Brouwer's rotating vessel I: stabilization

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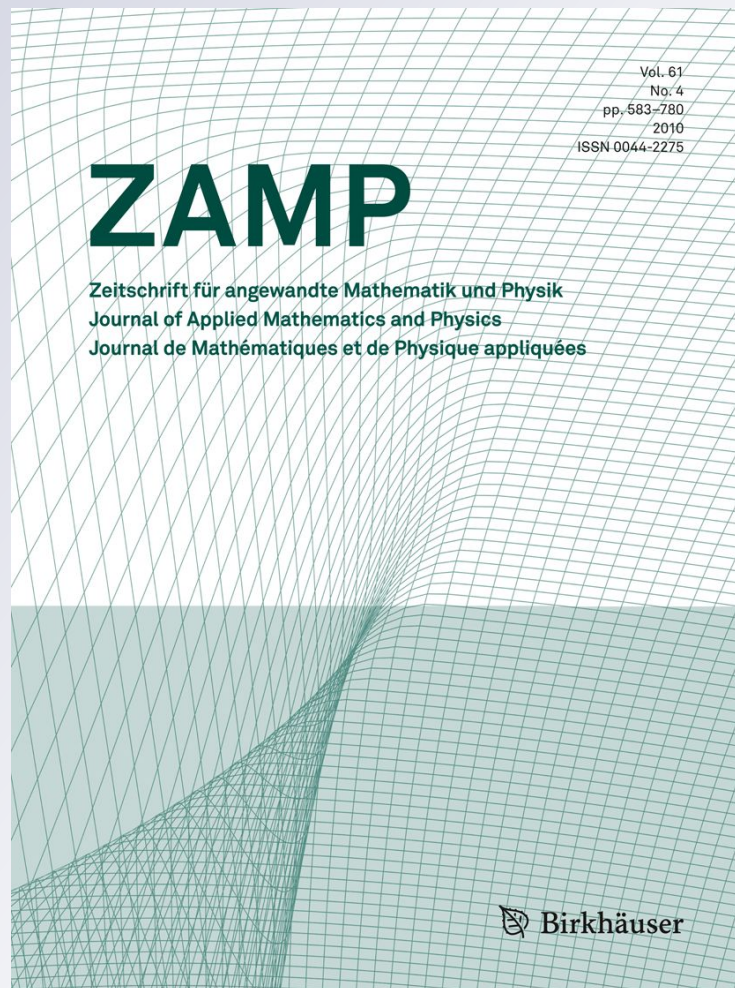Springer

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Abstract. The paradox of destabilization of a conservative or non-conservative system by small dissipation, or Ziegler's paradox (1952), has stimulated an interest in the sensitivity of reversible and Hamiltonian systems with respect to dissipative perturbations. We discuss the motion of a particle in Brouwer's rotating vessel, a typical gyroscopic system, that has an unstable equilibrium caused by internal damping for a wide range of rotation velocities. Using quasi-periodic averaging-normalization by MATHEMATICA, we find that modulation of the rotation frequency in the cases of single-well and saddle equilibria stabilizes the system for a number of combination resonances, thus producing quenching of the unstable motion.

Mathematics Subject Classification (2000). 34D · 70J.

Keywords. Dissipation-induced instabilities · Whitney's umbrella · Brouwer's rotating vessel · Gyroscopic quenching.

1. Introduction

A fascinating category of mechanical and physical systems exhibits the following paradoxical behavior: when modeled as systems without damping, they possess stable equilibria or stable steady motions, but when small damping is introduced, some of these equilibria or steady motions become unstable.

The paradoxical effect of damping on dynamic instability was noticed first for rotor systems, which have stable steady motions for a certain range of speed, but which become unstable when the speed is changed to a value outside the range. An example can be found in the classic by Thomson and Tait [19] who showed that a statically unstable conservative system which has been stabilized by gyroscopic forces could be destabilized again by the introduction of small damping forces. Later examples are given by Kimball [10] who studied the destabilization of a flexible rotor in stable rotation at a speed above the critical speed for resonance, see also Smith [18] and Kapitsa [9].

In mechanical engineering, the phenomenon later became known as *Ziegler's paradox*; see [22,23], papers that became classical and widely known in the community of mechanical engineers; it also attracted the attention of mathematicians.

The crucial ideas for the explanation of these destabilization phenomena were formulated by Bottema in [1,2]. In the context of bifurcation theory, a geometrical explanation can be given in terms of the so-called Whitney umbrella [21] and singularities of maps from $\mathbb{R}^n$ into $\mathbb{R}^m$ with $m = 2n-1$; see Fig. 1. This relation was pointed out in [8]. In the case of two degrees of freedom with an equilibrium (singularity) at the origin, the bifurcation manifold is locally described by the cubic surface

$$y_1 = x_1^2, \quad y_2 = x_2, \quad y_3 = x_1 x_2$$

or by elimination

$$y_1 y_2^2 - y_3^2.$$

As shown in [1,2] and [8], using the eigenvalue equation in the case of a gyroscopic system with damping, the same geometric structure determines the stability transitions of an equilibrium in such a mechanical system. The part of x (or y) is played by the natural parameters of the system.
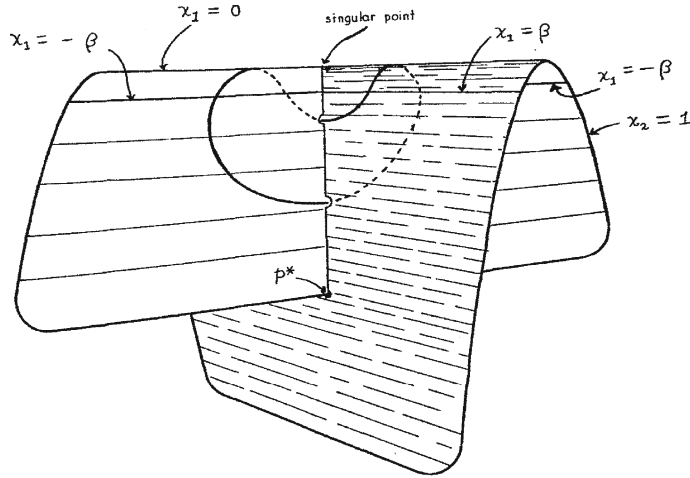
FIG. 1. Whitney's original sketch of the umbrella [21] which indicates the bifurcation set

An important field of application where these destabilization phenomena arise is gyroscopic dynamics. Investigation into the stability of equilibria of the Hauger's [7] and Crandall's [6,15] gyropendulums as well as of the Tippe Top [4,13] and the Rising Egg [4] leads to the system of linear equations known as the modified Maxwell–Bloch equations. For the history and a survey of the literature see [11].

The purpose of the present note is twofold. We start with introducing a physically very simple model studied by Brouwer [5] that can serve as a simple illustration of destabilization by damping. Note, however that, although simple looking, the equations that arise are familiar from other applications, see for instance [12,14,17]. It is important to have mechanisms that enable us to stabilize again a gyroscopic system. For a typical case of Brouwer's rotating vessel, we show that quenching of the instability can be achieved by small modulations of the rotation speed. Several resonances play a part here.

Secondly, we are interested in what is happening when instability sets in and nonlinear terms must be taken into account. This will be the subject of a subsequent paper.

2. Brouwer's rotating vessel

In 1918, Brouwer [5] studied the equilibrium position of a point mass in a vessel, described by a smooth surface S, rotating around a vertical axis with constant angular velocity ω. With the equilibrium chosen on the vertical axis at $(x, y) = (0, 0)$ on S, the linearized equations of motion without dissipation are

$$\ddot{x} - 2\omega\dot{y} + (gk_1 - \omega^2)x = 0, \tag{1}$$
$$\ddot{y} + 2\omega\dot{x} + (gk_2 - \omega^2)y = 0. \tag{2}$$

The constants k_1 and k_2 are the x, y-curvatures of S at $(0, 0)$, g is the gravitational constant. If $k_1 < 0$, $k_2 < 0$, we have already instability without damping; we leave out this case. To avoid too many parameters, we will consider symmetric curvatures; these boundary cases got little attention up till now.

2.1. Stability and instability without damping

Assuming there is no damping, there are the following two cases:
- $k_1 = k_2 = k$, $k > 0$ (single-well at equilibrium).
  Lyapunov stability with two eigenvalues zero if $\omega^2 = gk$.

- $k_1 = -k_2 = k$, $k > 0$ (saddle at equilibrium).
  Lyapunov stability if $\omega^2 \geq gk$ (fast rotation) with two eigenvalues zero if $\omega^2 = gk$.
  Instability if $0 < \omega^2 < gk$.

2.2. Destabilization by damping

On adding constant (Coulomb) damping, Brouwer [5] finds a number of instability cases. Bottema [3] obtains similar results when introducing internal and external damping. In our study, we will focus on internal damping, considering symmetric curvatures. The equations of motion become in this case:

$$\ddot{x} - 2\omega\dot{y} + c\dot{x} + (gk_1 - \omega^2)x = 0,$$
$$\ddot{y} + 2\omega\dot{x} + c\dot{y} + (gk_2 - \omega^2)y = 0,$$

with k_1, k_2 as above. The damping constant c is positive. The characteristic (eigenvalue) equation takes the form:

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

with $a_1 = 2c$,

$a_2 = 2gk + 2\omega^2 + c^2$, single-well,
$a_2 = 2\omega^2 + c^2$, saddle,
$a_3 = 2c(gk - \omega^2)$, single-well,
$a_3 = -2c\omega^2$, saddle,
$a_4 = (gk - \omega^2)^2$, single-well,
$a_4 = \omega^4 - (gk)^2$, saddle.

Analyzing the cases again, we find:

- Single-well: asymptotic stability if $0 < \omega^2 < gk$.
  The fast rotation branch $\omega^2 > gk$ has become unstable.
- Saddle: The requirement $a_3 > 0$ is never satisfied, so the saddle is always unstable for any size of positive damping.

Note: repeating the calculations after introducing asymmetric damping, c_1 and c_2 instead of c, does not change the stability results.

3. Quenching of the unstable single-well motion

In the context of engineering, quenching of instabilities by a practical physical mechanism is important. Intuitively, it is not clear what to propose, but based on earlier studies (see [20]), we choose *modulation of the vessel rotation* as this might produce the combination resonances that may change the stability characteristics. We will focus first on the case of rotation of a single-well equilibrium. This is unstable by damping for large enough rotational velocity

$$\omega^2 > gk.$$

The damping coefficient is supposed to be small, we replace c by εc with ε a small, positive parameter. The modulation with constant amplitude a is given by

$$\omega_\varepsilon^2 = \omega^2 + 2a\varepsilon \cos \nu t$$

with again ω a constant. The equations of motion become with these assumptions:

$$\ddot{x} - 2\omega\dot{y} - (\omega^2 - gk)x = \varepsilon F_1 + O(\varepsilon^2),$$
$$\ddot{y} + 2\omega\dot{x} - (\omega^2 - gk)y = \varepsilon F_2 + O(\varepsilon^2),$$

with

$$F_1 = -c\dot{x} + \frac{2a}{\omega}\dot{y}\cos\nu t + 2ax\cos\nu t,$$

$$F_2 = -c\dot{y} - \frac{2a}{\omega}\dot{x}\cos\nu t + 2ay\cos\nu t.$$

Putting $\varepsilon = 0$ (no damping, no frequency modulation), we obtain for the frequencies near the origin:

$$\lambda_1 = \omega + \sqrt{gk}, \quad \lambda_2 = \omega - \sqrt{gk}.$$

3.1. The variational equations

Using the independent solutions of the unperturbed ($\varepsilon = 0$) system, we introduce variation of constants by the transformation $x, \dot{x}, y, \dot{y} \to A, B, C, D$:

$$x = A\cos\lambda_1 t + B\sin\lambda_1 t + C\cos\lambda_2 t + D\sin\lambda_2 t, \tag{3}$$

$$\dot{x} = -\lambda_1 A\sin\lambda_1 t + \lambda_1 B\cos\lambda_1 t - \lambda_2 C\sin\lambda_2 t + \lambda_2 D\cos\lambda_2 t, \tag{4}$$

$$y = -A\sin\lambda_1 t + B\cos\lambda_1 t - C\sin\lambda_2 t + D\cos\lambda_2 t, \tag{5}$$

$$\dot{y} = -\lambda_1 A\cos\lambda_1 t - \lambda_1 B\sin\lambda_1 t - \lambda_2 C\cos\lambda_2 t - \lambda_2 D\sin\lambda_2 t. \tag{6}$$

Introducing the transformation into the equations of motion and using matrix inversion, we find:

$$\dot{A} = -\frac{\varepsilon}{\lambda_1 - \lambda_2}(F_1\sin\lambda_1 t + F_2\cos\lambda_1 t),$$

$$\dot{B} = \frac{\varepsilon}{\lambda_1 - \lambda_2}(F_1\cos\lambda_1 t - F_2\sin\lambda_1 t),$$

$$\dot{C} = \frac{\varepsilon}{\lambda_1 - \lambda_2}(F_1\sin\lambda_2 t + F_2\cos\lambda_2 t),$$

$$\dot{D} = \frac{\varepsilon}{\lambda_1 - \lambda_2}(-F_1\cos\lambda_2 t + F_2\sin\lambda_2 t).$$

In the following, we will omit the $O(\varepsilon^2)$ terms to obtain $O(\varepsilon)$ error estimates on the timescale $1/\varepsilon$.

3.2. The combination resonance $\lambda_1 - \lambda_2 = \nu$

Substituting our choices of F_1 and F_2 and transforming by Eqs. (3–6), we have to make assumptions about the frequencies $\lambda_1, \lambda_2, \nu$ before applying averaging-normalization. It turns out that for the unstable single-well, there is only one (combination) resonance leading to synchronization:

$$\lambda_1 - \lambda_2 = \nu \quad \text{or} \quad 2\sqrt{gk} = \nu.$$

Keeping, with some abuse of notation the same variables $A, \dots, D$, the averaged-normal form equations become in the case of this resonance:

$$\dot{A} = \frac{\varepsilon}{(\lambda_1 - \lambda_2)\omega}(-c\lambda_1\omega A + a(\lambda_2 - \omega)D) + O(\varepsilon^2),$$

$$\dot{B} = \frac{\varepsilon}{(\lambda_1 - \lambda_2)\omega}(-c\lambda_1\omega B - a(\lambda_2 - \omega)C) + O(\varepsilon^2),$$

$$\dot{C} = \frac{\varepsilon}{(\lambda_1 - \lambda_2)\omega}(a(-\lambda_1 + \omega)B + c\lambda_2\omega C) + O(\varepsilon^2),$$

$$\dot{D} = \frac{\varepsilon}{(\lambda_1 - \lambda_2)\omega}(a(\lambda_1 - \omega)A + c\lambda_2\omega D) + O(\varepsilon^2).$$

Omitting the $O(\varepsilon^2)$ terms and solving the resulting normal form equations, we find approximations of A, B, C, D with an $O(\varepsilon)$ error-estimate valid on the timescale $1/\varepsilon$; in the case of asymptotic stability of
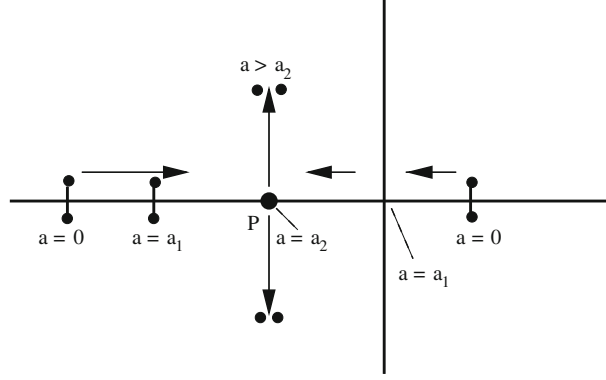
FIG. 2. Movement of the eigenvalues near equilibrium at the single-well with dissipation and small modulation of the rotation speed ω. The eigenvalues have multiplicity two; for $a = 0$ (no modulation), we have two coincident positive and two coincident negative eigenvalues (double eigenvalues are indicated by two nearby *dots*). At $a = a_1 = c^2\omega^2\frac{\omega^2-gk}{gk}$, the two positive eigenvalues arrive at the origin. For $a_1 < a < a_2(= \frac{c^2\omega^4}{gk})$, four eigenvalues are real and negative. At P (for $a = a_2$), we have 4 equal eigenvalues, for $a > a_2$ the eigenvalues move into the complex plane with negative real part

equilibrium, the estimate is valid for all time (see [16]). The variations of constants transformation (3-6) carry this error-estimate over to the variables $x, \dot{x}, y, \dot{y}$.

For the equilibrium at the origin, we find from the normal form equations double eigenvalues of the form:

$$\frac{1}{2}\left(c\omega(\lambda_2 - \lambda_1) \pm \sqrt{c^2\omega^2(\lambda_1 + \lambda_2)^2 + 4a^2(\lambda_1 - \omega)(\lambda_2 - \omega)}\right),$$

or by using the expressions for λ_1, λ_2, we have

$$-c\omega\sqrt{gk} \pm \sqrt{c^2\omega^4 - a^2gk}.$$

The equilibrium at the origin is asymptotically stable if

$$a > a_1\left(= c^2\omega^2\frac{\omega^2 - gk}{gk}\right).$$

See Fig. 2 for the movements of the eigenvalues in the complex plane.

Outside the combination resonance If we have no resonance at first-order averaging, the off-diagonal elements vanish to $O(\varepsilon)$, so that the equilibrium remains unstable. Higher-order averaging will introduce new combination resonances but this does not change the stability picture as the additional terms are $O(\varepsilon^2)$. The tuning into the combination resonance $\lambda_1 - \lambda_2 = \nu$ $(2\sqrt{gk} = \nu)$ is essential to stabilize the single-well system.

4. Quenching the unstable saddle motion

In this section, we will consider *modulation of the vessel rotation* in the unstable case of the rotation of a saddle (unstable by damping for any rotational velocity ω) while being stable without dissipation. In the case $0 < \omega^2 < gk$, we have already without damping a hyperbolic instability case; this cannot be stabilized by an $O(\varepsilon)$ perturbation, so we choose $\omega^2 > gk$. As before, we put

$$k_1 = k > 0 \quad \text{and} \quad k_2 = -k.$$

Remarkably enough, we will find that this saddle motion can be stabilized.

We put for the damping coefficient $c_1 = c_2 = \varepsilon c$; the modulation is given again by

$$\omega_\varepsilon^2 = \omega^2 + 2a\varepsilon \cos \nu t$$

with ω and ν positive constants. Most of the conclusions will be valid in an open set of parameter space.

The equations of motion become with small damping and small ω modulation:

$$\ddot{x} - 2\omega\dot{y} - \varepsilon \frac{2a}{\omega} \cos \nu t \dot{y} + \varepsilon c\dot{x} - \beta^2 x - 2\varepsilon ax \cos \nu t = O(\varepsilon^2),$$

$$\ddot{y} + 2\omega\dot{x} + \varepsilon \frac{2a}{\omega} \cos \nu t \dot{x} + \varepsilon c\dot{y} - \alpha^2 y - 2\varepsilon ay \cos \nu t = O(\varepsilon^2),$$

with

$$\alpha^2 = \omega^2 + gk, \quad \beta^2 = \omega^2 - gk.$$

More in general

$$\ddot{x} - 2\omega\dot{y} - \beta^2 x = \varepsilon F_1 + O(\varepsilon^2),$$

$$\ddot{y} + 2\omega\dot{x} - \alpha^2 y = \varepsilon F_2 + O(\varepsilon^2),$$

with in our problem

$$F_1 = -c\dot{x} + \frac{2a}{\omega}\dot{y} \cos \nu t + 2ax \cos \nu t,$$

$$F_2 = -c\dot{y} - \frac{2a}{\omega}\dot{x} \cos \nu t + 2ay \cos \nu t.$$

4.1. The variational equations

To perform averaging-normalization, we transform (variation of constants)

$$x = A \cos \alpha t + B \sin \alpha t + C \cos \beta t + D \sin \beta t,$$

$$\dot{x} = -\alpha A \sin \alpha t + \alpha B \cos \alpha t - \beta C \sin \beta t + \beta D \cos \beta t,$$

$$y = -\frac{\omega}{\alpha} A \sin \alpha t + \frac{\omega}{\alpha} B \cos \alpha t - \frac{\beta}{\omega} C \sin \beta t + \frac{\beta}{\omega} D \cos \beta t,$$

$$\dot{y} = -\omega A \cos \alpha t - \omega B \sin \alpha t - \frac{\beta^2}{\omega} C \cos \beta t - \frac{\beta^2}{\omega} D \sin \beta t.$$

If $\varepsilon = 0$, A, B, C and D will be constants.

Using the transformation in the equations of motion and omitting the $O(\varepsilon^2)$ terms, we find the conditions for $\dot{A}, \dot{B}, \dot{C}, \dot{D}$:

$$\dot{A} \cos \alpha t + \dot{B} \sin \alpha t + \dot{C} \cos \beta t + \dot{D} \sin \beta t = 0,$$

$$-\dot{A} \frac{\omega}{\alpha} \sin \alpha t + \dot{B} \frac{\omega}{\alpha} \cos \alpha t - \dot{C} \frac{\beta}{\omega} \sin \beta t + \dot{D} \frac{\beta}{\omega} \cos \beta t = 0,$$

$$-\dot{A}\alpha \sin \alpha t + \dot{B}\alpha \cos \alpha t - \dot{C}\beta \sin \beta t + \dot{D}\beta \cos \beta t = \varepsilon F_1,$$

$$-\dot{A}\omega \cos \alpha t - \dot{B}\omega \sin \alpha t - \dot{C} \frac{\beta^2}{\omega} \cos \beta t - \dot{D} \frac{\beta^2}{\omega} \sin \beta t = \varepsilon F_2.$$

Inversion of the algebraic equations for the derivatives produces the variational equations:

$$\dot{A} = -\varepsilon \frac{\alpha \sin \alpha t}{\alpha^2 - \omega^2} F_1 + \varepsilon \frac{\omega \cos \alpha t}{\beta^2 - \omega^2} F_2,$$

$$\dot{B} = \varepsilon \frac{\alpha \cos \alpha t}{\alpha^2 - \omega^2} F_1 + \varepsilon \frac{\omega \sin \alpha t}{\beta^2 - \omega^2} F_2,$$

$$\dot{C} = \varepsilon \frac{\omega^2 \sin \beta t}{\beta(\alpha^2 - \omega^2)} F_1 + \varepsilon \frac{\omega \cos \beta t}{\omega^2 - \beta^2} F_2,$$

$$\dot{D} = -\varepsilon \frac{\omega^2 \cos \beta t}{\beta(\alpha^2 - \omega^2)} F_1 + \varepsilon \frac{\omega \sin \beta t}{\omega^2 - \beta^2} F_2.$$

4.2. Averaging-normalization

Applying averaging-normalization, we have to make assumptions again about the frequencies α, β, ν. It turns out that the saddle presents a much more complicated picture than the single-well. Considering the variational equations, we have the following five resonances:

- $2\alpha = \nu$. This is a Mathieu resonance that, as we will show, does not contribute to stabilization.
- $2\beta = \nu$, also a Mathieu resonance making matters worse.
- The sum-resonance $\alpha = \beta + \nu$.
- The sum-resonance $\alpha + \beta = \nu$.
- Special resonance $\alpha = 3\beta = \beta + \nu$.

Each of these cases correspond with different synchronization scenarios.

If we assume other frequency relations than these five, we have to perform averaging-normalization to higher order, but this does not lead to stabilization. To establish the stability characteristics in the five resonance cases, we consider the normal forms of the righthand sides of $(\dot{A}, \dot{B}, \dot{C}, \dot{D})$. We will abbreviate $K = gk/\omega^2$ $(0 < K < 1)$.

4.3. The Mathieu-resonances

The case $2\alpha = \nu$ produces the normal form equation

$$\begin{pmatrix} \dot{A} \\ \dot{B} \\ \dot{C} \\ \dot{D} \end{pmatrix} = \varepsilon \begin{pmatrix} -\frac{c(2+K)}{2K} & \frac{a}{2\omega\sqrt{1+K}} & 0 & 0 \\ \frac{a}{2\omega\sqrt{1+K}} & -\frac{c(2+K)}{2K} & 0 & 0 \\ 0 & 0 & \frac{c(2-K)}{2K} & 0 \\ 0 & 0 & 0 & \frac{c(2-K)}{2K} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$$

with eigenvalues of the matrix of coefficients

$$\frac{c(2-K)}{2K}, \frac{c(2-K)}{2K}, -c\frac{2+K}{2K} + \frac{a}{2\omega\sqrt{1+K}}, -c\frac{2+K}{2K} - \frac{a}{2\omega\sqrt{1+K}}.$$

There are at least two positive eigenvalues, so we have instability.

The case $2\beta = \nu$ produces the normal form matrix of coefficients

$$\begin{pmatrix} -\frac{c(2+K)}{2K} & 0 & 0 & 0 \\ 0 & -\frac{c(2+K)}{2K} & 0 & 0 \\ 0 & 0 & \frac{c(2-K)}{2K} & -\frac{a}{2\omega\sqrt{1-K}} \\ 0 & 0 & -\frac{a}{2\omega\sqrt{1-K}} & \frac{c(2-K)}{2K} \end{pmatrix}$$

with eigenvalues

$$-\frac{c(2+K)}{2K},\; -\frac{c(2+K)}{2K},\; \frac{c(2-K)}{2K}+\frac{a}{2\omega\sqrt{1-K}},\; \frac{c(2-K)}{2K}-\frac{a}{2\omega\sqrt{1-K}}.$$

There is at least one positive eigenvalue, so we have instability.

4.4. The combination resonance $\alpha = \beta + \nu$

The normal form matrix becomes

$$\begin{pmatrix} -\frac{c(2+K)}{2K} & 0 & 0 & -\frac{a}{2\omega}\sqrt{1+K} \\ 0 & -\frac{c(2+K)}{2K} & \frac{a}{2\omega}\sqrt{1+K} & 0 \\ 0 & -\frac{a}{2\omega\sqrt{1+K}} & \frac{c(2-K)}{2K} & 0 \\ \frac{a}{2\omega\sqrt{1+K}} & 0 & 0 & \frac{c(2-K)}{2K} \end{pmatrix}$$

We find two eigenvalues with multiplicity two:

$$-\frac{c}{2}\pm\frac{1}{2K}\sqrt{4c^2-\frac{a^2}{\omega^2}K^2}.$$

The modulation coefficient a produces stability with four real negative real eigenvalues, if

$$\omega c\frac{\sqrt{4-K^2}}{K}<a<\frac{2c\omega}{K}.$$

The eigenvalues are complex with negative real part, implying stability, if

$$a>\frac{2c\omega}{K}\quad\left(=\frac{2c\omega^3}{gk}\right).$$

4.5. The combination resonance $\alpha + \beta = \nu$

The normal form matrix becomes

$$\begin{pmatrix} -\frac{c(2+K)}{2K} & 0 & 0 & \frac{a}{2\omega}\sqrt{1+K} \\ 0 & -\frac{c(2+K)}{2K} & \frac{a}{2\omega}\sqrt{1+K} & 0 \\ 0 & -\frac{a}{2\omega\sqrt{1+K}} & \frac{c(2-K)}{2K} & 0 \\ -\frac{a}{2\omega\sqrt{1+K}} & 0 & 0 & \frac{c(2-K)}{2K} \end{pmatrix}$$

We find the same double eigenvalues as in the case $\alpha = \beta + \nu$ and so the same stability conclusions. The two combination resonances enable us to stabilize the gyroscopic system. This is in agreement with the general considerations regarding the stabilization of linear systems given in [17].

In the case of the combination resonances $\alpha = \beta + \nu$ and $\alpha + \beta = \nu$, the value

$$a = \omega c\frac{\sqrt{4-K^2}}{K}$$

corresponds with two real negative eigenvalues and two eigenvalues zero. For the linear system, this boundary case gives instability. On adding nonlinear terms, we will have a two-dimensional center manifold. For both combination resonances, we conclude that for asymptotic stability the amplitude of the frequency modulation has to be larger than $2c\omega^3/gk$.

4.6. The special resonance $\alpha = 3\beta$, $\nu = 2\beta$

In this case $K = 4/5$ and the normal form matrix becomes

$$
\begin{pmatrix}
-\frac{7c}{4} & 0 & 0 & -\frac{3a}{2\omega\sqrt{5}} \\
0 & -\frac{7c}{4} & \frac{3a}{2\omega\sqrt{5}} & 0 \\
0 & -\frac{\sqrt{5}a}{6\omega} & \frac{3c}{4} & -\frac{\sqrt{5}a}{2\omega} \\
\frac{\sqrt{5}a}{6\omega} & 0 & -\frac{\sqrt{5}a}{2\omega} & \frac{3c}{4}
\end{pmatrix}
$$

The eigenvalues are

$$
\frac{1}{2}\left(-4\gamma - 5\delta \pm \sqrt{100\gamma^2 - 100\gamma\delta + 5\delta^2}\right), \; \frac{1}{2}\left(-4\gamma + 5\delta \pm \sqrt{100\gamma^2 + 100\gamma\delta + 5\delta^2}\right)
$$

with positive constants

$$
\gamma = \frac{c}{4}, \; \delta = \frac{a}{2\sqrt{5}\omega}.
$$

At least one of the eigenvalues is positive so we have instability.

5. Conclusions

We have revisited the rotating vessel problem formulated by Brouwer and the result of Bottema who in 1956 considered again destabilization by small damping; this can be interpreted by what is now called the Whitney umbrella singularity. The equations of motion are typical for gyroscopic two degrees of freedom mechanics. Interestingly, this gyroscopic problem can be stabilized again, both in the case of an unstable single-well and the unstable saddle, by introducing a small modulation of the rotation speed while tuning into an appropriate combination resonance. In a subsequent paper, we will consider the influence of nonlinear terms.

We considered symmetric curvatures and symmetric damping, but the hyperbolicity of the resulting stable equilibria implies that the results are valid in an open set of parameter space.

Acknowledgments

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