equation

$$u_{tt} - u_{xx} + u = \varepsilon \left( u_t - \frac{1}{3} u_t^3 \right), \; t \geq 0, 0 < x < \pi, \tag{8}$$

with boundary conditions $u(0,t) = u(\pi,t) = 0$ and initial values $u(x,0) = \phi(x), u_t(x,0) = \psi(x)$ that are supposed to be sufficiently smooth. Apart from being a classical example, the equation plays an essential part in modeling self-excited vibrations of waves produced by an external wind field or other types of fluid flow perturbations.
The authors of [8] use multiple timing to first order, which yields the same results as averaging. We have for the eigenfunctions and eigenvalues

$$v_n(x) = \sin(nx), \; \lambda_n = \omega_n^2 = n^2 + 1, \; n = 1, 2, \cdots,$$

and to perform our averaging-normalization scheme,we propose to expand the solution of the initial boundary value problem in a Fourier series with respect to these eigenfunctions. Substituting the expansion $\sum u_n(t) v_n(x)$ into the differential equation, we have

$$\sum_{n=1}^{\infty} \ddot{u}_n \sin nx + \sum_{n=1}^{\infty} (n^2+1) u_n \sin nx = \varepsilon \sum_{n=1}^{\infty} \dot{u}_n \sin nx - \frac{\varepsilon}{3} (\sum_{n=1}^{\infty} \dot{u}_n \sin nx)^3.$$

When taking $L_2$-inner products with $\sin mx, m = 1, 2, \cdots$, it is shown in [14] that we find the system

$$\ddot{u}_m + \omega_m^2 u_m = \varepsilon \left( \dot{u}_m - \frac{1}{4} \dot{u}_m^3 - \frac{1}{2} \sum_{i \neq m}^{\infty} \dot{u}_i^2 \dot{u}_m \right) + \varepsilon \cdots, \; m = 1, 2, \cdots$$

where the dots stand for nonresonant terms.
This infinite system of ordinary differential equations is equivalent to the original problem. For the variation of constants transformation we have to avoid amplitude-phase variables as the transformation is singular for normal modes. We have checked however, that amplitude-phase variables produce the same results. To start with, we use the transformation

$$u_n(t) = a_n(t) \cos \omega_n t + b_n(t) \sin \omega_n t, \tag{9}$$

$$\dot{u}_n(t) = -\omega_n a_n(t) \sin \omega_n t + \omega_n b_n(t) \cos \omega_n t. \tag{10}$$

After averaging, more insight is obtained by using amplitude-phase variables $r_n, \psi_n$ from (4). Putting

$$a_n^2 + b_n^2 = r_n^2 = E_n, \; n = 1, 2, \cdots,$$

we find

$$\dot{E}_n = \varepsilon E_n \left( 1 + \frac{n^2+1}{16} E_n - \frac{1}{4} \sum_{k=1}^{\infty} (k^2+1) E_k \right) + O(\varepsilon^2), \; \dot{\psi}_n = O(\varepsilon^2).$$

We have kept the same notation for the variables after normalization. To obtain asymptotic approximations we omit the $O(\varepsilon^2)$ terms, replacing $E_n, \psi_n$ by their approximations $\bar{E}_n, \bar{\psi}_n$. We have immediately a nontrivial result: starting in a mode with zero energy, this mode will not be excited on a timescale $1/\varepsilon$. Another observation is that if we have initially only one nonzero mode (a normal mode), say for $n = m$, the equation for $E_m$ becomes

$$\dot{E}_m = \varepsilon E_m \left(1 - \frac{3}{16}(m^2+1)E_m\right) + O(\varepsilon^2).$$

We conclude that in the one-mode case we have stable equilibrium at the value

$$E_m^* = \frac{16}{3(m^2+1)} + O(\varepsilon).$$

The results until this point can be found in the literature, usually without error estimates.

We note, that the averaging theorem formulated above, yields that the approximate solutions have precision $o(\varepsilon)$ on the timescale $1/\varepsilon$. It can be shown that if we start with initial conditions in a finite number of modes, the error is $O(\varepsilon)$ (see [14]).
Slow manifold theory enables us to formulate stronger results. It follows from the normal hyperbolicity of the normal modes ($E_m^*$ has eigenvalue $-\varepsilon$, the infinite number of other modes grow with eigenvalue $+\varepsilon$), that for the original wave equation (8), for $\varepsilon > 0$ but small, an infinite number of two-dimensional, unstable Lyapunov manifolds exist $\varepsilon$-close to the normal mode coordinate planes, containing the normal mode solutions in the case $\varepsilon = 0$.
It is not difficult to extend this to cases with more dimensions. Consider for instance the case of excitation of two modes, $k$ and $m$. Putting $E_n = 0$, $n \neq k, m$, the system becomes

$$\begin{aligned} \dot{E}_k &= \varepsilon E_k \left(1 - \frac{3}{16}(k^2+1)E_k - \frac{1}{4}(m^2+1)E_m\right) + O(\varepsilon^2), \\ \dot{E}_m &= \varepsilon E_m \left(1 - \frac{3}{16}(m^2+1)E_m - \frac{1}{4}(k^2+1)E_k\right) + O(\varepsilon^2). \end{aligned}$$

Considering the system for $E_k, E_m$, we find four critical points and three heteroclinic connections: $(0,0)$ unstable node; $(\frac{16}{3(k^2+1)}, 0)$ stable node; $(0, \frac{16}{3(m^2+1)})$ stable node; $(E_k^*, E_m^*)$ with $E_k^* E_m^* > 0$, saddle corresponding with an unstable 2-torus in the original system.
The saddle point has heteroclinic connections with the other three critical points.
The two degrees of freedom system in the full system is normally hyperbolic. Again we conclude from slow manifold theory, that for the original wave equation an infinite number of corresponding higher-dimensional manifolds exist.

# 5 Discussion

In the example of the Rayleigh wave equation we have found an infinite number of finite-dimensional slow manifolds. They are unstable and difficult to observe. In a number of models, an asymmetric force field plays a part, for instance in the case of galloping motion of hanging cables in a wind field. For Eq. (8) this means the addition of a term $\varepsilon f(u)$ with $f(u)$ an even function that can be expanded in a Taylor series in $u$. In particular we have $f(0) = 0, df/du(0) = 0$ with examples like $f(u) = u^2$, $u^2 - u^4$ or $u \sin u$. The normal form analysis to first order does not change and we draw the same conclusions as in section 4.
To deal with other wave equations, the difficulty is often the number of possible resonances. For instance, making the Rayleigh wave equation nondispersive (omitting $u$ on the lefthand side in Eq. (8)) produces a very complicated normal form. Valid asymptotic approximations can be obtained by truncation methods, but one loses then the results on the existence of slow manifolds.
The parametrically excited wave equation mentioned in section 3 is even more interesting as it shows bifurcations for rather small values of $\varepsilon$ involving the interaction of a finite number of modes as found experimentally in [11]. This analysis can be found in [10].

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