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Averaging Techniques and the Oscillator-Flywheel Problem

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We discuss normalisation and averaging techniques, which play a part in the analysis of nonlinear dynamical systems, with special attention to the technique of averaging over two or more angles. Under rather general assumptions we show that in certain sets in phase-space, the resonance manifolds, the flow is to first-order determined by a pendulum equation. The theory is illustrated by an analysis of the oscillator-flywheel problem.

1. Introduction

There are many problems in engineering and in physics where small parameters arise in a natural way. One can think of weak coupling between modes, the effect of small friction, slow variation of coefficients in the equations corresponding with variation of quantities as mass, length etc. We first consider a simple example: a one-dimensional oscillating spring with small damping described for $t \geqslant 0$ by the equation of motion

$$\ddot{x} + \epsilon\mu\dot{x} + \omega^2 x + \epsilon f(x) = 0 . \tag{1}$$

The parameter ϵ will always be small and positive; μ is a positive constant independent of ϵ and $f(x)$ characterizes the nonlinearity of the spring. The constant $\epsilon\mu$ is often called the damping constant; ω is the frequency of the unperturbed ($\epsilon=0$) oscillator which, in this first example, is a constant independent of ϵ. It is convenient to introduce a coordinate transformation $x,\dot{x}\to r,\psi$ by

$$x = r\cos(\omega t+\psi),\ \dot{x} = -r\omega\sin(\omega t+\psi) . \tag{2}$$

The equations for r and ψ become

$$\begin{aligned}\dot{r} &= -\frac{\epsilon}{\omega}\sin(\omega t+\psi)[\mu r\omega\sin(\omega t+\psi)-f(r\cos(\omega t+\psi))]\\ \dot{\psi} &= -\frac{\epsilon}{\omega r}\cos(\omega t+\psi)[\mu r\omega\sin(\omega t+\psi)-f(r\cos(\omega t+\psi))]\end{aligned} \tag{3}$$

for which we may add appropriate initial values. System (3) for the amplitude

r and phase ψ is characterized by a slowly-varying righthand side (slowly because of the small parameter ϵ). Note that the periodic character of eq. (3) arises from the introduction of the coordinate transformation (2) which was inspired by the unperturbed ($\epsilon=0$) solution of eq. (1). System (3) does not appear to be simpler to handle than eq. (1); however, it turns out that systems like (3) admit the application of a special technique, averaging, which produces approximations in a relatively simple way. First we shall demonstrate this for the more general system with prescribed initial value

$$\dot{y} = \epsilon F(t,y),\ y(0) = y_0 \tag{4}$$

with $y,y_0,F\in\mathbb{R}^n,F$ is T-periodic in t. System (3) is of the form (4) with $n=2,\ T=2\pi/\omega$.

We introduce the averaged vector function

$$F^\circ(y) = \frac{1}{T}\int_0^T F(t,y)dt\ .$$

Note that y has been kept fixed during the averaging process. Suppose that we can solve the averaged equation with the prescribed initial value

$$\dot{\tilde{y}} = \epsilon F^\circ(\tilde{y}),\ \tilde{y}(0) = y_0\ .$$

Under certain conditions, see [8] Chapter 2, we have the estimate

$$y(t)-\tilde{y}(t) = O(\epsilon),\ \text{for } \epsilon\to 0,\ 0\leqslant \epsilon t \leqslant C$$

with C a positive constant independent of ϵ. This is often stated as: $\tilde{y}(t)$ approximates $y(t)$ to $O(\epsilon)$ on the time-scale $1/\epsilon$.

Returning now to system (3) we average the system over t, keeping the variables r and ψ fixed. Assuming that f is a smooth function we find after an elementary calculation for the first equation

$$\dot{\tilde{r}} = -\epsilon\frac{\mu}{2}\tilde{r}$$

which integrates to

$$\tilde{r}(t) = r(0)e^{-\epsilon\frac{\mu}{2}t}\ . \tag{5}$$

It turns out that $\tilde{r}(t)$ approximates $r(t)$ on a larger interval of time than predicted by the theory, in fact with error $O(\epsilon)$ for $\epsilon\to 0$, $t\in[0,\infty)$; see [8], Chapter 4.

We omit the discussion of the phase ψ and return to eq. (1). Suppose now, as a second example, that the frequency ω is not a constant but changes slowly with time: $\omega=\omega(\epsilon t)$. We introduce a coordinate transformation $x,\dot{x}\to r,\phi$ for $r>0$ by

$$x = r\sin\phi,\qquad \dot{x} = \omega r\cos\phi\ . \tag{6}$$

Putting $\tau=\epsilon t$ we find from eq. (1)

$$\begin{aligned} \dot{r} &= \epsilon[-\mu r\cos^2\phi - \frac{r}{\omega(\tau)}\cos^2\phi\frac{d\omega(\tau)}{d\tau} - \frac{\cos\phi}{\omega(\tau)} f(r\sin\phi)] \\ \dot{\tau} &= \epsilon \\ \dot{\phi} &= \omega(\tau) + O(\epsilon)\,. \end{aligned} \tag{7}$$

System (7) contains the dependent variables r, ϕ, τ and the righthand side is periodic in ϕ. Assuming that $\omega(\tau)$ is bounded away from zero by a constant, independent of ϵ, the angle ϕ is a fast moving variable with respect to the amplitude variable r and the variable τ. So we can treat ϕ as a time-like variable and it is natural to average the slowly part of system (7), i.e. the equations for r and τ, over ϕ. We find for the averaged amplitude equation

$$\dot{\tilde{r}} = -\tfrac{1}{2}\epsilon[\mu\tilde{r} + \frac{\tilde{r}}{\omega}\frac{d\omega}{d\tau}]\,.$$

Integration produces

$$\tilde{r}(t)\omega^{\frac{1}{2}}(\epsilon t) = r(0)\omega^{\frac{1}{2}}(0)e^{-\epsilon\frac{\mu}{2}t}\,. \tag{8}$$

Eq. (8) includes the result of eq. (5). If we take $\mu=0$, the relation given by eq. (8) represents a well-known adiabatic invariant of the system. In systems with slowly varying coefficients, adiabatic invariants play a part analogous to the part played by first integrals in classical mechanics. For a discussion see [1], [6] and [7]. In general the accuracy of such an invariant is $O(\epsilon)$ on the time-scale $1/\epsilon$; see [8] Chapter 5.

A large number of rather different problems, for instance gyroscopic systems in mechanical engineering or gravitating systems in celestial mechanics, can be written in a form like system (7):

$$\begin{aligned} \dot{x} &= \epsilon X(\phi,x) + O(\epsilon^2), \quad x\in D\subset\mathbb{R}^n \\ \dot{\phi} &= \Omega(x) + O(\epsilon), \qquad \phi\in T^m. \end{aligned} \tag{9}$$

T^m is the m-torus; system (9) contains n slowly varying variables $x_1, \ldots, x_n$ and m angles $\phi_1, \ldots, \phi_m$.

In the Sections 2-4 we shall survey a number of techniques and theorems while adding some new observations. In Section 5 we shall discuss an application to an oscillator flywheel system, which is one of the prototype problems of nonlinear mechanical engineering. The appendix has been added as it contains some useful technical details which have been omitted in [8]. The emphasis is, apart from the appendix, on the discussion of methods and ideas. Details can be found in the literature given in the sections.

2. Normalisation and averaging

The concept of the (or a) normal form of a vector field or any other mathematical object is based on the idea that by a suitable transformation one can put such an object in a simpler form. One may hope that the object in normal form is easier to study than before transforming. An example from linear algebra is a nonsingular matrix with distinct eigenvalues which we can put into diagonal form. A simple example of a differential equation put into normal form shows the possibilities. Consider the two-dimensional system

$$\dot{x}_1 = 2x_1 + a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 + \cdots$$
$$\dot{x}_2 = x_2 + b_1x_1^2 + b_2x_1x_2 + b_3x_2^2 + \cdots$$

The righthand side is analytic in a neighbourhood of (0,0) and we have written out the first terms of the expansion. Introducing a near-identity transformation of the form

$$x_1 = y_1 + h_1(y_1,y_2)$$
$$x_2 = y_2 + h_2(y_1,y_2)$$

we can give a power series expansion for h_1 and h_2 with coefficients such that the system for (y_1,y_2) becomes of the form

$$\dot{y}_1 = 2y_1 + cy_2^2$$
$$\dot{y}_2 = y_2$$

The constant c depends on the coefficients a_1,b_1, etc. This is a remarkable simplification.

An introduction to normalisation techniques can be found in [10], more details in [1] and [8]. We shall demonstrate the technique for system (9) with $m=1$, while using a near-identity transformation

$$x = y + \epsilon u(\phi,y) \tag{10}$$

However, $u(\phi,y)$ will not be given in the form of a power series. We substitute (10) into system (9) assuming that the vectorfields are smooth enough to admit expansion with respect to ϵ. We find

$$\dot{x} = \dot{y} + \epsilon\frac{\partial u}{\partial \phi}[\Omega(y+\epsilon u) + O(\epsilon)] + \epsilon\frac{\partial u}{\partial y}\dot{y} = \epsilon X(\phi, y+\epsilon u) + O(\epsilon^2)$$

and after expansion with respect to ϵ

$$\dot{y} = \epsilon X(\phi,y) - \epsilon\frac{\partial u}{\partial \phi}\Omega(y) + O(\epsilon^2)\,. \tag{11}$$

One chooses u in the transformation (10) such that eq. (11) becomes simpler than the first equation of system (9). We can achieve this by putting

$$u(\phi,y) = \frac{1}{\Omega(y)}\int^{\phi}(X(\theta,y) - X^{\circ}(y))d\theta$$

with averaged vector field

$$X^\circ(y) = \int_{S^1} X(\phi,y)d\phi$$

Equation (11) becomes with this choice

$$\begin{aligned} \dot{y} &= \epsilon X^\circ(y) + O(\epsilon^2) \\ \dot{\phi} &= \Omega(y) + O(\epsilon). \end{aligned} \tag{12}$$

Note that, because of the choice of $u(\phi,y)$, we have to stay away from the zeros of $\Omega(y)$; also that we still have some freedom left in the choice of $u(\phi,y)$ as we have introduced an indefinite integral; we omit the details. System (12) can be called the normal form to first order of system (9).

REMARK 1. Near-identity transformation in terms of formal power series as we indicated for the two-dimensional example is usually called normalisation according to Poincaré and Dulac. For system (9) we used normalisation by averaging. Both procedures can be put in the same formal frame-work (which can be technically rather complicated) and they lead in a certain context to equivalent normal forms; see [1] and [8].

REMARK 2. One of the main advantages of averaging is, that it is an explicit normalisation technique which makes it very useful as an operational tool.

3. ESTIMATES OF ACCURACY OUTSIDE THE RESONANCE MANIFOLD

The theory of approximation of the solutions of systems like (4) or (9) runs along the following lines. First one calculates a suitable normal form of the system; secondly one truncates the normalised system at a certain order and solves the remaining system. Estimates of accuracy are then usually based on contraction (Gronwall's lemma), or equivalently, the implicit function theorem. A first result is this:

THEOREM 1. *Consider the system*

$$\begin{aligned} \dot{x} &= \epsilon X(\phi,x) + O(\epsilon^2), \quad x(0),\ x \in D \subset \mathbb{R}^n \\ \dot{\phi} &= \Omega(x) + O(\epsilon),\ \phi(0),\ \phi \in S^1 . \end{aligned}$$

We assume that the righthand side of the system is smooth and that

$$0 < a \leqslant \inf_{x \in D} |\Omega(x)| \leqslant \sup_{x \in D} |\Omega(x)| \leqslant b < \infty$$

where a and b are ϵ-independent constants. Let (y,ψ) be the solution of

$$\begin{aligned} \dot{y} &= \epsilon X^\circ(y),\ y(0) = x(0) \\ \dot{\psi} &= \Omega(y),\ \psi(0) = \phi(0) \end{aligned}$$

then, if $y(t)$ remains in an interior subset of D

$$x(t) = y(t) + O(\epsilon) \text{ on the time-scale } 1/\epsilon ,$$

$$\phi(t) = \psi(t) + O(\epsilon t e^{\epsilon t}) .$$

PROOF. See [8] Chapter 5.

Theorem 1 can be extended to obtain higher-order accuracy i.e. an $O(\epsilon^2)$ approximation for x and an $O(\epsilon)$ approximation for ϕ on the time-scale $1/\epsilon$. The theorem is valid in domains which do not contain zeros of $\Omega(x)$. The equation $\Omega(x)=0$ defines the resonance manifold(s) in $\mathbb{R}^n$. In and near a resonance manifold we have to apply a local analysis as in boundary layer theory, see [2].

The theory becomes rapidly more complicated if we use averaging over more angles, for instance in the case of the 2-torus ($m=2$). We put

$$\begin{aligned} \dot{x} &= \epsilon X(\phi_1,\phi_2,x), \quad x \in D \subset \mathbb{R}^n \\ \dot{\phi}_1 &= \Omega_1(x) \qquad\quad \phi_1 \in S^1 \\ \dot{\phi}_2 &= \Omega_2(x) \qquad\quad \phi_2 \in S^1 \end{aligned}$$

where X is 2π-periodic in ϕ_1 and ϕ_2. So the torus T^2 is the product of two circles $S^1 \times S^1$, in this case with length 2π. Complex Fourier expansion of X yields

$$X(\phi_1,\phi_2,x) = \sum_{k,l=-\infty}^{\infty} c_{kl}(x) e^{i(k\phi_1 + l\phi_2)} .$$

Averaging over the angles produces

$$\frac{1}{(2\pi)^2} \int_{\phi_2=0}^{2\pi} \int_{\phi_1=0}^{2\pi} X(\phi_1,\phi_2,x) d\phi_1 d\phi_2 = c_{oo}(x)$$

provided that $k\phi_1 + l\phi_2 \neq 0$. In this case the averaged equation is

$$\dot{y} = \epsilon c_{oo}(y), \qquad y \in D \subset \mathbb{R}^n . \tag{14}$$

What happens if $k\phi_1 + l\phi_2 = 0$?

Suppose, that for some k,l the coefficient $c_{kl}(x)$ is not identically zero. Combining the equations for ϕ_1 and ϕ_2 in (13) we observe that $k\phi_1 + l\phi_2$ can be zero (or small) for some interval of time in subsets where

$$k\dot{\phi}_1 + l\dot{\phi}_2 = 0$$

or

$$k\Omega_1(x) + l\Omega_2(x) = 0, \quad k,l \in \mathbb{Z}, \; x \in \mathbb{R}^n \tag{15}$$

If equation (15) has solutions in $\mathbb{R}^n$, the combination angle $k\phi_1 + l\phi_2$ locally varies slowly. Equation (15) defines the resonance manifolds in $\mathbb{R}^n$ of system (13). The set of resonance manifolds may have an infinite number of components (solutions of equation (15)); sometimes this set is called the Arnold web.

The solutions of the equation for x outside the resonance manifolds can be approximated by the solutions of equation (14). A straight-forward generalisation of Theorem 1 shows that outside the resonance manifolds, the solutions of equation (14) with appropriate initial values approximate the solutions of the equation for x in (13) to $O(\epsilon)$ on the time-scale $1/\epsilon$. The approximation is carried out to the second-order in the appendix.

EXAMPLE. Consider the system

$$\begin{aligned} \dot{x} &= \epsilon X(\phi_1,\phi_2,x), \quad x\in\mathbb{R} \\ \dot{\phi}_1 &= x, \qquad\qquad \phi_1\in S^1 \\ \dot{\phi}_2 &= 1, \qquad\qquad \phi_2\in S^1 \end{aligned}$$

The initial values $x(0)$, $\phi_1(0)$ and $\phi_2(0)$ are supposed to be given; X is 2π-periodic in ϕ_1 and ϕ_2.

We perform Fourier expansion of X as before. The resonance manifolds are given by equation (15) which becomes

$$kx + l = 0, \quad k,l\in\mathbb{Z},\ x\in\mathbb{R}\ .$$

The resonance manifolds in this case are represented by rational numbers. If the Fourier expansion contains a finite number of terms only, it is easy to define 'inside and outside of resonance manifolds'. For instance take the case

$$\dot{x} = \tfrac{1}{2}\epsilon - \epsilon\cos(\phi_1-\phi_2)\ .$$

Since $\dot{\phi}_1-\dot{\phi}_2 = x-1$, there is one resonance manifold: $x=1$. Outside a neighbourhood of $x=1$ we average to obtain $\dot{y}=\frac{1}{2}\epsilon$. So

$$y(t) = x(0) + \tfrac{1}{2}\epsilon t\ .$$

Outside a neighbourhood of $x=1$ we have $x(t)-y(t)=O(\epsilon)$ on the time-scale $1/\epsilon$. It is interesting to note that this approximation does break down if one allows for values of x in the resonance manifold; two one-parameter families of exact solutions of the system are

$$\begin{aligned} &x(t) = 1,\ \phi_1(t) = \phi_1(0) + t,\ \phi_2(t) = \phi_2(0) + t \\ &\phi_1(0) - \phi_2(0) = \pm\tfrac{1}{3}\pi \end{aligned}$$

so for these initial values $y(t) - x(t) = \frac{1}{2}\epsilon t$.

Solutions with $x(0)<1$ will approach a neighbourhood of the resonance manifold but in this example with one resonance manifold, these solutions cannot be trapped. This can be shown as follows. Putting $\psi=\phi_1-\phi_2$ we derive the equation

$$\ddot{\psi} + \epsilon\cos\psi = \tfrac{1}{2}\epsilon\ .$$

This equation has no attractors. Several solutions $x(t)$, starting in a

neighbourhood of $x(0)=0$, are shown in Figure 1.

Returning now to the case in which the Fourier expansion of X contains an infinite number of terms, one observes that there can exist an accumulation of resonance manifolds. For instance if one has $c_{kl}(x)$ not identically zero for $k=0,1,2,...$, and some finite l, the point $x=0$ is a point of accumulation of resonance manifolds. Near $x=0$, the averaged equation has no clear meaning as regarding the original equation for x.

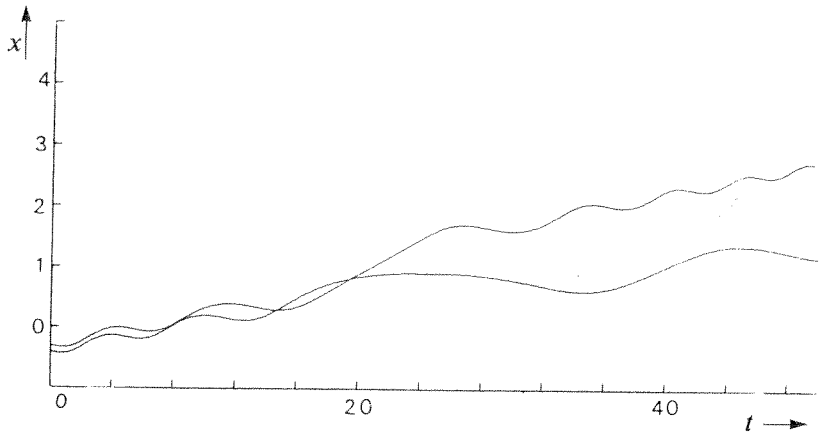

FIGURE 1.
Two solutions of $\dot{x}=\frac{1}{2}\epsilon-\epsilon\cos(\phi_1-\phi_2)$, $\dot{\phi}_1=x$, $\dot{\phi}_2=1$; $\epsilon=.1$; $\phi_1(0)=\phi_2(0)=0$, $x(0)=-.3$ and $x(0)=-.4$; $0\leqslant t\leqslant 50$.

4. THE FLOW IN THE RESONANCE MANIFOLD

As we have seen, resonance manifolds can accumulate; they can also intersect. We shall consider now an isolated resonance manifold M which by a change of coordinates has been placed at $x_1=0$ in $\mathbb{R}^n$. To be more explicit, consider the system

$$\begin{aligned} \dot{x} &= \epsilon X(\phi,x) \quad , \quad x\in\mathbb{R}^n \\ \dot{\phi} &= \Omega(x) \quad , \quad \phi\in T^m \end{aligned} \tag{16}$$

where $\Omega=(\Omega_1,\ldots,\Omega_m)$, $x=(x_1,\ldots,x_n)$ and with M defined by

$$k_1\Omega_1(x)+\cdots k_m\Omega_m(x)=0,\ x\in\mathbb{R}^n \text{ and certain } k_1,\ldots,k_m\in\mathbb{Z}$$

with isolated solution $x_1=0$.

A natural approximation scheme for the flow in and near M runs as follows. Consider an $O(\delta(\epsilon))$ neighbourhood of M with $\delta(\epsilon)=o(1)$. This is a boundary layer in the sense of singular perturbation theory, see [2], where one uses a local variable ξ defined by

$$x_1=\delta(\epsilon)\xi\,.$$

We put $x=(x_1,\eta)$ with $\eta\in\mathbb{R}^{n-1}$ and we replace ϕ by $\psi=k_1\phi_1+\cdots k_m\phi_m$ and $(m-1)$ independent angles, say $\phi_2,\ldots,\phi_m$. System (16) becomes

$$\begin{aligned}
\delta(\epsilon)\dot{\xi} &= \epsilon X_1(\psi,\phi_2,\ldots,\phi_m,\delta(\epsilon)\xi,\eta)\\
\dot{x}_i &= \epsilon X_i(\psi,\phi_2,\ldots,\phi_m,\delta(\epsilon)\xi,\eta),\quad i=2,\ldots,n\\
\dot{\psi} &= \sum_{i=1}^{m} k_i\Omega_i(\delta(\epsilon)\xi,\eta)\\
\dot{\phi}_i &= \Omega_i(\delta(\epsilon)\xi,\eta)\quad i=2,\ldots,m\,.
\end{aligned}$$

Supposing that the vector functions are sufficiently smooth, we can expand to obtain

$$\begin{aligned}
\delta(\epsilon)\dot{\xi} &= \epsilon X_1(\psi,\phi_2,\ldots,\phi_m,0,\eta)+O(\epsilon\delta(\epsilon))\\
\dot{x}_i &= \epsilon X_i(\psi,\phi_2,\ldots,\phi_m,0,\eta)+O(\epsilon\delta(\epsilon)),\ i=2,\ldots,n\\
\dot{\psi} &= \delta(\epsilon)\sum_{i=1}^{m} k_i\frac{\partial\Omega_i}{\partial x_i}(0,\eta)\xi+O(\delta^2(\epsilon))\\
\dot{\phi}_i &= \Omega_i(0,\eta)+O(\delta(\epsilon)),\ i=1,2,\ldots,m
\end{aligned}$$

A significant degeneration (for the terminology see again [2]) arises if $\delta(\epsilon)=\epsilon^{\frac{1}{2}}$. In an $O(\epsilon^{\frac{1}{2}})$ neighbourhood of M the equations can then be written as

$$\begin{aligned}
\dot{\xi} &= \epsilon^{\frac{1}{2}}X_i(\psi,\phi_2,\ldots,\phi_m,0,\eta)+O(\epsilon)\\
\dot{\eta} &= O(\epsilon)\\
\dot{\psi} &= \epsilon^{\frac{1}{2}}\sum_{i=1}^{m} k_i\frac{\partial\Omega_i}{\partial x_i}(0,\eta)\xi+O(\epsilon)\\
\dot{\phi}_i &= \Omega_i(0,\eta)+O(\epsilon^{\frac{1}{2}}),\ i=2,\ldots,m
\end{aligned}$$

This system is in the form of system (9) or (16) but with small parameter $\epsilon^{\frac{1}{2}}$, slow variables ξ,η,ψ and fast variables $\phi_2,\cdots,\phi_m$. Fourier expansion of X_1 and averaging over the $(m-1)$ fast variables produce the system

$$\begin{aligned}
\dot{\tilde{\xi}} &= \epsilon^{\frac{1}{2}}(c_1(\tilde{\eta})+c_2(\tilde{\eta})\cos\tilde{\psi}+c_3(\tilde{\eta})\sin\tilde{\psi})\\
\dot{\tilde{\eta}} &= 0\\
\dot{\tilde{\psi}} &= \epsilon^{\frac{1}{2}}\sum_{i=1}^{m} k_i\frac{\partial\Omega_i}{\partial x_i}(0,\tilde{\eta})\tilde{\xi}\\
\dot{\tilde{\phi}}_i &= \Omega_i(0,\tilde{\eta}),\ i=2,\ldots,m\,.
\end{aligned}$$

In an $O(\epsilon^{\frac{1}{2}})$ neighbourhood of M the solutions $\tilde{\xi},\tilde{\eta}$ and $\tilde{\psi}$ with appropriate initial values, approximate ξ,η and ψ with error $O(\epsilon^{\frac{1}{2}})$ on the time-scale $1/\epsilon^{\frac{1}{2}}$. As $\tilde{\eta}$ is a constant in this approximation, we can derive for $\tilde{\psi}$ the following

fundamental equation by differentiation

$$\ddot{\tilde{\psi}} + \epsilon a(\tilde{\eta})\cos\tilde{\psi} + \epsilon b(\tilde{\eta})\sin\tilde{\psi} = \epsilon c(\tilde{\eta}) \tag{17}$$

with a,b and c constants depending on $\tilde{\eta}$ only.

It is easy to show that if equation (17) contains critical points, these points are saddles or centers. This means that in this general set-up, to first order the flow in the resonance manifold is described in a structurally unstable way: the $(n-1)$ variables $x_2, \cdots, x_m$ are frozen, the combination angle ψ satisfies a second-order pendulum equation without friction (notwithstanding the amount of dissipation which may be present in the original system (16)).

The implication is, that to describe the flow in the resonance manifold adequately, one generally needs at least a second-order calculation. Such a calculation does not only add more precision, a second-order calculation is essential to obtain a correct qualitative and quantitative picture of the phenomena.

If equation (17) contains a center, the system will also have a homoclinic orbit or saddle loop. In general system (16) will contain dissipation terms and the second-order result will cause a break-up of the homoclinic orbit such that we have the possibility of trapping of orbits or of chaos in the resonance manifold. It turns out that the first possibility has been realised in the oscillator-flywheel problem, Section 5.

We finally note that it is easy to give examples where one has an intersection of resonance manifolds. Suppose p resonance manifolds, corresponding with combination angles $\psi_1, \cdots, \psi_p$, intersect and one introduces local variables near this set of intersection. One can repeat the calculation of this section to obtain a coupled system of the form

$$\ddot{\psi}_i + \epsilon \sum_{k=1}^{p} (a_{ik}\cos\psi_i + b_{ik}\sin\psi_i) = \epsilon \sum_{k=1}^{p} c_{ik},\ i = 1, \cdots, p\ .$$

The $2p$-dimensional phase-flow is divergence-free and again, a second-order calculation is necessary.

5. The oscillator-flywheel problem

To illustrate the difficulties which arise in the case of two angles, i.e. $m=2$ in system (13), we analyse a model of the oscillator-flywheel. A wheel, which is slightly excentric because of the mass m (see Fig. 2), is mounted on a spring such that it can move up and down during rotation. The vertical displacement of the wheel is measured by the variable x, while the position of mass m, causing the excentricity, is indicated by the variable ϕ.

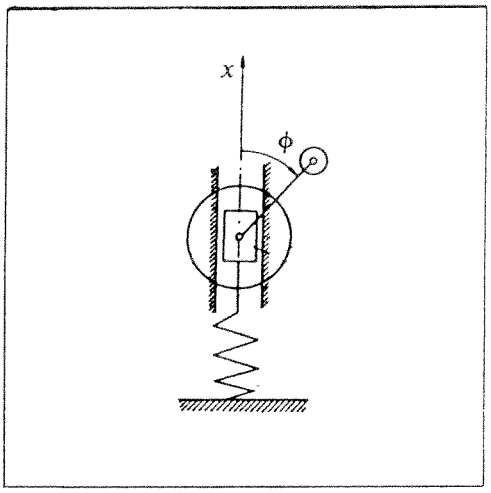

FIGURE 2.
The oscillator-flywheel

The equations describing this system are:

$$\begin{aligned} \ddot{x} + x &= \epsilon[-x^3 - \dot{x} + \dot{\phi}^2 \cos\phi] + O(\epsilon^2), \\ \ddot{\phi} &= \epsilon[\tfrac{1}{4}(2-\dot{\phi}) + (1-x)\sin\phi] + O(\epsilon^2). \end{aligned} \tag{18}$$

Here ϵ is a small number depending on m and the mass of the flywheel. For a detailed formulation of this problem see [3] or [4]. In these books a preliminary discussion of this problem has been given based on numerical techniques.

We will present an analytical method to solve approximately system (18), using second-order averaging theory over two angles. This analytical approach has the advantage that general insight in the behaviour of the solutions can be obtained.

The first step is to bring (18) in the standard form for averaging. Introducing a transformation $x = r\sin\phi_2$, $\dot{x} = r\cos\phi_2$, $\phi = \phi_1$, and $\dot{\phi}_1 = \Omega$, $r > 0$ and $\Omega > 0$, system (18) takes the form:

$$\begin{aligned} \dot{r} &= \epsilon\cos\phi_2[-r^3\sin^3\phi_2 - r\cos\phi_2 + \Omega^2\cos\phi_1] + O(\epsilon^2), \\ \dot{\Omega} &= \epsilon[\tfrac{1}{4}(2-\Omega) + \sin\phi_1 - r\sin\phi_1\sin\phi_2] + O(\epsilon^2), \\ \dot{\phi}_1 &= \Omega, \\ \dot{\phi}_2 &= 1 + \epsilon[r^2\sin^4\phi_2 + \tfrac{1}{2}\sin 2\phi_2 - \frac{\Omega^2}{r}\cos\phi_1\sin\phi_2] + O(\epsilon^2). \end{aligned} \tag{19}$$

These equations are in the form of system (13). The resonance condition (15) produces: $k\cdot\Omega + l = 0$ for $k, l \in \mathbb{Z}$. We can write the right-hand side of (19) as a

function of $\phi_1-\phi_2$, $\phi_1+\phi_2$ and ϕ_2. Because $\Omega>0$ the only values of k and l, which produce a combination angle $\psi=k\phi_1+l\phi_2$, which is slowly varying, are $k=1$ and $l=-1$. In this case the resonance occurs if $\Omega=1$. Other values of k and l will produce combination angles $\psi=k\phi_1+l\phi_2$, which do not appear in the right-hand side of (19). So the resonance manifold M is defined by $\Omega=1$.

This has important implications for the averaging technique, in which the equations are averaged with respect to all variables with an $O_s(1)$- derivative; after averaging, these "faster moving" variables have been eliminated from the right-hand side (O_s-estimates 'are 0 but not o').

Averaging

We have to distinguish between two regions: the resonance manifold M defined by $\Omega=1$ and its neighbourhood *and* the region away from M (the outer region). We know that the resonance region is an $O(\sqrt{\epsilon})$-neighbourhood of $\Omega=1$ ([8], § 5.5.2).

In the outer region we can average (19) with respect to $\phi_1-\phi_2$ and ϕ_2 (or ϕ_1 and ϕ_2) and obtain (compare with (14)):

$$\begin{aligned} \dot{r} &= -\tfrac{1}{2}\epsilon r + O(\epsilon^2) \\ \dot{\Omega} &= \tfrac{1}{4}\epsilon(2-\Omega) + O(\epsilon^2) \end{aligned} \tag{20}$$

We see that in the outer region r will be decreasing for all time and that Ω increases towards its equilibrium value 2. In the resonance region we average system (19) with respect to ϕ_2 and obtain

$$\begin{aligned} \dot{r} &= \epsilon[-\tfrac{1}{2}r+\tfrac{1}{2}\Omega^2\cos\psi] + O(\epsilon^2)\,, \\ \dot{\Omega} &= \epsilon[\tfrac{1}{4}(2-\Omega)-\tfrac{1}{2}r\cos\psi] + O(\epsilon^2)\,, \\ \dot{\psi} &= \Omega-1-\epsilon[\frac{3}{8}r^2+\frac{1}{2}\frac{\Omega^2}{r}\sin\psi] + O(\epsilon^2)\,. \end{aligned} \tag{21}$$

System (21) contains two critical points;
$(r,\Omega,\psi)=(\frac{1}{2}\sqrt{2}+O(\epsilon),\quad 1+O(\epsilon),\ \frac{\pi}{4}+O(\epsilon))$ which is unstable and $(\frac{1}{2}\sqrt{2}+O(\epsilon),\ 1+O(\epsilon),\ -\frac{\pi}{4}+O(\epsilon))$ which is an attractor.

Taking the initial values $r(0)=1.0$, $\Omega(0)=0.5$ and varying $\psi(0)$, we expect the solutions of system (19) to be attracted into the resonance region for some values of $\psi(0)$. For some $\psi(0)$ the solution will stay in the resonance region for all time (Ω near to 1 for all t) and for other $\psi(0)$ the solution will move through the resonance region (Ω tends to 2 for large t).

The physical meaning of these two situations can be made clear with the help of Figure 2. The non-resonance situations implie for large t that $\Omega\approx 2$ and $r\approx 0$, which means that there is no vertical motion and a fast rotating fly-wheel (angular velocity 2). The resonance situation implies an equilibrium with

$r \approx \frac{1}{2}\sqrt{2}$ and $\Omega \approx 1$, which means a slower rotation of the wheel combined with a notable vertical motion. In practice this phenomenon could damage the construction.

As a numerical example we take $\epsilon = 0.01$ and $r(0) = 1.0$, $\Omega(0) = 0.5$ and we vary $\psi(0) = \psi_0$ in system (19). If there is a finite T_p, defined by $\Omega(T_p) = 1.2$, the solution has passed through the resonance region; if not, we expect the solution to be attracted by the asymptotically stable equilibrium point. We write $\psi_0 = \frac{k}{1000}.2\pi$, $k \in \mathbb{N}$, and obtain three intervals, leading to attraction, i.e. $k \in$ [167:196], [452:485] and [758:774] (see Fig. 3).

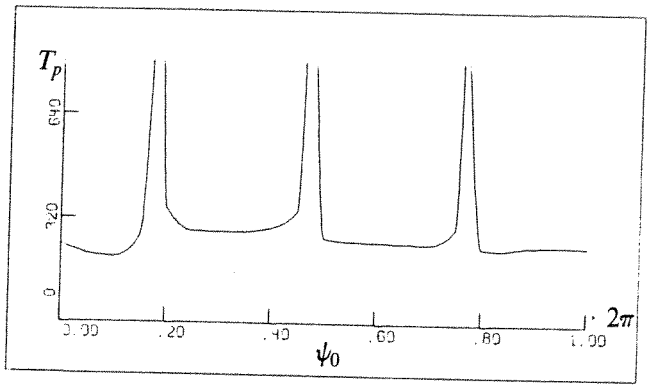

FIGURE 3.
T_p defined by $\Omega(T_p) = 1.2$ for each $\psi_0 \in [0, 2\pi]$

In the following we shall show how these intervals can be approximated in an analytical way.

We will distinguish two situations: Ω is in an $O(\sqrt{\epsilon})$-neighbourhood of 1 (the resonance region) and Ω is not in such a region (the outer region). If Ω is not in the resonance region we use system (19).

If Ω is in the resonance region, we introduce a local variable $\omega = \frac{\Omega - 1}{\sqrt{\epsilon}}$ in system (19) and we shall neglect the $O(\epsilon\sqrt{\epsilon})$-terms. The result is:

$$\begin{aligned} \dot{r} &= \epsilon[-\tfrac{1}{2}r + \tfrac{1}{2}\cos\psi] \\ \dot{\omega} &= \sqrt{\epsilon}[\tfrac{1}{4} - \tfrac{1}{2}r\cos\psi] - \tfrac{1}{4}\epsilon\omega \\ \dot{\psi} &= \sqrt{\epsilon}\omega + \epsilon[-\tfrac{3}{8}r^2 - \tfrac{1}{2}\frac{\sin\psi}{r}] \end{aligned} \tag{22}$$

We shall look for attraction in the resonance region, which is now given by $\omega = 0$. First, we approximate the solutions in the outer region. There are two fast-moving variables in (19) and using second order averaging theory (for a

theorem on second order averaging in the case of two angles see the appendix) one can obtain $O(\epsilon^2)$ approximations of r and Ω and $O(\epsilon)$ approximations for ψ and ϕ_2, valid on the time scale $\frac{1}{\epsilon}$.

This approximation will move in the direction of the resonance region. We will determine a time t^*, such that for $t=t^*$ the solution is "on the boundary" of the outer and the resonance region, i.e. has a distance of $O(\sqrt{\epsilon})$ to $\Omega=1$. If $\Omega=1-k\sqrt{\epsilon}$ $(k>0)$ we expect that the approximation of the solutions of the equations of the outer region still have some asymptotic validity, see [8], § 5.5.3. If we use this endpoint $(r(t^*),\omega(t^*) = \frac{\Omega(t^*)-1}{\sqrt{\epsilon}}, \psi(t^*))$ of the outer equations as a starting point for the inner equations, we expect a good approximation of the solution inside the resonance region, because the starting point is in a $O(\sqrt{\epsilon})$-neigbourhood of $\Omega=1$. (The asymptotic validity in the outer and inner region can both be proven; the only problem left is the matching of the approximations.)

We now have to solve the following problem: for which initial values $(r(t^*),\omega(t^*),\psi(t^*))$ will the solution of the inner equations (22) be attracted by the attractor? (Remember that $(r(t^*),\omega(t^*),\psi(t^*))$ depends on ψ_0 and ϵ.) We rewrite (22) to obtain

$$\begin{aligned} \frac{d^2\psi}{d\tau^2} + \tfrac{1}{2}r\cos\psi &= \tfrac{1}{4} - \sqrt{\epsilon}\frac{d\psi}{d\tau}(\tfrac{1}{4}+\tfrac{1}{2}\frac{\cos\psi}{r}) + O(\epsilon)\,, \\ \frac{dr}{d\tau} &= \sqrt{\epsilon}[-\tfrac{1}{2}r+\tfrac{1}{2}\cos\psi]\,, \\ \dot{\tau} &= \sqrt{\epsilon}\,. \end{aligned} \tag{23}$$

If $\epsilon=0$ we have a center and a saddle (we have seen this already in Section 4), for $\epsilon>0$ a saddle and an attractor (see Fig. 4). For $\epsilon=0$ we can define an energy function: $E=\frac{1}{2}(\frac{d\psi}{d\tau})^2-\frac{1}{4}\psi + \frac{1}{2}r\sin\psi+c$, with c a constant. This energy function is shown in Fig. 4c.

REMARK. If $\epsilon=0$ system (23) is a conservative system, in essence described by a two-dimensional phase-space, which is shown in Fig. 4a. However, for $\epsilon>0$, r is a slowly moving variable and the phase-space is three-dimensional. So the phase portrait which is shown in Fig. 4b, is only a projection in the $(\psi,\dot{\psi})$-plane.

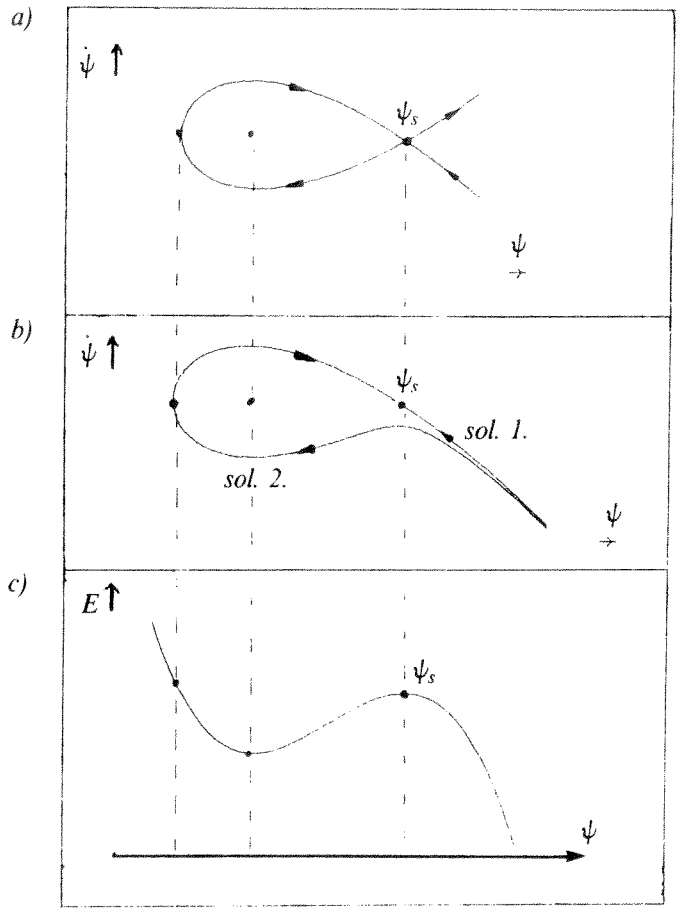

FIGURE 4.
System (23) for $\epsilon=0$ (a) and $\epsilon>0$ (b).
In (c) the energy corresponding to (a) is shown.
ψ_s is the saddle.

For $\epsilon>0$ we take $c=\frac{1}{4}\psi_s-\frac{1}{2}r\sin\psi_s$, so that the trajectory of Fig. 4a corresponds to $E=0$ (note that c is slowly varying because of r). Inspired by work of HABERMAN ([5]) we differentiate

$$\frac{dE}{d\tau} = -\sqrt{\epsilon}\left(\frac{d\psi}{d\tau}\right)^2\left(\frac{1}{4}+\frac{1}{2}\frac{\cos\psi}{r}\right) + \frac{1}{2}\frac{dr}{d\tau}(\sin\psi-\sin\psi_s) + O(\epsilon),$$

and with the help of this expression we approximate the energy dissipated by solution 1 and solution 2 of Fig. 4b (E_1 and E_2 resp.). We then conclude that the solution will be attracted if its initial energy is between E_1 and E_2.

Details of the calculation of E_1 and E_2 are given in [9].

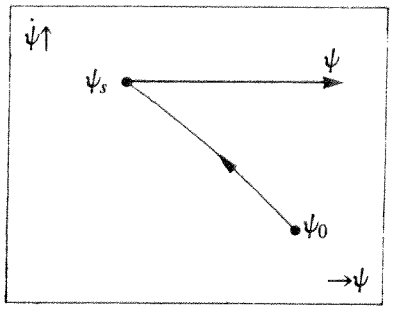

FIGURE 5.
Trajectory corresponding to E_1

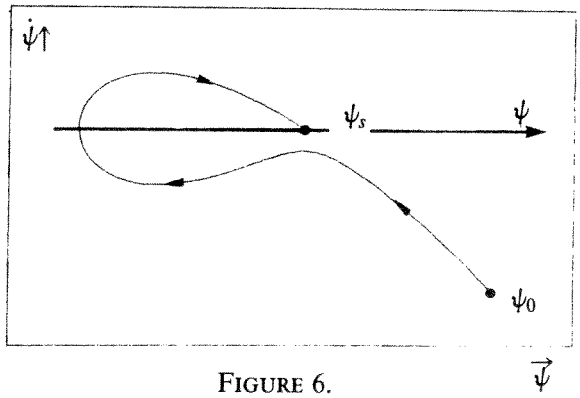

FIGURE 6.
Trajectory corresponding to E_2

Now we have to determine those ψ_0, for which the starting point of the inner equations $(r(t\star),\omega(t\star),\psi(t\star))$ is on the trajectory defined by $E=0$, as it is for those ψ_0 that the solution will move towards the saddle. It turns out that for $\epsilon=0.01$ there are three such values of ψ_0, which are listed in the table with their corresponding E_1 and E_2.

ψ_0	E_1	E_2
2.28	0.02	0.15
2.44	0.02	0.15
2.61	0.02	0.15

So there are three different solutions, leading to the saddle; for other values of ψ_0 the solution $(r(t\star),\omega(t\star),\psi(t\star))$ is not on the trajectory $E=0$. Now we have to solve $0.02\leqslant E(0)\leqslant 0.15$, and the ψ_0 that solves this inequality corresponds to a solution that will be in resonance for large t (see again [9] for details).

The result is that according to this first approximation, attraction will occur

for $k \in$ [161:175], [497:506] or [791:801], with $\psi_0 = \frac{k}{1000} \cdot 2\pi$. These values should be compared with the values given earlier which have been obtained by numerical integration.

Appendix

Second Order Averaging over Two Angles

B. van den Broek

We start with 2 lemmas.

LEMMA 1. *Consider system (1)*

$$\begin{aligned} \dot{x} &= \epsilon X(\phi,x) \quad , x(0)=x_0 \;\; ; x\in D\subset\mathbb{R}^n \\ \dot{\phi} &= \Omega_1(x)+\epsilon\Omega_2(\phi,x) \;\; , \phi(0)=\phi_0 \;\; ; \phi\in T^2 \end{aligned} \tag{1}$$

We assume that the vectorfields are twice continuously differentiable with bounded derivatives in the domains considered. We write $\phi=\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$, $\Omega_1=\begin{bmatrix}\Omega_{11}\\ \Omega_{12}\end{bmatrix}$ *and* $\Omega_2=\begin{bmatrix}\Omega_{21}\\ \Omega_{22}\end{bmatrix}$.

Consider the Fourier expansion $X(\phi,x)=\sum_{k,l=-\infty}^{\infty} c_{kl}e^{i(k\phi_1+l\phi_2)}$. *Restricting oneself to the pairs* (k,l) *which arise non-trivially (i.e.* c_{kl} *not identically zero in D) one assumes that there are no subsets in D such that* $k\Omega_{11}(x)+l\Omega_{12}(x)=0$; *(this means that the combination angles* $k\phi_1+l\phi_2$ *are not permitted to vary slowly in D).*

Then the solutions of (1) can be written as

$$\begin{aligned} x(t) &= y(t) + \epsilon u^{(1)}(\psi,y) + \epsilon^2 u^{(2)}(\psi,y) \\ \phi(t) &= \psi(t) + \epsilon v(\psi,y) \,, \end{aligned}$$

where $y(t)$ *and* $\psi(t)$ *are solutions of*

$$\dot{y} = \epsilon X^{01}(y) + \epsilon^2 X^{02}(y) + \epsilon^3 R_1,\; y(0) = y_0 \tag{2}$$

$$\dot{\psi} = \Omega_1(y) + \epsilon\Omega_2^0(y) + \epsilon^2 R_2,\ \psi(0) = \psi_0.$$

Here $\Omega_2^0(y) = \int_{T^2} \Omega_2(\phi, y)d\phi$, R_1 *and* R_2 *are uniformly bounded in* $D \times T^2$; $X^{01}, X^{02}, u^{(1)}, u^{(2)}$ *and* v *will be defined later on.*

PROOF. The spatial derivative with respect to the variable x or y will be indicated by ∇; so $\dfrac{\partial X}{\partial x} = \nabla X$.

We have

$$\dot{\phi} = \dot{\psi} + \epsilon \frac{\partial v}{\partial \psi} \cdot \dot{\psi} + \epsilon \nabla v \cdot \dot{y}$$

$$= \Omega_1(y) + \epsilon\Omega_2^0(y) + \epsilon \frac{\partial v}{\partial \psi} \Omega_1(y) + O(\epsilon^2)$$

and on the other hand

$$\dot{\phi} = \Omega_1(y + \epsilon u^{(1)} + \epsilon^2 u^{(2)}) + \epsilon\Omega_2(\psi + \epsilon v, y + \epsilon u^{(1)} + \epsilon^2 u^{(2)})$$

$$= \Omega_1(y) + \epsilon \nabla\Omega_1 \cdot u^{(1)} + \epsilon\Omega_2(\psi, y) + O(\epsilon^2)$$

Comparing these two expressions for $\dot{\phi}$ we have

$$\frac{\partial v}{\partial \psi} \cdot \Omega_1 = \nabla\Omega_1 \cdot u^{(1)} + \Omega_2 - \Omega_2^0 \tag{3}$$

and therefore

$$\Omega_{11} \cdot \frac{\partial v}{\partial \psi_1} + \Omega_{12} \cdot \frac{\partial v}{\partial \psi_2} = \nabla\Omega_1 \cdot u^{(1)} + \Omega_2 - \Omega_2^0.$$

Note that the average of the right-hand side for (3) equals zero, so v is a bounded function. Furthermore we have

$$\dot{x} = \dot{y} + \epsilon \frac{\partial u^{(1)}}{\partial \psi} \cdot \dot{\psi} + \epsilon \nabla u^{(1)} \cdot \dot{y} + \epsilon^2 \frac{\partial u^{(2)}}{\partial \psi} + \epsilon^2 \nabla u^{(2)} \cdot \dot{y}$$

$$= \epsilon X^{01}(y) + \epsilon^2 X^{02}(y) + \epsilon \frac{\partial u^{(1)}}{\partial \psi} (\Omega_1 + \epsilon\Omega_2^0) + \epsilon^2 \nabla u^{(1)} \cdot X^{01} + \epsilon^2 \frac{\partial u^{(2)}}{\partial \psi} \Omega_1 + O(\epsilon^3)$$

$$= \epsilon[X^{01}(y) + \frac{\partial u^{(1)}}{\partial \psi} \cdot \Omega_1] + \epsilon^2[X^{02}(y) + \frac{\partial u^{(1)}}{\partial \psi} \cdot \Omega_2^0 + \nabla u^{(1)} \cdot X^{01} + \frac{\partial u^{(2)}}{\partial \psi} \cdot \Omega_1] + O(\epsilon^3)$$

and on the other hand

$$\dot{x} = \epsilon X(\psi + \epsilon v,\ y + \epsilon u^{(1)} + \epsilon^2 u^{(2)})$$

$$= \epsilon X(\psi, y) + \epsilon^2 \frac{\partial X}{\partial \psi} \cdot v + \epsilon^2 \nabla X \cdot u^{(1)} + O(\epsilon^3).$$

Comparing the two expressions for $\dot{x}$ we have

$$\Omega_{11} \cdot \frac{\partial u^{(1)}}{\partial \psi_1} + \Omega_{12} \cdot \frac{\partial u^{(1)}}{\partial \psi_2} = X(\psi, y) - X^{01}(y),$$

$$\Omega_{11}\cdot\frac{\partial u^{(1)}}{\partial\psi_1} + \Omega_{12}\frac{\partial u^{(2)}}{\partial\psi_2} = \nabla X\cdot u^{(1)} - \nabla u^{(1)}\cdot X^{01} - X^{02} - \frac{\partial u^{(1)}}{\partial\psi}\Omega_2^0 \quad (4)$$

$$X^{01}(y) = \int_{T^2} X(\psi,y)d\psi.$$

These expressions define $u^{(1)},u^{(2)}$ and X^{01}. We require $u^{(2)}$ to have average zero and have

$$X^{02}(y) = \int_{T^2}\frac{\partial X}{\partial\psi}\cdot v + \nabla X\cdot u^{(1)} - \nabla u^{(1)}\cdot X^{01}d\psi, \quad (5)$$

which defines X^{02}. □

LEMMA 2. *Consider the equation (2)*

$$\begin{aligned}\dot{y} &= \epsilon X^{01}(y)+\epsilon^2 X^{02}(y)+\epsilon^3 R_1 \quad ,y(0)=y_0\\ \dot{\psi} &= \Omega_1(y)+\epsilon\Omega_2^0(y)+\epsilon^2 R_2 \qquad ,\psi(0)=\psi_0.\end{aligned}$$

Suppose that (z,ξ) is the solution of (2) truncated as follows

$$\begin{aligned}\dot{z} &= \epsilon X^{01}(z)+\epsilon^2 X^{02}(z) \quad , z(0)=z_0\\ \dot{\xi} &= \Omega_1(z)+\epsilon\Omega_2^0(z) \qquad , \xi(0)=\zeta_0,\end{aligned} \quad (3)$$

then we have

$$\|y-z\| \leqslant (\|y_0-z_0\| + \epsilon^3 t\|R_1\|)e^{\epsilon Lt}$$

where L is the Lipschitz-constant of $X^{01}+\epsilon X^{02}$. If $y_0-z_0=O(\epsilon^2)$ we have $y=z+O(\epsilon^2)$ on the time-scale $\frac{1}{\epsilon}$.

Furthermore $\|\psi-\xi\|\leqslant\|\psi_0-\xi_0\|+M\|y-z\|t+\epsilon^2 t\|R_2\|$, where M is the Lipschitz-constant of $\Omega_1+\epsilon\Omega_2^0$. If $z_0-y_0=O(\epsilon^2)$ and $\psi_0-\xi_0=O(\epsilon)$ we have

$$\psi = \xi + O(\epsilon) \text{ on the time-scale } \frac{1}{\epsilon}.$$

PROOF. We have $\dot{y}-\dot{z}=\epsilon(X^{01}(y)-X^{01}(z)) + \epsilon^2(X^{02}(y)-X^{02}(z))+\epsilon^3 R_1$. Putting L for the Lipschitz-constant of $X^{01}+\epsilon X^{02}$ we get $\|\dot{y}-\dot{z}\|\leqslant\epsilon L\|y-z\| + \epsilon^3\|R_1\|$. Using Gronwall's lemma, we have

$$\|y-z\|\leqslant(\|y_0-z_0\|+\epsilon^3 t\|R_1\|)e^{\epsilon Lt}$$

For the angular variables we find

$$\begin{aligned}\dot{\psi}-\dot{\xi} &= \Omega_1(y) + \epsilon\Omega_2^0(y) - \Omega_1(z) - \epsilon\Omega_2^0(z) + \epsilon^2 R_2\\ &\leqslant M\|y-z\| + \epsilon^2 R_2\end{aligned}$$

where M is the Lipschitz-constant of $\Omega_1+\epsilon\Omega_2^0$.
So we have $\|\psi-\xi\|\leqslant\|\psi_0-\xi_0\| + \epsilon^2\|R_2\|\cdot t$. This produces an $O(\epsilon)$ estimate for the angular variables if $y-z=O(\epsilon^2)$ on the time-scale $\frac{1}{\epsilon}$. Now we can formu-

late the following

THEOREM. *Consider system (1)*

$$\begin{aligned} \dot{x} &= \epsilon X(\phi,x) && ,x(0)=x_0,\ x\in D\subset\mathbb{R}^n, \\ \dot{\phi} &= \Omega_1(x)+\epsilon\Omega(\phi,x) && ,\phi(0)=\phi_0,\ \phi\in T^2. \end{aligned} \tag{1}$$

and the associated system (3)

$$\begin{aligned} \dot{z} &= \epsilon X^{01}(z)+\epsilon^2 X^{02}(z) && ,z(0)=x_0-\epsilon u^{(1)}(\phi_0,x_0) \\ \dot{\xi} &= \Omega_1(z)+\epsilon\Omega_2^0(z) && ,\psi(0)=\phi_0. \end{aligned} \tag{3}$$

Consider the Fourier expansion $X(\phi,x)=\sum_{k,l=-\infty}^{\infty} c_{kl}e^{i(k\phi_1+l\phi_2)}$. *Restricting oneself to the pairs* (k,l) *which arise non-trivially (i.e.* c_{kl} *not identically zero in D) one assumes that there are no subsets in D such that* $k\Omega_{11}(x)+l\Omega_{12}(x)=0$; *(this means that the combination angles* $k\phi_1+l\phi_2$ *are not permitted to vary slowly in D). If this condition is satisfied* $x(t)=z(t)+\epsilon u^{(1)}(\psi(t),z(t))+O(\epsilon^2)$ *and* $\phi(t)=\psi(t)+O(\epsilon)$, *valid on time-scale* $\frac{1}{\epsilon}$. *Here is*

$$\Omega_{11}\cdot\frac{\partial u^{(1)}}{\partial\psi_1}+\Omega_{12}\cdot\frac{\partial u^{(1)}}{\partial\psi_2}=X(\psi,y)-X^{01}(y),$$

$$X^{01}(y)=\int_{T^2}X(\phi,y)d\phi,$$

$$\Omega_2^0(y)=\int_{T^2}\Omega_2(\phi,y)d\phi,$$

$$X^{02}(y)=\int_{T^2}\nabla X\cdot u^{(1)}+\frac{\partial X}{\partial\psi}\cdot v^{(1)}-\nabla u^{(1)}\cdot X^{01}d\psi,$$

$$\Omega_{11}\frac{\partial v^{(1)}}{\partial\psi_1}+\Omega_{12}\frac{\partial v^{(1)}}{\partial\psi_2}=\nabla\Omega_1\cdot u^{(1)}+\Omega_2-\Omega_2^0.$$

PROOF. This is a straightforward application of Lemma 1 and Lemma 2.

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