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APPROXIMATIONS OF HIGHER ORDER RESONANCES WITH AN APPLICATION TO CONTOPOULOS' MODEL PROBLEM

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SUMMARY

Higher order resonances in two degrees of freedom Hamiltonian systems are studied by using Birkhoff normalization. The normal forms can be used as a starting point to develop a theory of asymptotic approximations on the natural time-scale of the resonances. The asymptotic expressions are used to obtain a geometric picture of the flow in 4-space. An application of the theory is found in the model problem of Contopoulos for the Hamiltonian $H = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \frac{1}{2}(\omega_1^2x^2+\omega_2^2y^2) - \varepsilon xy^2$. A comparison with numerical results obtained earlier yields excellent agreement and we put Contopoulos' formal «third» integral in a new perspective.

INTRODUCTION

We consider systems with two degrees of freedom which can be described by a set of Hamilton equations. The Hamiltonian H depends on two configuration coordinates (q_1,q_2) and two momentum coordinates (p_1,p_2). The usual procedure is to expand the Hamiltonian near an equilibrium position in four-dimensional phase-space and we assume that we can write the expansion as follows

$$H(q,p) = \tfrac{1}{2}\omega_1(q_1^2 + p_1^2) + \tfrac{1}{2}\omega_2(q_2^2 + p_2^2) + \dots \qquad (1)$$

The dots denote higher order terms of the Taylor-expansion of H; we took $H(0,0) = 0$ which is no restriction of generality; ω_1 and ω_2 are real constants.

One can summarize this by saying: the Hamiltonian is in Birkhoff normal form until degree two (see for instance [1], appendix 7).

In the neighbourhood of an equilibrium point one can always transform a Hamiltonian into Birkhoff normal form until degree two unless $|\omega_1/\omega_2|=1$. The 1:1 resonance is not a higher-order resonance so that we omit this exceptional case here.

Some examples of mechanical systems which can be described by such a Hamiltonian have been treated in the thesis of Van der Burgh [2]; we shall discuss an example later on.

The higher order terms of the Taylor expansion of H produce nonlinear terms in the equations of motion. Before discussing nonlinear problems

it is useful to understand the linear problem completely; we shall study the phase-flow of the linearized system in section 2.
The starting point for the description of the nonlinear problem is to put the Hamiltonian into Birkhoff normal form (section 3); it is then possible to obtain asymptotic approximations of the phase-flow (section 4). It is useful to try to obtain a geometric picture of the flow. Pictures of the flow in 4-space are fairly complicated and in section 5 we present two different 'projections' or visualizations of the flow (figures 2 and 3). In section 6 a third way of projection is presented in figure 4.
Starting with Jeans, there have been long discussions in astrophysics on the so-called third integral of the galaxy. To illustrate some new developments in the theory, Contopoulos formulated a simple model problem; the simplicity reduces the computing effort but is otherwise deceptive as the problem contains all the essential difficulties and richness of structure of nonlinear Hamiltonian mechanics. In sections 6 and 7 we discuss the Contopoulos model problem using the theory of asymptotic approximations and we compare with results obtained earlier.

2. THE PHASE-FLOW OF THE LINEAR SYSTEM

In the linearized system the energy in each of the separate degrees of freedom is conserved; the existence of these *two independent integrals* cause the system to be *integrable*. If we put

$$\tau_i = \tfrac{1}{2}(q_i^2 + p_i^2), \qquad i=1,2 \tag{2}$$

the integrals are $\tau_i = E_i$ (constant), $i=1,2$.
The following construction consists of trivial computations but it makes the part played by the integrals transparant. We introduce action-angle variables by the canonical transformation

$$q_i = \sqrt{2\tau_i}\,\sin\phi_i, \qquad p_i = \sqrt{2\tau_i}\,\cos\phi_i, \qquad i=1,2$$

The Hamiltonian H_1 corresponding with the linearized system becomes in these variables

$$H_1 = \omega_1\tau_1 + \omega_2\tau_2$$

and the equations of motion

$$\dot\phi_i = \omega_i, \qquad \dot\tau_i = 0, \qquad i=1,2$$

The solution is

$$\binom{q}{p} = \begin{pmatrix} \sqrt{2\tau_1(0)}\ \sin(\phi_1(0)+\omega_1 t) \\ \sqrt{2\tau_2(0)}\ \sin(\phi_2(0)+\omega_2 t) \\ \sqrt{2\tau_1(0)}\ \cos(\phi_1(0)+\omega_1 t) \\ \sqrt{2\tau_2(0)}\ \cos(\phi_2(0)+\omega_2 t) \end{pmatrix}$$

One obvious question will be: is this solution periodic? Of course this depends on ω_1/ω_2 being rational or not. The second question might be: is there a way of getting a geometric picture of the phase-flow? Since phase-space is four-dimensional, the flow is difficult to visualize. The existence of the two integrals, however, hands us the necessary tools.

First of all, the energy $E_0 = \omega_1\tau_1+\omega_2\tau_2$ is invariant under the flow and its level surface is for $E_0 > 0$ a three-sphere S^3. Note: the reader should not be confused by the word is; of course the energy manifold is actually an ellipsoïd but any surface in $\mathbb{R}^n$ which is topologically equivalent with an (n-1)-sphere will be identified with S^{n-1}.

Secondly both τ_1 and τ_2 are conserved quantities. What does the, in general two-dimensional, surface

$$\omega_1\tau_1+\omega_2\tau_2 = E_0$$
$$\tau_1 = E_1$$

look like?

Considering again the coordinate transformation $q,p \rightarrow \tau,\phi$, we see that if the τ_i are fixed, the ϕ_i are left to be varying and will describe the surface which we are looking for. So this is a torus T^2.

There is a way of degeneration of the torus: if E_1 or $E_2 = 0$, T^2 reduces to a circle S^1. These two circles are the *normal modes* of the linear system. Note that the coordinate transformation is degenerate in these two cases.

The picture becomes as follows: looking at S^3 we see that it is foliated in a continuum of tori, with two circles as extreme cases.

Consider one of these circles, lying in three-space

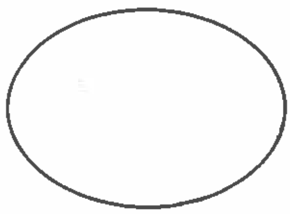


Figure 1a

The other circle passes through the centre of the first one, because the centre of the circle corresponds to a point where τ_1, say, is zero, and therefore τ_2 is maximal, i.e. it belongs to the normal mode. We have the following situation

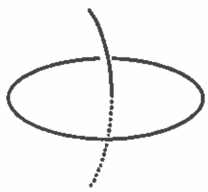


Figure 1b

On the other hand, if we draw the second circle first, the picture is the same, be it in another plane. This leads to

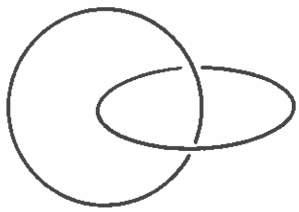


Figure 1c (Two normal modes)

The normal modes are linked. They are the centre lines for the tori which exist in the general case ($E_1E_2 \neq 0$). It may be difficult to visualize a torus having two circles as centre lines but remember that the tori are imbedded in a three-sphere. This is not an Euclidean space; there is no inside and outside here, only two 'insides' divided by a torus.

If ω_1/ω_2 is rational, the solutions are periodic and the orbits on the invariant tori are closed (the orbit is S^1). If ω_1/ω_2 is irrational each orbit is densely imbedded in an invariant torus (the orbit is $\mathbb{R}$).

3. BIRKHOFF-TRANSFORMATION OF THE NONLINEAR PROBLEM

At this point we have a fairly complete picture of what is happening in the linear case. How much of this picture does survive if we take into account the higher-order terms, i.e. if we consider nonlinear Hamiltonian systems with two degrees of freedom? In this paper an answer to this question will be given by constructing asymptotic approximations of the solutions corresponding with an approximate phase-flow.

Consider the Hamiltonian in expanded form (1)

$$H = H_2 + H_3 + H_4 + \dots$$

in which $H_2 = \frac{1}{2}\omega_1(q_1^2 + p_1^2) + \frac{1}{2}\omega_2(q_2^2 + p_2^2)$

and H_s is a homogeneous polynomial of degree $s \in \mathbb{N}$ in p,q.

Since we consider the phase-flow in the neighbourhood of an equilibrium point it is natural to scale the coordinates $q_i = \varepsilon\bar{q}_i$, $p_i = \varepsilon\bar{p}_i$, $i=1,2$ where ε is a small, positive parameter. Dividing the induced Hamiltonian by ε^2 (to keep the scaling process canonical) and omitting the bars we obtain

$$H = H_2 + \varepsilon H_3 + \varepsilon^2 H_4 + \dots$$

We shall say that we have a higher-order resonance if we can write $\omega_1 = m\lambda$ *and* $\omega_2 = n\lambda$ *with* $m,n \in \mathbb{N}$ *and relatively prime; moreover we require* $|m/n| \neq \frac{1}{3},\frac{1}{2},1,2,3.$

The lower-order resonance cases which we excluded have been treated in [2] and [3] and by many other authors. The starting point in most resonance problems is to put the Hamiltonian in Birkhoff normal form. A canonical perturbation theory which contains the same elements has been worked out in remarkable detail by Born (see [16] chapter 4). The technique was introduced by Birkhoff [4] in a rather implicit way; see also [1], appendix 7. A small excursion into the complex domain facilitates the computations. We follow the analysis in [5] and put

$$x_j = q_j + ip_j \qquad\qquad j=1,2.$$
$$y_j = q_j - ip_j$$

We find $\tau_j = x_j y_j$, $j=1,2$; Birkhoff-transformation is a canonical transformation such that the new Hamiltonian is a function of τ_j or $x_j y_j$ alone to a degree as high as possible. H_2 is in so-called Birkhoff normal form (we denote the new variables by the same symbols as the old ones)

$$H_2 = i\lambda(mx_1y_1 + nx_2y_2).$$

A near-identity transformation does not change H_2, the terms of higher degree are of the form

$$(x_1y_1)^{m_1}(x_2y_2)^{m_2}(x_1^n y_2^m)^{m_3}(y_1^n x_2^m)^{m_4}\ , \quad m_i \in \mathbb{N},\ \ i=1,\dots,4$$

in which m_3 or m_4 can be taken equal to zero.

Because of the condition of higher-order resonance $m+n \geq 5$ and we have for the transformed Hamiltonian $\tilde{H}$

$$\tilde{H} = i[\lambda(mx_1y_1+nx_2y_2) + \varepsilon^2(\tfrac{1}{2}Ax_1^2y_1^2 + Bx_1y_1x_2y_2 + \tfrac{1}{2}Cx_2^2y_2^2) + \ldots$$

$$\ldots + \varepsilon^{m+n-2}(Dx_1^ny_2^m + \bar{D}y_1^nx_2^m) + \ldots] \qquad (3)$$

The dots stand for terms which are in normal form and terms which are of higher order; $A,B,C \in \mathbb{R}$, $D \in \mathbb{C}$, $\bar{D}$ is the complex conjugate of D. The original purpose of Birkhoff's transformations was to obtain a new Hamiltonian which is a function of τ_j or x_jy_j only; such a system is integrable. One can succeed in this, at least formally, by taking m/n irrational. The corresponding transformation takes the form of a series which in general, however, diverges. The phenomenon is known in the literature as the small divisor problem. Our purpose will be to use Birkhoff-transformation as far as possible and analyse the remaining terms by the theory of asymptotic approximations.

The induced equations of motion of (3) are

$$\dot{x}_1 = i[mx_1 + \varepsilon^2(Ax_1y_1 + Bx_2y_2)x_1 + \ldots + \varepsilon^{m+n-2}n\bar{D}y_1^{n-1}x_2^m + \ldots]$$

$$\dot{x}_2 = i[nx_2 + \varepsilon^2(Bx_1y_1 + Cx_2y_2)x_2 + \ldots + \varepsilon^{m+n-2}mDx_1^ny_2^{m-1} + \ldots]$$

(there are four real equations as x_1 and x_2 are complex valued). We will absorb the absolute value of D into ε, so we may as well put $D = e^{i\alpha}$ with $\alpha \in S^1$. Two equivalent real systems will turn out to be useful.

With $a,b \in \mathbb{R}$, $a,b > 0$ and $\phi,\psi \in S^1$ we transform

$$x_1 = ae^{i\phi}$$

$$x_2 = be^{i\psi}$$

The induced equations are

$$\dot{a} = \varepsilon^{m+n-2}na^{n-1}b^m \sin(n\phi-m\psi+\alpha) + \ldots$$

$$\dot{b} = -\varepsilon^{m+n-2}ma^nb^{m-1} \sin(n\phi-m\psi+\alpha) + \ldots$$

$$\dot{\phi} = m + \varepsilon^2(Aa^2 + Bb^2) + \ldots + \varepsilon^{m+n-2}na^{n-2}b^m \cos(n\phi-m\psi+\alpha) + \ldots \qquad (4)$$

$$\dot{\psi} = n + \varepsilon^2(Ba^2 + Cb^2) + \ldots + \varepsilon^{m+n-2}ma^{n}b^{m-2} \cos(n\phi-m\psi+\alpha) + \ldots$$

At the same time we use the variables defined by

$$a = \frac{E}{\sqrt{m}} \cos \gamma \qquad E \in \mathbb{R}, \quad E > 0$$

$$\gamma \in (0, \pi/2)$$

$$b = \frac{E}{\sqrt{n}} \sin \gamma$$

$$\chi = n\phi - m\psi + \alpha$$

$$\phi = \phi$$

System (4) becomes (N can be taken high enough)

$$\dot{E} = 0 + O(\varepsilon^N)$$

$$\dot{\gamma} = -\frac{(\varepsilon E)^{m+n-2}}{m^{(n/2)-1}\, n^{(m/2)-1}} \cos^{n-1}\gamma \sin^{m-1}\gamma \sin \chi + O(\varepsilon^N)$$

$$\dot{\chi} = (\varepsilon E)^2[\tfrac{1}{m}(nA-mB)\cos^2\gamma + \tfrac{1}{n}(nB-mC)\sin^2\gamma] + \ldots \tag{5}$$

$$\ldots + \frac{(\varepsilon E)^{m+n-2}}{m^{(n/2)-1}\, n^{(m/2)-1}} \cos^{n-2}\gamma \sin^{m-2}\gamma (n \sin^2\gamma - m \cos^2\gamma) \cos \chi +$$

$$+ O(\varepsilon^N)$$

$$\dot{\phi} = m + (\varepsilon E)^2[\tfrac{A}{m} \cos^2\gamma + \tfrac{B}{n} \sin^2\gamma] + \ldots +$$

$$\ldots + \frac{(\varepsilon E)^{m+n-2}}{n^{(m/2)-1}\, m^{(n/2)}} \cos^{n-2}\gamma \sin^{m}\gamma \cos \chi + O(\varepsilon^N)$$

In system (5) we recognize the remarkable fact that the equations for E, γ and χ to a high order in ε do not depend on the variable ϕ. This fact becomes less remarkable if one realizes the close relation between Birkhoff normalization and averaging. This means that to a certain approximation in ε we can approximate E, γ and χ while ignoring ϕ and after that calculate an approximation for ϕ by direct integration of the last equation of (5). This procedure leads to $O(\varepsilon)$ approximations on the time-scale $1/\varepsilon^2$; see [5] or in a somewhat less general setting [3]. One of the conclusions of these calculations is that on this time-scale and to this order of accuracy no exchange of energy between the two degrees of freedom (internal resonance) takes place; the amplitudes a and b in system (4) are approximated by their initial values.

We shall look now for approximations on a longer time-scale which provide us with new qualitative information on the flow.

4. APPROXIMATIONS ON A LONGER TIME-SCALE.

The results stated in this section were derived in [5]. On considering the equations for E, γ and χ in system (5) one observes that with respect to the variables E and γ the angle χ is rapidly varying. This suggests that we can obtain approximations by averaging the right-hand sides over χ. Of course this procedure breaks down if the right-hand side of the equation for χ becomes small; this happens in a neighbourhood of values for which

$$\frac{nA-mB}{m}\cos^2\gamma + \frac{nB-mC}{n}\sin^2\gamma = 0 \tag{6}$$

The manifold M in four dimensional phase-space defined by equation (6) is called the resonance manifold. Having fixed E and γ with equation (6), i.e. having fixed the amplitudes a and b, the two angles ϕ and ψ are still varying in S^1; so the restriction of the resonance manifold to a surface with E = constant is a torus.
We expect the form of the approximations and the flow to be different in two domains:

D_R, the neighbourhood of the resonance manifold M; introducing the distance $d(x,M)$ for a point x in 4-space to the manifold M we have

$$D_R = \{x \mid d(x,M) = O\left(\varepsilon^{\frac{m+n-4}{3}}\right)\}$$

We call D_R the resonance domain.

D_O, the remaining part of 4-space in which we study the Hamiltonian system. We call D_O the outer domain.

We then have the following result

THEOREM 1.
Consider equations (5) in the outer domain D_O and the equations

$$\dot{\tilde{E}} = 0\ ,\quad \dot{\tilde{\gamma}} = 0$$
$$\dot{\tilde{\chi}} = (\varepsilon E)^2[\tfrac{1}{m}(nA-mB)\cos^2\tilde{\gamma} + \tfrac{1}{n}(nB-mC)\sin^2\tilde{\gamma}] \tag{7}$$

with the same initial values as for E, γ and χ. Then we have

$$E-\tilde{E},\ \gamma-\tilde{\gamma},\ \chi-\tilde{\chi} = O\left(\varepsilon^{\frac{m+n-4}{6}}\right) \quad \textit{on the time-scale} \quad \varepsilon^{-\frac{m+n}{2}} .$$

In this approximation the behaviour of the flow in the outer domain is 'quasi-linear', there is no exchange of energy between the two degrees of freedom. If $m+n = 5$, the error of the approximation is of $O(\varepsilon^{1/6})$ on the time-scale $\varepsilon^{-5/2}$; this is the worst possible case. In section 3 we found an $O(\varepsilon)$ approximation on the time-scale ε^{-2}.

The behaviour of the flow is different in the resonance domain D_R. Suppose that equation (6) has a solution, say $\gamma = \gamma_r$. We associate with equation (5) the system valid in D_R

$$\dot{\tilde{E}} = 0$$

$$\dot{\tilde{\gamma}} = -C_{nm}(\varepsilon\tilde{E})^{m+n-2} \cos^{n-1}\gamma_r \sin^{m-1}\gamma_r \sin\tilde{\chi} + O(\varepsilon^{m+n} \sin\tilde{\chi}) \qquad (8)$$

$$\dot{\tilde{\chi}} = 2(\varepsilon\tilde{E})^2(\tilde{\gamma}-\gamma_r)C_0 \sin\gamma_r \cos\gamma_r + O(\varepsilon^4)$$

in which $C_{nm} = n^{1-m/2} m^{1-n/2}$, $C_0 = 2nmB-n^2A-m^2C$; the right-hand sides have been obtained by expansion of equation (5) in a Taylor series near $\gamma = \gamma_r$. From the equation for $\tilde{\gamma}$ and $\tilde{\chi}$ we find

$$\ddot{\tilde{\chi}} + 2(\varepsilon E)^{m+n} (C_0 C_{nm} \sin^m\gamma_r \cos\gamma_r + O(\varepsilon^2)) \sin\tilde{\chi} = 0. \qquad (9)$$

Since equation (9) is the pendulum equation we have for each value of E two periodic solutions at $\gamma = \gamma_r$ and $\tilde{\chi} = 0$ or $\tilde{\chi} = \pi$; one is elliptic and the other one is hyperbolic. The asymptotic estimates are given by

THEOREM 2.

We associate with equations (5) in the resonance domain D_R *the equations (8) and (9) with the same initial values as for* E, γ *and* χ. *We have*

$$\gamma-\tilde{\gamma} = O\left(\varepsilon^{\frac{2}{3}(m+n-4)}\right),\ E-\tilde{E},\ \chi-\tilde{\chi} = O\left(\varepsilon^{\frac{m+n-4}{6}}\right)$$

on the time-scale $\varepsilon^{-\frac{m+n}{2}}$.

In the resonance domain $\varepsilon^{-\frac{m+n}{2}}$ is the natural time-scale of the resonance. The theory should be completed by a discussion of the part played by the constants arising in the equations and a bifurcation analysis of the normal modes. For these technically complicated questions and the proofs of theorem 1 and theorem 2 the reader is referred to [5]; explicit examples of higher-order resonance are analysed in the next sections.

5. A GEOMETRIC PICTURE OF THE FLOW.

The usual procedure in quantitative analysis is to construct a two-dimensional surface of section, which can be interpreted as a Poincaré mapping of the flow for a fixed value of the energy. To visualize the flow it can be helpful to construct surfaces of section in different directions, e.g. the q_1,p_1-plane or the q_2,p_2-plane. Here we shall try to visualize the complete flow on the energy manifold, which is diffeo-

morphic to the 3-sphere S^3. One may wonder, why does one bother at all about a global geometric picture? The answer is basically a philosophical one. We take the point of view that to understand what is going on in a dynamical system it is not enough to produce only numbers or analytical expressions. To understand what is going on it is essential to obtain a geometric picture which is as complete as possible and for which the quantitative results are necessary prerequisites.
In [5] one of the authors presented a picture of the flow on the energy manifold; this picture has been reproduced as figure 2. Think of S^3 as consisting of two solid tori, linked together along their common boundary; the normal modes are the centre lines of these solid tori. Around

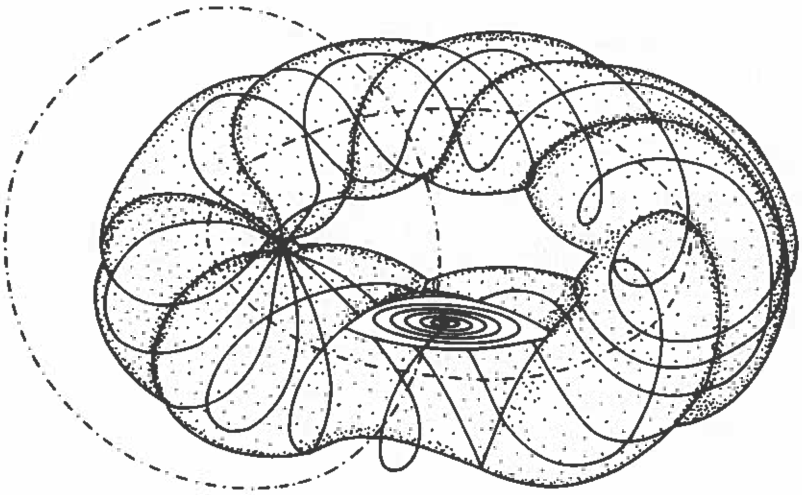
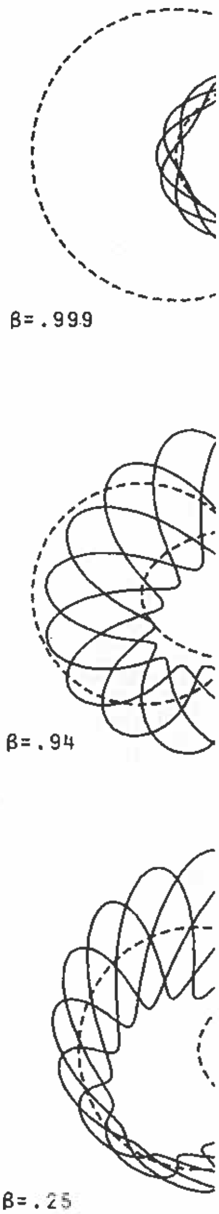


Figure 2. The 2:9 resonance. The -.-.- lines are the linked normal modes around which we find an elliptic and an hyperbolic periodic solution ——. The dotted surface is the stable and unstable manifold of the hyperbolic solution. There is one section transversal to the elliptic orbit to show what the inside looks like. (Courtesy D.Reidel Publ.Co.)

the elliptic orbit we find in the resonance manifold the surrounding invariant tori (shown in a transversal section) which together constitute the solid tori. The size of the resonance domain increases with the energy, which is $O(\varepsilon^2)$, and decreases with the order of resonance, characterized by m+n. For reasons of comparison we consider a similar picture constructed by a polynomial mapping $\mathbb{R}^4 \to \mathbb{R}^2$ of periodic functions of the form

$$x_1 = \alpha \cos mt \qquad x_3 = \beta \cos nt \qquad m,n \in Q$$
$$x_2 = \alpha \sin mt \qquad x_4 = \beta \sin nt \qquad \alpha,\beta \in \mathbb{R}$$



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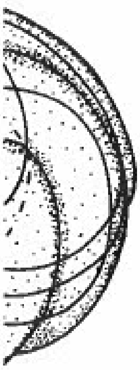

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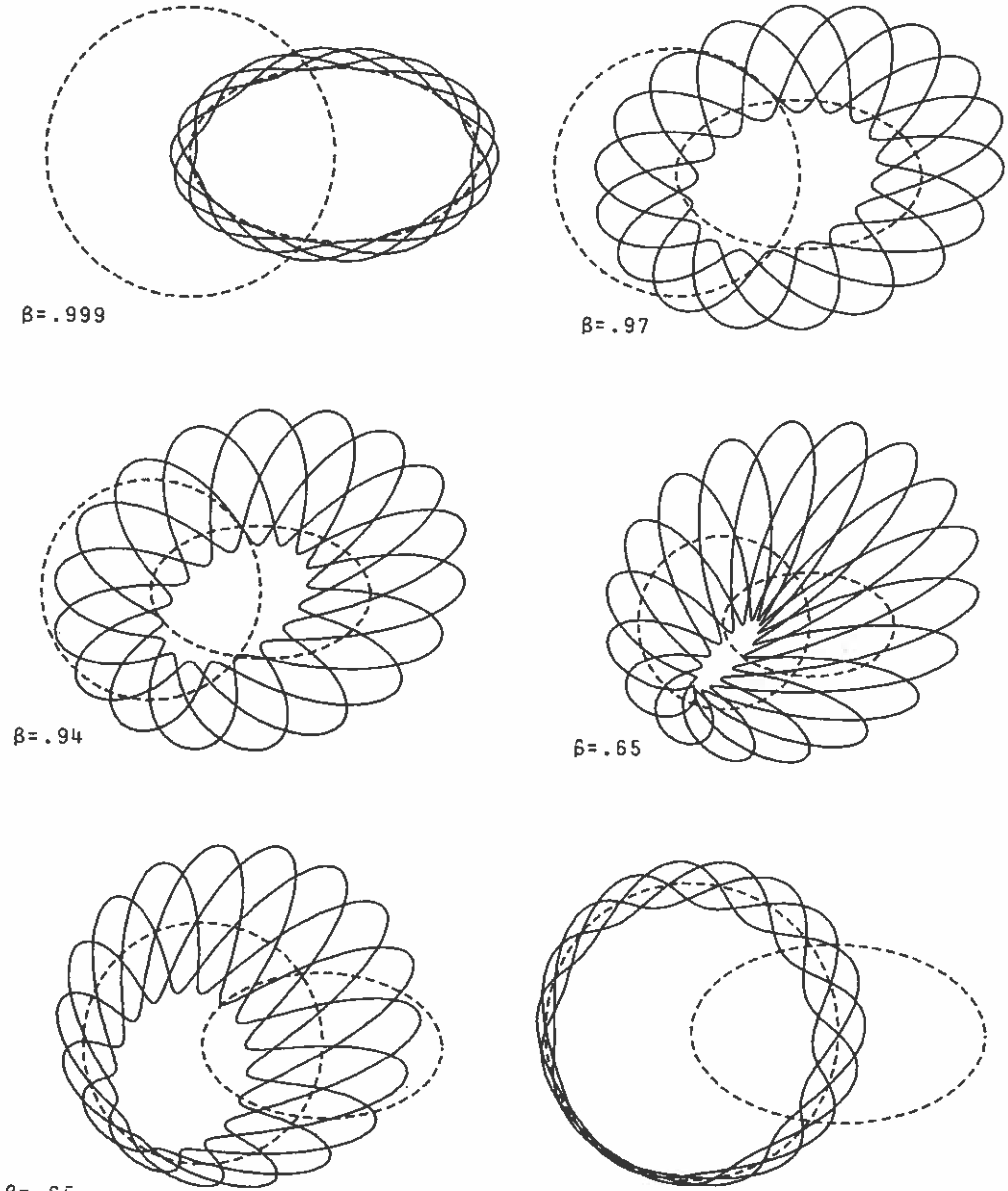

Figure 3. Periodic solutions mapped from $\mathbb{R}^4$ into $\mathbb{R}^2$ by a polynomial mapping. The two normal modes have been indicated by ------ and correspond with $\alpha=0$ *and* $\beta=0$; $m=2$, $n=9$. *We have a 'conservation law' of the form* $\alpha^2+\beta^2=1$.

6. THE CONTOPOULOS MODEL PROBLEM.

In this section we shall illustrate the preceding theory of higher-order resonance by treating an explicit example; moreover we shall generalize the theory somewhat. The example is a model problem for resonances in axi-symmetric galaxies which was formulated by Contopoulos [8]. For numerical explorations in more realistic models of galaxies the reader is referred to [6] and [7].

Consider the real Hamiltonian

$$H = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2}(\omega_1^2x^2 + \omega_2^2y^2) - \varepsilon xy^2 \tag{10}$$

with ε a small parameter, ω_1 and ω_2 positive real numbers. If $\omega_1/\omega_2 \in Q$, the preceding theory of higher-order resonance applies. If $\omega_1/\omega_2 \in \mathbb{R}$ each neighbourhood of ω_1/ω_2 contains infinitely many rational numbers and we shall use this fact in our perturbation theory.

The corresponding equations of motion are

$$\begin{aligned} \ddot{x} + \omega_1^2 x &= \varepsilon y^2 \\ \ddot{y} + \omega_2^2 y &= 2\varepsilon xy \end{aligned} \tag{11}$$

In [3] a discussion has been given of this problem in a more general context for the case of first and second order resonances in which ω_1/ω_2 is near 1/1, 1/2, 2/1, 1/3 (the case 3/1 has been omitted). We assume here that ω_1/ω_2 is not near these first and second order resonance values.

We express the near-rationality of ω_1/ω_2 as follows

$$\frac{\omega_2^2}{\omega_1^2} = \frac{n^2}{m^2}(1 + \delta(\varepsilon)) \tag{12}$$

in which $n,m \in \mathbb{N}$, $(n,m) = 1$; $\delta(\varepsilon)$ is called the detuning. We introduce the time-scale τ by

$$\omega_1 t = m\tau$$

and the transformation

$$\begin{aligned} x &= a\cos(m\tau+\phi) & y &= b\cos(n\tau+\psi) \\ \frac{dx}{d\tau} &= -am\sin(m\tau+\phi) & \frac{dy}{d\tau} &= -bn\sin(n\tau+\psi) \end{aligned} \tag{13}$$

The equations of motion become

$$\begin{aligned} \frac{d^2x}{d\tau^2} + m^2x &= \varepsilon\frac{m^2}{\omega_1^2}y^2 \\ \frac{d^2y}{d\tau^2} + n^2y &= 2\varepsilon\frac{m^2}{\omega_1^2}xy - n^2\delta(\varepsilon)y \end{aligned} \tag{11a}$$

Note that in the context of the theory of higher order resonance we shall look for periodic solutions with x,y-frequency ratio m/n while the x,y-frequency ratio of the linearized system equals ω_1/ω_2.
The detuning does not interfere with the Birkhoff-transformation of section 3 if $\delta(\varepsilon) = O(\varepsilon^2)$; we shall assume from now on that this estimate holds. Birkhoff-transformation of the Hamiltonian and introducing (13) or using averaging techniques (see [3]) produces, after lengthy calculations

$$\frac{da}{d\tau} = o + O(\varepsilon^3) \qquad \frac{db}{d\tau} = o + O(\varepsilon^3)$$

$$\frac{d\phi}{d\tau} = \frac{\varepsilon^2 m^3}{\omega_1^4(m^2-4n^2)}\, b^2 + O(\varepsilon^3) \tag{14}$$

$$\frac{d\psi}{d\tau} = \frac{\varepsilon^2 m^4}{\omega_1^4 n(m^2-4n^2)}\, a^2 + \varepsilon^2\, \frac{m^2(8n^2-3m^2)}{4\omega_1^4 n(m^2-4n^2)}\, b^2 + \frac{n}{2}\delta(\varepsilon) + O(\varepsilon^3).$$

Omitting the $O(\varepsilon^3)$ terms in equation (14) one obtains $O(\varepsilon)$ approximations of the amplitudes and phase-angles on the time-scale $1/\varepsilon^2$. To describe the higher-order resonances as explained in sections 3 and 4 we have to calculate the terms of $O(\varepsilon^{m+n-2})$. This leads to the determination of the constant α in the expression $\chi = n\phi - m\psi + \alpha$. Because of the particular form of the Hamiltonian we can predict the result without long calculations. This follows from

THEOREM 3.
Consider the real Hamiltonian

$$H = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) + U(x,y)$$

in which U(x,y) *is analytic near* (0,0) *and has a Taylor-expansion which starts with quadratic terms:* $U(x,y) = \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2) + \ldots$, $|\omega_1/\omega_2|$ *not near* 1:2, 2:1, 1:1, 1:3, 3:1. *Then Birkhoff-transformation of this Hamiltonian into the normal form (3) implies that the constant* D *is a real number.*

The proof will be given by one of the authors (F.V.) in a subsequent paper.

Theorem 3 has as a consequence that $\alpha = 0$ and

$$\chi = n\phi - m\psi.$$

For the angle χ we find from (14) the equation

$$\frac{d\chi}{d\tau} = \frac{\varepsilon^2 m^5}{\omega_1^4 n(m^2-4n^2)} \left[\frac{3m^2-4n^2}{4m^2}\, b^2 - a^2\right] - \frac{nm}{2}\,\delta(\varepsilon) + O(\varepsilon^3) \tag{15}$$

As in section 4 we obtain the resonance manifold by putting the right-hand side of equation (15) equal to zero. So the resonance manifold can be approximated by the expression

$$\frac{m^4}{\omega_1^4 n(m^2-4n^2)}\left[\frac{3m^2-4n^2}{4m^2}\tilde{b}^2 - \tilde{a}^2\right] = \frac{n}{2}\,\frac{\delta(\varepsilon)}{\varepsilon^2} \tag{16}$$

The approximate energy integral is

$$\tfrac{1}{2}\omega_1^2\tilde{a}^2 + \tfrac{1}{2}\omega_2^2\tilde{b}^2 = E_0 \tag{17}$$

From equations (16) and (17) we can compute the amplitudes of the periodic solutions in the resonance manifold and the conditions of existence. An example is given in

LEMMA 1.

For the Hamiltonion (10) with exact resonance, i.e. $\omega_1/\omega_2 = m/n$ *with* $m,n \in \mathbb{N}$, $(m,n) = 1$ *(so* $\delta(\varepsilon) = 0$*), periodic solutions with frequency ratio* m/n *exist if and only if* $\omega_1/\omega_2 > 2\sqrt{3} + O(\varepsilon)$. *In that case the amplitudes* a_r, b_r *of the periodic solutions are approximated by*

$$\tilde{a}_r^2 = \frac{3m^2-4n^2}{3m^2\omega_1^2}\,2E_0 \qquad\qquad \tilde{b}_r^2 = \frac{8}{3\omega_1^2}\,E_0$$

Proof.

Putting $\delta(\varepsilon) = 0$ in equation (16) produces the relation

$$\frac{3m^2-4n^2}{4m^2}\,\tilde{b}_r^2 = \tilde{a}_r^2 \quad .$$

This relation together with equation (17) yields the expressions for $\tilde{a}_r$ and $\tilde{b}_r$. The conditions of existence are with the energy integral

$$0 \leqq \omega_1^2\tilde{a}_r^2 \leqq 2E_0 \quad\text{and}\quad 0 \leqq \omega_2^2\tilde{b}_r^2 \leqq 2E_0 .$$

This together with the expressions for $\tilde{a}_r$ and $\tilde{b}_r$ leads directly to the inequality given in the lemma.

In [9] Contopoulos conjectured the non-existence of periodic solutions with frequency ratio m/n at exact resonance ($\delta(\varepsilon) = 0$) when $m/n < 1$. This conjecture was based on numerical experiments. From lemma 1 we know this conjecture to be true and moreover that the conjecture holds for $m/n < 2/\sqrt{3}$.

Another straightforward application of our theory is the following. In a number of numerical studies projections of orbits into x,y-space have been given.

Certain periodic and quasi-periodic orbits have been called 'tube orbits' by Ollongren [6]; in [8] tube orbits have been studied in the dynamical system (11). Using our results we identify the tube orbits with the orbits around the stable periodic orbit; the positions of the stable and unstable periodic solutions are given by the following lemma.

LEMMA 2.

The projections into x,y-space of the approximations of the periodic solutions of system (11) which are lying in the resonance manifold are represented by algebraic curves.

Proof.

Substituting $\tilde{a}_r$ and $\tilde{b}_r$ into equation (14) and putting $\tilde{\chi} = 0$ we obtain an approximation for the stable periodic solution in the resonance manifold

$$\tilde{x}(\tau) = \tilde{a}_r \cos(m[\tau+\tilde{\phi}(\tau)/m])$$
$$\tilde{y}(\tau) = \tilde{b}_r \cos(n[\tau+\phi(\tau)/m]) \qquad (18a)$$

The right-hand sides of these equations can be written as polynomials in $\cos(\tau+\tilde{\phi}(\tau)/m)$ of degree m and n respectively. Elimination of the time-dependent cosine produces an algebraic relation between $\tilde{x}(\tau)$ and $\tilde{y}(\tau)$. The same reasoning applies to the unstable periodic solution where

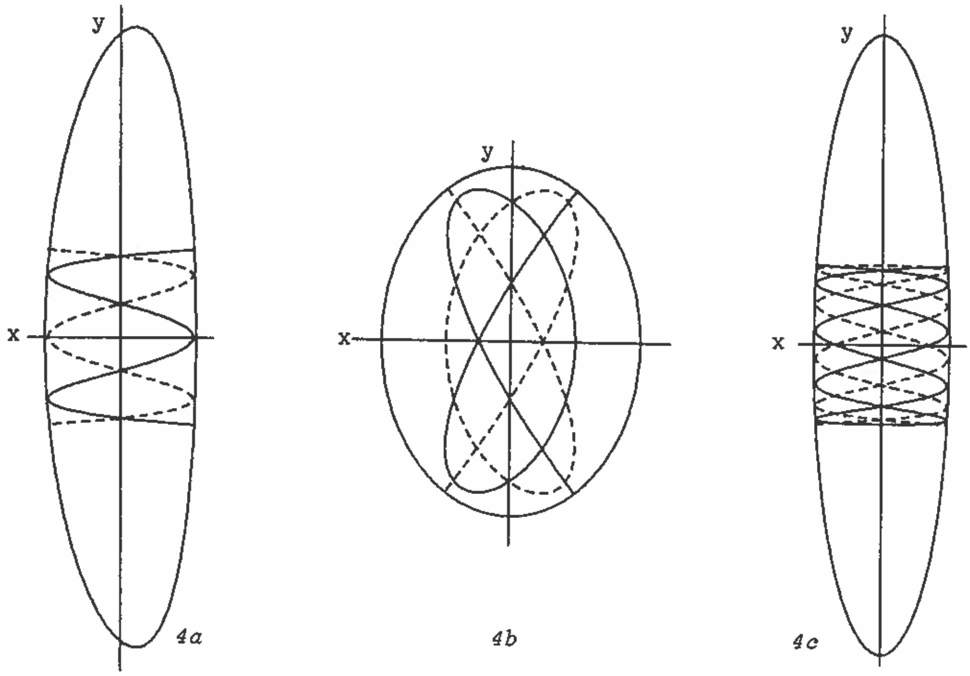

Figure 4. Projections of the stable and unstable (dotted) periodic solutions lying in the resonance manifold obtained from equations (18a-b). The closed boundary is the curve of zero-velocity. In fig.4a-b m:n = 4:1 and m:n = 4:3; they correspond with Contopoulos [10] fig.3 and 5. In fig.4c m:n = 9:2.

$$\tilde{x}(\tau) = \tilde{a}_r \cos(m[\tau+\tilde{\phi}(\tau)/m])$$
$$\tilde{y}(\tau) = \tilde{b}_r \cos(n[\tau+\tilde{\phi}(\tau)/m] - \pi/m) \qquad (18b)$$

The projections of these periodic solutions have been plotted in figure 4; the cases m:n = 4:1 and 4:3 have been studied by Contopoulos, [10, figures 3 and 5].

Another application concerns the effect of detuning. From the equation for the resonance manifold (16) and equation (12) we find

$$\frac{\omega_2}{\omega_1} = \frac{n}{m} + \frac{m}{n} \frac{m^2}{\omega_1^4(m^2-4n^2)} \left[\frac{3m^2-4n^2}{4m^2}\tilde{b}^2 - \tilde{a}^2\right] \varepsilon^2 + O(\varepsilon^4) \qquad (19)$$

Putting $\tilde{a} = 0$ or $\tilde{b} = 0$ and eliminating $\tilde{b}$ or $\tilde{a}$ with the energy integral (17) produces extreme values of the $O(\varepsilon^2)$ term. It is then possible, by fixing ω_2, to calculate the interval of ω_1 values for which periodic solutions with m/n frequency ratio exist. The result will depend on the energy E_0.
Contopoulos chooses in [11]: $E_0 = .00765$, $\omega_2^2 = .9$ and looks for periodic solutions with m = 2, n= 3. Note that it follows from lemma 1 that these periodic solutions do not exist if $\omega_1^2 = .4$. Contopoulos finds existence of these periodic solutions in [11] if

$$.39989 \leq \omega_1^2 \leq .39993.$$

Using equation (19) we find precisely the same numbers.

We present another consequence of equation (19):

LEMMA 3.
Let a higher-order resonance with frequency ratio n : m *exist in the interval* I; *in order that a higher-order resonance with frequency ratio* $\bar{n} : \bar{m}$ *exists in* I, $\bar{n} : \bar{m}$ *is necessarily an approximation of* n : m *in the sense*

$$\left|\frac{n}{m} - \frac{\bar{n}}{\bar{m}}\right| \leq \frac{3E_0\varepsilon^2}{2(\omega_1\omega_2)^3} \frac{\omega_1^2}{|\omega_1^2-4\omega_2^2|}$$

The proof follows directly from equation (19) by estimating the extreme values of the $O(\varepsilon^2)$ term with the energy integral (17).
It has been checked that in the case of the 2 : 3 resonance with $E_0 = .00765$, $\omega_2^2 = .9$ there are no other resonances $\bar{n} : \bar{m}$ in I with $\bar{n}+\bar{m} \leq 100$. In fact a stronger conclusion holds: for higher-order resonances m : n with E_0 and ω_2 as before and m+n $\leq$ 100 ω_1-intervals exist where we have existence of n : m periodic solutions but in none of these cases other resonances $\bar{n} : \bar{m}$ with $\bar{n}+\bar{m} \leq 100$ can be found in these intervals.

7. CONTOPOULOS' « THIRD » INTEGRAL.

The problem of the existence of a second integral of motion independent of the energy (in astrophysics a third integral because of the existence of an additional integral : angular momentum) for two degrees of freedom Hamiltonian systems constitutes a problem with a long and interesting history; see Ollongren [6] for a survey and references and Contopoulos [12] for a recent survey.

Contopoulos has given an expansion for this second integral of motion for Hamiltonians of the form (10) and slightly more general. The expansion has only a formal character, i.e. no proof of convergence has been given but there is good agreement with numerical results in [8-12].

In [3] the asymptotic character (in the mathematical sense of the expression) of Contopoulos' second integral has been demonstrated at the main resonance $\omega_1 : \omega_2 \approx 1 : 1$.

For higher-order resonances we can understand these formal results as follows. The total energy E of the system is conserved. If the nonlinear coupling constant ε equals zero, the energy of the motion in the two separate degrees of freedom is conserved. In that case these energies E_1 and E_2 (or E and E_1) correspond with two independent integrals of motion. If $\varepsilon \neq 0$, we have the total energy as one integral and the Contopoulos' formal integral of the form

$$\phi_1 = E_1 + \text{higher-order terms.}$$

We remark that $E_1 = \frac{1}{2}(\dot{x}^2 + \omega_1^2 x^2)$, the higher-order terms start with a cubic polynomial in $x, \dot{x}, y, \dot{y}$.

The asymptotic method gives us asymptotic integrals, i.e. functions I_i, $i=1,2$, such that

$$I_i(\bar{x}(t)) - I_i(\bar{x}(o)) = O(\varepsilon^N) \qquad \text{on } 0 \leq \varepsilon^M t \leq L$$

in which N and M are certain constants; $\bar{x}(t)$ is the solution of the differential equation (such as (14)) obtained after the normalizing (averaging) coordinate transformations.

If one wants to use these integrals in the original coordinates, then one has to substitute $\bar{x}$ as an asymptotic expansion in ε and functions of x, the original coordinate, into the I_i's, up to the order of accuracy one seeks to obtain.

Our discussion of the topology of phase-space in sections 4 and 5 implies that Contopoulos' formal integral corresponds with asymptotic approximations in the sense described in these sections. This replaces the 'formal character' of ϕ_1 by 'asymptotic character in the mathematical

sense'. For the understanding of the nature of ϕ_1 we add two remarks.
First it is clear that in the resonance manifold (the domain of tube orbits) the higher-order terms are the most effective; here we have behaviour of the orbits different from linear behaviour.
Outside the resonance domain E_1 dominates in such a way that the orbits behave nearly as linear ones; these orbits have been called box orbits by Ollongren [6]. It is also understandable that there has been some misunderstanding in the literature about the question whether Contopoulos' «third» integral ϕ_1 is independent of the energy or not. For the independence is clear in the resonance domain, but to demonstrate the independence outside the resonance domain one needs rather precise calculations. One can formulate it differently: to a certain precision outside the resonance domain E is conserved and E_1 is an asymptotically conserved quantity, in the resonance domain E is conserved and ϕ_1 is an asymptotically conserved quantity.
Secondly one may wonder whether it is possible to prove the convergence of the expansion ϕ_1 thus producing an exact second integral of motion. The answer to this question is that one *cannot* expect the expansion to be convergent for all permitted initial values. Convergence can be expected for initial values on the invariant tori around the periodic solutions which we have found. In this field, normal forms of differential equations and the convergence of normalizing transformations, considerable progress has been made by the work of Brjuno [13-14]. In these papers one finds conditions for normalizing transformations to be convergent and examples of divergence; also one can find here remarks on the relation between various canonical normalizing transformations and on the history of the problem. An interesting application of normal form theory in celestial mechanics has been given by Brjuno in [15].
Returning to the Contopoulos' model problem, one may wonder what happens for the initial conditions, located between the invariant tori, for which the expansion ϕ_1 is divergent. From various analytical and numerical results one expects to find , between the invariant tori, higher-order resonances with m+n large. For increasing values of the energy these resonances became more and more prominent thus dissolving and complicating the basic picture of higher-order resonance which we sketched in this paper.

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The referenc
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