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Profits and Pitfalls of Timescales in Asymptotics*

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Abstract. The method of multiple timescales is widely used in engineering and mathematical physics. In this article we draw attention to the literature on the comparison of various perturbation methods. We indicate where we can obtain an advantage from the concept of timescales, and we present examples where the anticipation of timescales makes sense and cases where, because of resonances or bifurcations, the analysis is less straightforward. In a number of problems second order approximations are essential to understand the phenomena. We will conclude from these examples that the anticipation of timescales as in the multiple timescales method may give misleading results; methods that do not anticipate timescales a priori such as averaging and renormalization should be used in research problems.

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1. Introduction.

Many problems in physics and engineering can be formulated as a perturbation problem, i.e., as a small perturbation of a problem that we know how to solve. Usually a small, positive parameter ε plays a part in the formulation. We start with some examples to illustrate the concept of timescales. First we discuss a few simple, traditional problems. In later sections we will see that in many research problems the anticipation of timescales is not so easy.

Example 1. *Consider the harmonic equation with a slight perturbation of the frequency* 1*:*

$$\ddot{x} + (1+\varepsilon)x = 0.$$

In this case it is easy to solve the perturbed equation, and we find the general solution

$$x(t) = A\cos(\sqrt{1+\varepsilon}t) + B\sin(\sqrt{1+\varepsilon}t)$$

with arbitrary constants A and B which are, for instance, determined by initial conditions. Expanding with respect to ε in a Taylor series, we find

$$\cos(\sqrt{1+\varepsilon}t) = \cos t - \frac{\varepsilon t}{2}\sin t + \frac{\varepsilon^2 t}{8}\sin t - \frac{\varepsilon^2 t^2}{8}\cos t + \varepsilon^3 \cdots$$

and for $\sin t$ a similar expression. The exact solution is periodic with respect to t, but the expansion with respect to ε is not. In fact, the expansion contains terms that are

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unbounded with time, so-called secular *terms. These secular terms assume different forms and are called timelike variables or* timescales. *In this elementary problem, the timescales* $t, \varepsilon t$, *and* $\varepsilon^2 t$ *all play a part.*

EXAMPLE 2. *A well-known example is the damped harmonic oscillator*

$$\ddot{x} + \mu\dot{x} + x = 0,\ \mu > 0.$$

Usually one chooses μ *rather small to avoid quenching the oscillation too quickly. Suppose now that we are considering a mechanical process where, for some reason, the damping slowly increases from (say)* $\mu = \varepsilon$ *to* $\mu = 2\varepsilon$. *For this oscillator, we propose the equation*

$$\ddot{x} + \varepsilon(2 - e^{-\varepsilon t})\dot{x} + x = 0.$$

Note that in the equation a timescale, εt, *is already present, but maybe the dynamics of this oscillator will produce more timescales. If* $t = 0$, *we have the damped oscillator given above for* $\mu = \varepsilon$; *if we let* t *tend to infinity, we have this oscillator with* $\mu = 2\varepsilon$. *What happens for the time in between? If* $\varepsilon = 0$, *the independent variable is time* t. *It is natural to assume that as the damping varies with* εt, *an approximation of the problem can be achieved by assuming that two* timescales *play a part:* t *and* εt. *We will show how to handle such a problem.*

The picture of timescales is not always so simple as in the examples above, and there are many other types of perturbation problems. Consider, for instance, the following example of the classical Euler equation.

EXAMPLE 3.

$$t^2\ddot{x} - t\dot{x} + (1 + \varepsilon)x = 0.$$

The so-called Euler-index λ *is obtained by substituting* $x(t) = t^\lambda$. *This produces the index equation*

$$\lambda^2 - 2\lambda + 1 + \varepsilon = 0,$$

with

$$\lambda = 1 \pm i\sqrt{\varepsilon}.$$

Thus, independent solutions are $t\cos(\sqrt{\varepsilon}\ln t)$ *and* $t\sin(\sqrt{\varepsilon}\ln t)$ *with timescales* t *and* $\sqrt{\varepsilon}\ln t$. *However, ignoring the exact solution and putting* $\varepsilon = 0$ *in the equation gives the index equation*

$$\lambda^2 - 2\lambda + 1 = 0,$$

with double roots 1. *The independent solutions if* $\varepsilon = 0$ *are* t *and* $t\ln t$, *which do not have much in common with the perturbed solutions.*

This is a so-called bifurcation problem for the index equation, which has two coincident solutions that bifurcate to two different solutions when adding a small parameter. We will discuss bifurcations more generally in what follows.

The examples so far have all been concerned with linear equations; in some cases we could easily identify so-called "natural timescales," but sometimes a linear problem already has unexpected phenomena. The first two cases above are simple examples of problems we can handle systematically. In such problems we look for bounded or even periodic solutions, and it turns out that *secularity conditions* play an essential part. Example 3 concerns a typical bifurcation problem, but it contains unbounded solutions. The case of unbounded solutions is not well suited for averaging or multiple timescale methods. In this paper we will consider bounded solutions of ordinary differential equations (ODEs).

Two basic concepts will play a part in this article: normal forms (or normalization) and bifurcation theory. We know of normal forms from linear algebra, for instance, when putting an $n \times n$ matrix into diagonal form by a linear transformation if all the eigenvalues are different or into Jordan normal form if they are not. Normalization in this context means putting the mathematical object, in this case a matrix, into a simpler form by transformation. In the same spirit, perturbation methods for ODEs can also be seen as normal form methods, for the idea is always to transform the ODEs to equations that are simpler and can be solved or at least partly solved. For ODEs, normalization has, for instance, been formulated in [2], [3], [14], and [20]. A number of examples will demonstrate this. The reader will also find useful examples and discussions in the books [6] and [1]

Bifurcation theory plays a large part in the research of the mathematical sciences and it is a topic that extends far beyond perturbation theory. Bifurcations in a mathematical or physical system correspond to qualitative changes of the states of the system, for instance, stability changes or the emergence of a new type of solution like quasi-periodic or chaotic solutions. Usually, systems in mathematical physics are described by equations containing a number of parameters. Consider, for instance, the Mathieu equation of section 4 in the form

$$\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0.$$

It is a classical result that the trivial solution $(x, \dot{x}) = (0, 0)$ is unstable if $\omega = 1, 2, \dots$. Also, for certain values of ω, periodic solutions emerge or vanish. Such values of ω are called bifurcation values of the Mathieu equation, and the phenomena of transition from stability to instability, or the emergence or vanishing of periodic solutions, are called bifurcations.

In this article we will address the question of how to obtain asymptotic approximations based on the concept of timescales without a priori knowing them. Methods that anticipate timescales may produce erroneous results. A good second question is to ask in what sense we have obtained an approximation of the solution of the original problem. A number of tutorial aspects of this paper can also be found in [24].

The numerical explorations in this paper were carried out using Runge–Kutta 4 and checked by RADAU5.

2. The Basic Idea of Two (or Multiple) Timescales. A rather general perturbation problem is posed by ODEs that contain a small positive parameter ε as in

$$\dot{x} = f(t, x, \varepsilon),\ x \in \mathbb{R}^n, \tag{2.1}$$

depending smoothly to some order on x and t for $t_0 \leq t < \infty$ and ε for $0 \leq \varepsilon \leq \varepsilon_0$; the dot represents differentiation with respect to t. Thus, we can Taylor-expand:

$$\dot{x} = f_0(t, x) + \varepsilon f_1(t, x) + \varepsilon^2 f_2(t, x) + \varepsilon^3 \cdots . \tag{2.2}$$

For such a general problem, we usually cannot formulate explicit solutions of the equation in terms of elementary functions, but we can expect the presence of certain timescales $t, \varepsilon t, \varepsilon^2 t, \dots$, which we called "natural" in the introduction, on which approximate solutions depend; in some problems we have similar choices for spatial variables. Contrasted with this approach of anticipating timescales is averaging, a normal form method, where no a priori assumption on the form of time dependence is made. This contrasting approach also holds for the renormalization method.

The idea of anticipating timescales was introduced in Kiev by Krylov and Bogoliubov in 1935 [12]; the first application (as far as we are aware) was by Kuzmak in 1959 [13], but after this the Kiev school seems to have rejected the idea of multiple timescales (see [24]). After 1960, the idea was advocated and studied by Kevorkian [9], Cochran [4], and Nayfeh; see, for instance, [16]. The method, also called multiple timing, is intuitively clear and became very popular, especially in engineering.

Introductions to the multiple timescale method can be found in [2], [16], [20], and [23]. A comparison of averaging and multiple timing by a number of important examples can be found in [10]. The relation between multiple timing and the renormalization method was discussed in [2], [3], and [14]. In [17], Perko established the equivalence of the averaging method and multiple timing for standard equations like

$$\dot{x} = \varepsilon f(t, x)$$

on intervals of time of order $1/\varepsilon$. This was a major step forward. See also the extensive discussions in [15], [5], and [20].

Asymptotic equivalence of methods would imply that, considering a solution of a differential equation $x(t)$, expressions $\bar{x}_1(t)$ and $\bar{x}_2(t)$ obtained by different methods should both represent an approximation of $x(t)$ with error $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$ on the same interval of time (for instance, of size $1/\varepsilon$).

In what follows, we will often indicate that an approximation with error $\delta(\varepsilon)$ is valid on an interval of size $1/\varepsilon$. A more precise statement is that the error estimate is valid for $t_0 \leq \varepsilon t \leq t_0 + L$ with t_0, L constants independent of ε. It was shown in [17] that the approximations obtained by averaging and by multiple timing are equivalent to $O(\varepsilon)$ on intervals of time of order $1/\varepsilon$.

2.1. Two Timescales. Many small ε parameter problems are studied using timescales like $t, \varepsilon t$, $\varepsilon^2 t$, and, in general, $\varepsilon^n t$ with $n \in \mathbb{N}$. A simple but typical approach for two timescales runs as follows. Consider the equation

$$\dot{x} = \varepsilon f(t, x) \tag{2.3}$$

with $f(t, x)$ T-periodic in t; the initial value $x(0)$ is given. Such a slowly varying equation might be obtained by solving an equation like (2.2) for $\varepsilon = 0$ and then applying variation of constants for the equation with $\varepsilon > 0$. We will look for solutions of the form

$$x(t) = x_0(t, \tau) + \varepsilon x_1(t, \tau) + \varepsilon^2 + \cdots \tag{2.4}$$

with $\tau = \varepsilon t$; the dots represent the higher order expansion terms. As the unknown functions $x_0, x_1, \ldots$ are supposed to depend on two variables, we have to transform the differential operator. We return to Example 2 for an illustration.

EXAMPLE 4.

$$\ddot{x} + \varepsilon(2 - e^{-\varepsilon t})\dot{x} + x = 0.$$

Introducing $\tau = \varepsilon t$, *the differential operators, and the expansion* (2.4) *into the equation* *yields*

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial^2}{\partial t\partial\tau} + \varepsilon^2\frac{\partial^2}{\partial t^2} + \varepsilon^2 + \cdots\right)(x_0 + \varepsilon x_1 + \varepsilon^2 + \cdots)$$
$$+ \varepsilon(2 - e^{-\tau})\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial\tau}\right)(x_0 + \varepsilon x_1 + \varepsilon^2 + \cdots) + (x_0 + \varepsilon x_1 + \varepsilon^2 + \cdots) = 0.$$

Collecting equal powers of ε, we find to zero order

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0,$$

with general solution

$$x_0(t,\tau) = A(\tau)\cos t + B(\tau)\sin t.$$

The unknown functions $A(\tau), B(\tau)$ are determined at the next order of ε:

$$\frac{\partial^2 x_1}{\partial t^2} + 2\frac{\partial^2 x_0}{\partial t\partial\tau} + (2 - e^{-\tau})\frac{\partial x_0}{\partial t} + x_1 = 0.$$

Using the expression for x_0, we can write this as

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = 2\left(\frac{dA}{d\tau}\sin t - \frac{dB}{d\tau}\cos t\right) + (2 - e^{-\tau})(A\sin t - B\cos t).$$

The solutions of the inhomogeneous harmonic equation produce unbounded (secular) terms unless

$$2\frac{dA}{d\tau} + (2 - e^{-\tau})A = 0,\ 2\frac{dB}{d\tau} + (2 - e^{-\tau})B = 0.$$

These equations are called the (non)secularity conditions of the expansion. Solving the equations for A and B, we find to first order for $x(t)$

$$e^{-\tau - \frac{1}{2}e^{-\tau} + \frac{1}{2}}(A(0)\cos t + B(0)\sin t).$$

$A(0)$ and $B(0)$ are determined by the initial conditions. As expected, the damping increases.

In [5]*, damped second order ODEs were considered with the purpose of showing that in this case $O(\varepsilon)$ approximations on intervals of time of order $1/\varepsilon$ have a validity extension to $[0,\infty)$. With minor adjustments, that result applies to this example. Such an extension of the validity on intervals of time was considered in a more general framework in* [20].

2.2. The Averaging Method. It is interesting to compare the multiple timing result with the approximation obtained by averaging. A fairly modern account of averaging is [20]; introductions can be found in [22] and [23]. The result of Perko [17] predicts that the two methods both yield an $O(\varepsilon)$ approximation, valid on an interval of time of size $1/\varepsilon$.

The idea of averaging is as follows. Consider again the initial value problem for (2.3) with $f(t,x)$ T-periodic in t. Average the right-hand side over t while keeping x fixed. The resulting equation is

$$\dot{y} = \varepsilon f^0(y),\ f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt.$$

We have replaced x by y as, even with the same initial value, the averaged equation defines a different vector function. However, we have from [20] the theorem that if $x(0) - y(0) = O(\varepsilon)$, we have

$$x(t) - y(t) = O(\varepsilon)$$

on a time interval of size $1/\varepsilon$. The equation for y can be considered as a normal form of (2.3), because it is simpler as t has vanished from the equation. It is also clear that y depends on εt as we can rescale $\tau = \varepsilon t$ before solving the equation. As mentioned before, the averaging approximation for the initial value problem for (2.3) is asymptotically equivalent to the first order result obtained by multiple timing. We will see that divergences may arise when considering bifurcation problems and also when considering intervals of time longer than $1/\varepsilon$.

EXAMPLE 5. *To handle Example 4 we put $\tau = \varepsilon t$ in the equation and transform by variation of constants $x, \dot{x} \to A, B$:*

$$x = A\cos t + B\sin t,\ \ \dot{x} = -A\sin t + B\cos t.$$

Substituting this into the three-dimensional system

$$\ddot{x} + \varepsilon(2 - e^{-\tau})\dot{x} + x = 0,\ \ \dot{\tau} = \varepsilon,$$

and using that

$$\frac{dx}{dt} = \dot{A}\cos t - A\sin t + \dot{B}\sin t + B\cos t = -A\sin t + B\cos t,$$

we find by solving for $\dot{A}, \dot{B}$ that

$$\frac{dA}{dt} = \varepsilon(2 - e^{-\tau})(-A\sin t + B\cos t)\sin t,$$
$$\frac{dB}{dt} = -\varepsilon(2 - e^{-\tau})(-A\sin t + B\cos t)\cos t.$$

Averaging over t and rescaling $\tau = \varepsilon t$ in the equations, we find the same results for the approximations $\tilde{A}, \tilde{B}$ as were obtained by multiple timing. This is as expected, but note that we did not assume a priori that the approximations of $(x, \dot{x})$ are functions of t, τ. The averaging procedure yields this dependence for the first order approximation.

As an exercise, the reader may compute multiple timing and averaging approximations for the modulated harmonic equation

$$\ddot{x} + (1 + \varepsilon)x = 0$$

and the Van der Pol equation

$$\ddot{x} + x = \varepsilon\dot{x}(1 - x^2),\quad x(0) = r,\ \ \dot{x}(0) = 0.$$

For the Van der Pol equation, averaging leads to the well-known lowest order result

$$x_0(t, \tau) = \frac{re^{\frac{1}{2}\tau}}{(1 + \frac{r^2}{4}(e^{\tau} - 1))^{\frac{1}{2}}}\cos t,\ \ \tau = \varepsilon t.$$

If we initially have $r = 2$, the lowest order term is periodic. This lowest order term represents an $O(\varepsilon)$ asymptotic approximation, valid on the timescale $1/\varepsilon$. For the amplitude of the periodic solution r, the validity of the $O(\varepsilon)$ approximation can be extended to $[0, \infty)$ (see [20]); this is not the case for the corresponding phase.

2.3. The Renormalization Method. The renormalization (or renormalization group) method has been described in [2] and [3], where many more references can also be found; an introduction is given in [11]. The analysis in [2] is formal but demonstrates the technique clearly. The method bears some resemblance to the Poincaré–Lindstedt method for periodic solutions of ODEs (see [22]); in this classical method, which is based on the implicit function theorem, both amplitudes and phases (or, more generally, initial conditions) and the time variable are expanded with respect to the small parameter. This involves dummy timelike variables that have to be identified during the subsequent computation steps. To avoid unbounded terms in the expansions, nonsecularity conditions are applied as in the examples above. The formulation of the renormalization method is much more general than the Poincaré–Lindstedt method: it is applicable to nonperiodic solutions and even to boundary layer problems. It is related to the averaging method in one fundamental respect: timescales emerge by the nonsecularity conditions, and they are not imposed a priori.

In all perturbation methods there is some freedom of choice, but with the right choices it is shown in [3] that the first and second order approximation results of averaging and renormalization are equivalent for ODEs of standard form (2.3). It is shown in [2] and [3] that if one uses multiple timing and by luck and/or experience has guessed the correct timescales, the expressions agree with averaging and renormalization. Some examples in [2] show that multiple timing may ignore important timescales and is in such examples deficient; see also section 4.

3. Algebraic Timescales for Bifurcations. Analytic and numerical approximation theory gives us useful details of a dynamical system, but one of the basic questions of engineering and mathematical physics is how to obtain a global picture; this is tied to the study of qualitative changes when the parameters of the system pass certain critical values. As we shall see, it is important in these problems to avoid making a priori assumptions on timescales.

Readers who are not acquainted with bifurcations should have a look at standard examples: Consider the following equation with bifurcation parameter μ:

$$\dot{x} = \mu x - x^3.$$

It is not difficult to see that if $\mu < 0$, there exists only one (stable) equilibrium solution. When μ passes zero, we have for $\mu > 0$ three equilibria, two stable and one unstable. This equation characterizes the so-called pitchfork bifurcation.

In the analysis of bifurcations in applications, some form of approximation theory is used, combined with linearization and matrix calculations. A typical computation for an equation of the form $\dot{x} = f(x, t, \varepsilon)$ is to identify an equilibrium or special time-dependent solution and study the behavior of this solution as the parameters change; this leads to the calculation of eigenvalues, Lyapunov exponents, and characteristic multipliers. In its simplest form, bifurcation phenomena in ODEs lead by linearization near an equilibrium to studying systems like

$$\dot{x} = A(\varepsilon)x, \tag{3.1}$$

where we can expand the $n \times n$ matrix A as

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots .$$

The $n \times n$ matrices A_n do not depend on ε.

If we have started with the standard form (2.3), we will have $A_0 = 0$. More generally, A_0 is derived from the unperturbed problem, A_1 is produced by perturbation methods, by a special effort we may find A_2, and hopefully we will have some knowledge about higher order terms. An important question is then what the eigenvalues of A_0 and $A_0 + \varepsilon A_1$ tell us about the eigenvalues of $A(\varepsilon)$. This question is tied to the structural stability of the matrices and whether their eigenvalues are single or multiple. Failure of structural stability and the appearance of multiple eigenvalues are characteristic of bifurcation phenomena and so merit special attention.

An $n \times n$ matrix is called *structurally stable* if all eigenvalues have nonzero real part. If we have a zero eigenvalue or purely imaginary eigenvalues, we can expect bifurcations. The presence of multiple eigenvalues affects the form of the expansions and the timescales.

EXAMPLE 6. *We start with an example derived from an equation in standard form* (2.3)*, so* $A_0 = 0$*. The expansion of* $A(\varepsilon)$ *up to* A_2 *is*

$$A(\varepsilon) = \varepsilon \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The system of differential equations becomes

$$\dot{x} = \varepsilon^2 y,\ x(0) = 0,$$
$$\dot{y} = -\varepsilon x,\ y(0) = 1.$$

A_1 *has two zero eigenvalues, and* $\varepsilon A_1 + \varepsilon^2 A_2$ *has eigenvalues* $\pm\varepsilon^{\frac{3}{2}} i$*. The solution of the initial value problem is*

$$x(t) = \varepsilon^{\frac{1}{2}} \sin(\varepsilon^{\frac{3}{2}} t),\ y(t) = \cos(\varepsilon^{\frac{3}{2}} t).$$

As can be seen from the eigenvalues, the timescale $\varepsilon^{\frac{3}{2}} t$ *plays a part. Expanding the trigonometric functions on an interval of time of size* $1/\varepsilon$*, we find that the timescales* t *and* εt *can be used to obtain asymptotic estimates. On a longer interval of time, for instance,* $1/\varepsilon^2$*, we need the timescale* $\varepsilon^{\frac{3}{2}} t$ *to obtain asymptotic estimates.*

For bifurcations, local linearization leads to eigenvalue problems associated with (3.1), so algebraic timescales are natural phenomena. Consider the following nonlinear problem.

EXAMPLE 7.

$$\dot{x} = A(\varepsilon)x + \varepsilon f(x),$$

with $f(x)$ *a polynomial four-dimensional vector field starting with quadratic terms in a neigborhood of* $x = 0$*. We have* $A_0 = 0$ *and*

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To $O(\varepsilon)$ *we have the eigenvalues* $\pm\varepsilon i, 0, 0$*, which suggests a so-called Hopf bifurcation and the presence of an additional two-dimensional center manifold in the full nonlinear system. Including the* ε^2 *terms, we find the eigenvalues* $\pm\varepsilon i, \pm\sqrt{a}\varepsilon^{\frac{3}{2}}$*. If* $a > 0$*, we have from the linearized system periodic solutions in a two-dimensional manifold* *that is unstable with timescale* $\sqrt{a}\varepsilon^{\frac{3}{2}}t$*. If* $a < 0$*, we have from the linearized system quasi-periodic solutions for most values of* a*, and periodic solutions for a denumerable number of values.*

What will happen in the full system depends on the choice of the nonlinear vector field $f(x)$*.*

Can we predict the form $\varepsilon^q t$ with q rational of such algebraic timescales? The following questions and results are classical.

Consider the following matrix expansion obtained by a perturbation method:

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 + \cdots .$$

- Can the eigenvalues be expanded in a convergent series of the form

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2 + \cdots ,$$

  where λ_0 is an eigenvalue of the matrix A_0? If this is the case, we expect timescales of the form $t, \varepsilon t, \varepsilon^2 t \ldots$.
- If we are in the critical case of bifurcations where λ_0 is zero or purely imaginary, how do the perturbations affect the eigenvalues and thus the qualitative behavior of the solutions of the differential equations?

If A_0 vanishes, we extract ε and treat A_1 as a perturbed matrix. We refer to [23] for references and summarize the results.

- If λ_0 is single, we have

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2 + \cdots .$$

  If $\lambda_0 = 0$, this means we have an $O(\varepsilon)$ size eigenvalue.
- According to Newton and Puisieux, if λ_0 is multiple, the expansion is in fractional powers of ε.

We conclude this section with a modification of an exercise formulated by Hale [7].

Example 8. *Consider the system*

$$\begin{cases} \ddot{x} + x = \varepsilon(1 - x^2 - ay^2)\dot{x}, \\ \ddot{y} + 2y = \varepsilon(1 - y^2 - ax^2)\dot{y}, \end{cases} \tag{3.2}$$

with positive parameter a*. There exist periodic solutions in the normal mode planes* $x = \dot{x} = 0$ *and* $y = \dot{y} = 0$*, but at this point we are interested in solutions outside the normal mode planes. We use the variation of constants transformation* $x, \dot{x}.y, \dot{y} \to r_1, \phi_1, r_2, \phi_2$*,*

$$x = r_1 \cos(t + \phi_1), \qquad \dot{x} = -r_1 \sin(t + \phi_1), \tag{3.3}$$

$$y = r_2 \cos(\sqrt{2}t + \phi_2), \qquad \dot{y} = -\sqrt{2}r_2 \sin(\sqrt{2}t + \phi_2), \tag{3.4}$$

to find a slowly varying system of the form of (2.3) *for* r_1, ϕ_1, r_2, ϕ_2*. The right-hand side is* 2π*- and* $2\pi/\sqrt{2}$*-periodic in* t*. We average over these two periods to obtain the system*

$$\begin{cases} \dot{r_1} = \frac{\varepsilon}{2} r_1(1 - \frac{1}{4}r_1^2 - \frac{1}{2}ar_2^2), & \dot{\phi_1} = 0, \\ \dot{r_2} = \frac{\varepsilon}{2\sqrt{2}} r_2(1 - \frac{1}{4}r_2^2 - \frac{1}{2}ar_1^2), & \dot{\phi_2} = 0. \end{cases} \tag{3.5}$$

According to Theorem 11.2 *of* [23]*, averaging over two independent periods of a system in standard form produces an* $O(\varepsilon)$ *approximation on intervals of time* $O(1/\varepsilon)$*. With* *some abuse of language we have used the same symbols for the approximate amplitudes and phase angles. Consider the zeros of the right-hand side of the averaged equations; by elementary analysis of the two quadrics*

$$1 - \frac{1}{4}r_1^2 - \frac{1}{2}ar_2^2 = 0, \qquad 1 - \frac{1}{4}r_2^2 - \frac{1}{2}ar_1^2 = 0,$$

we find positive zeros $r_1 = R_1$, $r_2 = R_2$. *If* $a \neq 1/2$, *these stationary solutions of the averaged equations correspond with quasi-periodic motion with fixed amplitudes; geometrically this describes a torus in phase space; if* $a = 1/2$, *the dynamics is also quasi-periodic, but it takes place on the sphere*

$$r_1^2 + r_2^2 = 4.$$

It is evident that as the parameter a *is varied, the transition at* $a = 1/2$ *involves a bifurcation as the flow on a torus changes to a flow on a sphere. Because of some degenerations in the averaged equations, the existence of tori, respectively, spheres, for the original system* (3.2) *is not immediate, but the existence can be proved; see* [23].

Linearization of the averaged vector field at the nontrivial zeros produces the system

$$\dot{r_1} = -\frac{\varepsilon}{4}(R_1^2 r_1 + 2aR_1R_2r_2),$$
$$\dot{r_2} = -\frac{\varepsilon}{4\sqrt{2}}(2aR_1R_2r_1 + R_2^2 r_2),$$

with again some abuse of language as we are using the same symbols r_1, r_2. *Note that the trace of the matrix of coefficients is negative, so the flow is exponentially contracting locally. For the determinant we find*

$$\frac{\varepsilon^2}{4\sqrt{2}}R_1^2R_2^2\left(\frac{1}{4} - a^2\right).$$

It follows that if $a > 1/2$, *the torus is unstable; if* $0 < a < 1/2$, *the torus is stable.*

If $a = 1/2$, *we have a bifurcation into a sphere with one eigenvalue zero. To establish the existence and stability of this geometrical object we have to include* $O(\varepsilon^2)$ *terms in the equations. We restrict ourselves here to determining the timescales involved. Adding second order terms, the eigenvalue equation for* $a = 1/2$ *will be of the form*

$$\lambda^2 + (c_1\varepsilon + c_2\varepsilon^2)\lambda + c_3\varepsilon^2 = 0$$

with constants c_1, c_2, c_3 *independent of* ε. *The eigenvalues will have a power series expansion with respect to* ε *as predicted by the general theory discussed above in the case of single eigenvalues. The timescales involved in the approximation of the quasi-periodic sphere will be* $t, \varepsilon t, \varepsilon^2 t, \dots$.

As mentioned above, the tori and sphere are persistent in the full system; more results relating to this exercise can be found in [23].

As we have seen, a bifurcation with single eigenvalues, even in the case of eigenvalue zero, produces standard power series. In the next section we consider an application with a multiplicity two zero eigenvalue.

4. Application: The Mathieu Equation. We consider the Mathieu equation, which plays a part in many engineering problems,

$$\ddot{x} + (\omega^2(\varepsilon) + \varepsilon\cos\nu t)x = 0,$$

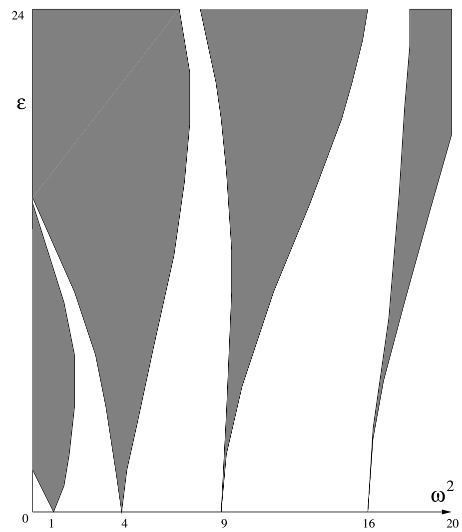

Fig. 4.1 *The gray Floquet tongues denote those parameter values ω and ε for which the trivial solution of the Mathieu equation is unstable. In our approximations we have described the lower part of the tongue emerging from $\omega = 1$ as in* (4.1).

in its fundamental 1:2-resonance with a slight detuning:

$$\ddot{x} + (1 + \varepsilon a + \varepsilon^2 b + \varepsilon \cos 2t)x = 0. \tag{4.1}$$

a and b are free parameters independent of ε, and $\omega^2 = 1 + \varepsilon a + \varepsilon^2 b$. We apply Lagrange variation of constants

$$x = y_1 \cos t + y_2 \sin t, \ \dot{x} = -y_1 \sin t + y_2 \cos t.$$

The slowly varying equations for (y_1, y_2) are, after averaging, of the form $\dot{y} = A(\varepsilon)y$ this (averaging) normal form approach produces to first order in ε

$$A(\varepsilon) = +\varepsilon \begin{pmatrix} 0 & \frac{1}{2}(a - \frac{1}{2}) \\ -\frac{1}{2}(a + \frac{1}{2}) & 0 \end{pmatrix} + O(\varepsilon^2).$$

The eigenvalues are

$$\lambda_{1,2} = \pm \frac{\varepsilon}{2} \sqrt{\frac{1}{4} - a^2},$$

and the two approximate independent solutions for (y_1, y_2) can be written as

$$e^{\pm \frac{1}{2} \sqrt{\frac{1}{4} - a^2}\, \varepsilon t}.$$

This leads to the well-known result that for $a^2 > \frac{1}{4}$ the solutions of the Mathieu equation are stable (the approximate solutions are trigonometric) and for $a^2 < \frac{1}{4}$ they are unstable. The approximations with appropriate initial values have error estimate $O(\varepsilon)$ on a long time interval $O(1/\varepsilon)$. In this approximation, the timescales for $x(t)$ are t and εt. The boundary of the instability domains, the Floquet tongues, are the bifurcation curves where the transition from unstable to stable solutions takes place in (ω^2, ε)-parameter space; see Figure 4.1.

4.1. What Happens at the Tongue Boundary? What happens at the transition values, for instance, at $\omega^2 = 1 + \varepsilon a$, where $a = \frac{1}{2}$? In this case, we have for the normal form to first order

$$A_1 = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix},$$

a typical degenerate matrix from bifurcation theory. Following [20] or [23] we perform a second order averaging normalization to find

$$A_2 = \begin{pmatrix} 0 & \frac{1}{64} + \frac{1}{2}b \\ \frac{7}{64} - \frac{1}{2}b & 0 \end{pmatrix}.$$

We find for the eigenvalues of $A(\varepsilon)$ to this order of approximation

$$\lambda^2 = -\frac{1}{4}\left(b + \frac{1}{32}\right)\varepsilon^3 + \frac{1}{4}\left(b + \frac{1}{32}\right)\left(\frac{7}{32} - b\right)\varepsilon^4.$$

The $O(\varepsilon^3)$ term dominates, and $b = -\frac{1}{32}$ produces a more precise location of the Floquet tongue. If $b > -\frac{1}{32}$, we have stability; if $b < -\frac{3}{32}$, we have instability.

The second order approximations of the solutions for (y_1, y_2) are a linear combination of $\exp.(+\lambda t)$ and $\exp.(-\lambda t)$. With appropriate initial values they yield approximations of the solutions of the Mathieu equation (4.1) with error estimate $O(\varepsilon^2)$ on a long time interval $O(1/\varepsilon)$.

It is remarkable that the timescale $\varepsilon^{\frac{3}{2}}t$ plays a part in this problem because near the boundary of the Floquet tongue we have that $\lambda^2 = O(\varepsilon^3)$. The timescales characterizing the flow near the Floquet tongue are until second order

$$t,\ \varepsilon t,\ \varepsilon^{\frac{3}{2}}t,\ \varepsilon^2 t.$$

The presence of the timescale $\varepsilon^{\frac{3}{2}}t$ was noted for the Mathieu equation in [2], using the renormalization method but without the explanation using bifurcation theory given here. It was also noted in [2] that, when using multiple timing with timescales $t, \varepsilon t, \varepsilon^2 t$, this extra timescale is not discovered.

5. Timescales for Resonance Manifolds. For a number of problems it is convenient to reformulate the problem in terms of actions (or amplitudes) and angles. Such angles can be timelike, i.e., they are monotonically increasing with time. If they are not timelike at certain values of parameters or variables, we will call this resonance. For the theory and more references we refer to [20] and [23]; here we will illustrate the phenomenon with examples.

5.1. Introductory Examples. Consider the following one-degree-of-freedom autonomous system.

EXAMPLE 9. *The equation to be studied is*

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}),$$

with (positive) constant frequency ω. Putting $\dot{x} = \omega y$ and introducing amplitude-angle variables $x, y \to r, \phi$ by

$$x = r\sin\phi,\ y = r\cos\phi,$$

we find the equations

$$\dot{r} = \varepsilon \frac{\cos\phi}{\omega} f(r\sin\phi, \omega r\cos\phi),$$
$$\dot{\phi} = \omega - \varepsilon \frac{\sin\phi}{\omega r} f(r\sin\phi, \omega r\cos\phi).$$

If $\omega > 0$, the angle ϕ is timelike; one observes that the right-hand sides are 2π-periodic in ϕ and a perturbation scheme can be started, for instance, by averaging over ϕ. Apply this, for instance, to the damped, Duffing equation, where $f(x, \dot{x}) = -a\dot{x} - bx^3$.

New phenomena may emerge in the case of more degrees of freedom.

EXAMPLE 10. *After suitable transformations in an application, we have obtained the system*

$$\begin{cases} \dot{x_1} = \varepsilon, \\ \dot{x_2} = \varepsilon\left(\cos(\phi_1 - \phi_2) + 2\cos(\phi_1 - 2\phi_2)\right), \\ \dot{\phi}_1 = x_1 + x_2, \\ \dot{\phi}_2 = x_2. \end{cases} \tag{5.1}$$

We have two amplitudes, x_1, x_2, and two angles, ϕ_1, ϕ_2; the combination angles $\psi = \phi_1 - \phi_2$, $\chi = \phi_1 - 2\phi_2$ are expected to play a part. We consider the angles ϕ_1, ϕ_2 as timelike variables and average over them; this is also called "averaging over a torus." The operation results in an average of zero for the right-hand side of the amplitude x_2, so there is hardly any change expected in the magnitude of this amplitude. Is this true and is it a correct strategy? The answer is affirmative in the cases that the angles are indeed timelike, but not in the domains where

$$\dot{\phi}_1 - \dot{\phi}_2 = \dot{\psi} \approx 0, \quad \dot{\phi}_1 - 2\dot{\phi}_2 = \dot{\chi} \approx 0.$$

These domains are called resonance manifolds or resonance zones; their size has to be established. We rewrite the system as

$$\begin{cases} \dot{x_1} = \varepsilon, \\ \dot{x_2} = \varepsilon(\cos\psi + 2\cos\chi), \\ \dot{\psi} = x_1, \\ \dot{\chi} = x_1 - x_2. \end{cases} \tag{5.2}$$

To study what happens in these domains, we consider the local dynamics in the case where $\dot{\psi}$, respectively, $\dot{\chi}$, is near to zero, and the case that both combination angles vary slowly.

The Resonance Zone near $\dot{\psi} = 0$. *For the first combination angle we find with $\psi(0) = 0$*

$$\frac{d\psi}{dt} = x_1(0) + \varepsilon t \to \psi(t) = x_1(0)t + \frac{1}{2}\varepsilon t^2.$$

So, if $x_1(0) > 0$, this resonance zone plays no part. If $x_1(0) \leq 0$, the solutions show a forced passage through the resonance zone. We will write down the solutions in this resonance zone explicitly, but first we need more insight into the asymptotic scales.

This resonance zone is located near $x_1 = 0$, which suggests the introduction of the local variable ξ_1 by

$$x_1 = \delta(\varepsilon)\xi_1, \ \delta(\varepsilon) = o(1).$$

System (5.2) *transforms to*

$$\begin{aligned} \delta(\varepsilon)\dot{\xi}_1 &= \varepsilon, \\ \dot{x_2} &= \varepsilon(\cos\psi + 2\cos\chi), \\ \dot{\psi} &= \delta(\varepsilon)\xi_1, \\ \dot{\chi} &= \delta(\varepsilon)\xi_1 - x_2. \end{aligned}$$

Choosing $\delta(\varepsilon) = \sqrt{\varepsilon}$, *the small terms of the transformed system have the same size; in the theory of singular perturbations this is called "a distinguished parameter," "a parameter producing a significant degeneration," or simply "balancing of equations." The system becomes*

$$\begin{aligned} \dot{\xi}_1 &= \sqrt{\varepsilon}, \\ \dot{x_2} &= \varepsilon(\cos\psi + 2\cos\chi), \\ \dot{\psi} &= \sqrt{\varepsilon}\xi_1, \\ \dot{\chi} &= \sqrt{\varepsilon}\xi_1 - x_2. \end{aligned}$$

The combination angle χ *is timelike in this resonance zone. Differentiating the equation for* ψ, *we find nothing new,*

$$\ddot{\psi} = \varepsilon,$$

but we know now by a systematic procedure the size of this resonance zone, $\sqrt{\varepsilon}$, *and the timescale in the resonance zone,* $\sqrt{\varepsilon}t$. *As indicated above, the solutions that start with* $x_1(0) < 0$ *are forced through the resonance zone with*

$$x_2(t) = x_2(0) + \varepsilon \int_0^t \cos\left(x_1(0)s + \frac{1}{2}\varepsilon s^2\right) ds + \varepsilon^2 + \cdots .$$

This is an interesting integral to evaluate. For instance, starting at $x_1(0) = 0$ *we can use the result*

$$\int_0^\infty \cos\left(\frac{1}{2}\varepsilon s^2\right) ds = \frac{1}{2}\sqrt{\frac{\pi}{\varepsilon}}.$$

If we expected for the passage of the resonance zone a time interval of order $1/\sqrt{\varepsilon}$, *this would cause an* $O(\sqrt{\varepsilon})$ *change of* $x_2(t)$. *We present in Figure* 5.1 *some of the solutions for* $x_2(t)$ *when passing through the resonance zone; the numerical calculation is based on the full system* (5.2).

The Resonance Zone near $\dot{\chi} = 0$. *To study the behavior of the solutions in this resonance zone, we rescale as follows:*

$$x_1 - x_2 = \delta(\varepsilon)\xi_2.$$

Here, ξ_2 *is a new, local variable, and again* $\delta(\varepsilon) = o(1)$ *as* $\varepsilon \to 0$. *Introducing* ξ_2 *into the equations produces*

$$\begin{aligned} \dot{x}_1 &= \varepsilon, \\ \delta(\varepsilon)\dot{\xi}_2 &= \varepsilon(1 - \cos\psi - 2\cos\chi), \\ \dot{\psi} &= x_1, \\ \dot{\chi} &= \delta(\varepsilon)\xi_2. \end{aligned}$$

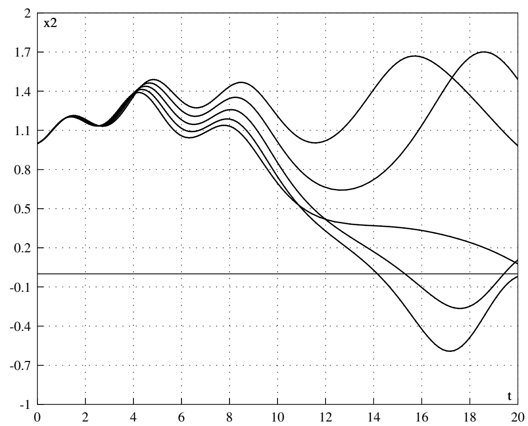

Fig. 5.1 *Dispersion of five orbits by passage through resonance in system* (5.2); $x_1(0) = -0.90, -0.95, -1.00, -1.05, -1.10$, $\varepsilon = 0.1$, *with* $x_2(0) = 1, \psi(0) = 2, \chi(0) = 1$ *for all orbits.*

The equations for ξ_2 *and* χ *have terms of the same size by choosing* $\delta(\varepsilon) = \sqrt{\varepsilon}$. *The equations become with this choice*

$$\begin{aligned} \dot{x}_1 &= \varepsilon, \\ \dot{\xi}_2 &= \sqrt{\varepsilon}(1 - \cos\psi - 2\cos\chi), \\ \dot{\psi} &= x_1, \\ \dot{\chi} &= \sqrt{\varepsilon}\xi_2. \end{aligned}$$

Excluding a neighborhood of $x_1 = 0$, ψ *is timelike, so we average over this variable producing the approximate equation*

$$\dot{\xi}_2 = \sqrt{\varepsilon}(1 - 2\cos\chi).$$

Differentiating the equation for χ, *we find for the leading terms the pendulum equation with constant forcing:*

$$\ddot{\chi} + 2\varepsilon\cos\chi = \varepsilon.$$

Equilibria correspond with $\cos\chi = 1/2$ *or* $\chi = \pm\pi/3$. *The equilibrium* $\chi = -\pi/3$ *is a centerpoint and* $+\pi/3$ *a saddle; the centerpoint is sensitive to higher order perturbations, so a second order calculation is necessary to establish its character. Stationary solutions of the resonance zone equation correspond in a number of problems with stable and unstable periodic solutions of the original system. See Figure* 5.2 *for an illustration; note that because of the unboundedness of* $x_1(t)$, *to remain in the resonance zone,* $x_2(t)$ *has to show approximately similar behavior.*

In this example we had to localize in space to size $O(\sqrt{\varepsilon})$; *the natural timescale in the resonance zone is* $\sqrt{\varepsilon}t$, *and outside the resonance zones it is* εt. *There are more interesting aspects to analyze, in particular, the calculation of a second order approximation, but we restrict ourselves to a few numerical illustrations; see Figure* 5.2,

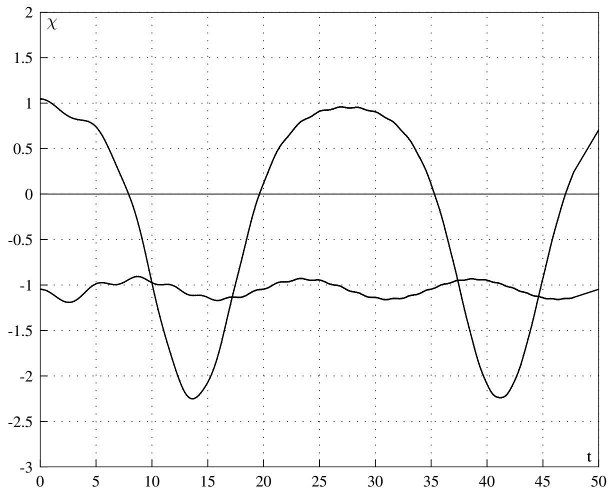

Fig. 5.2 *Numerical solutions of system* (5.2) *starting in the resonance zone* $\dot{\chi} = x_1 - x_2 = 0$, $x_1(0) = x_2(0) = 1, \psi(0) = 0, \varepsilon = 0.1$. *The solution starting from* $\chi(0) = -\pi/3$ *remains near this value, but the unstable case* $\chi(0) = \pi/3$ *shows strong oscillations. The solutions remain in the resonance zone.*

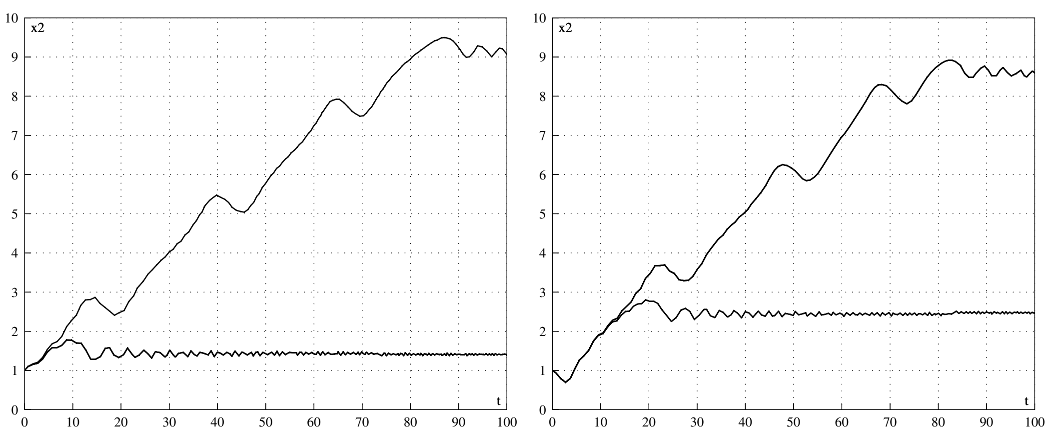

Fig. 5.3 *Numerical solutions of system* (5.2) *starting in the resonance zone* $\dot{\chi} = x_1 - x_2 = 0$, $\varepsilon = 0.1$, *but leaving the zone. In both figures* $x_1(0) = 1, x_2(0) = 1, \psi(0) = 1$. *In each figure,* $x_2(t)$ *tends to bounded oscillations but the amplitudes are very different. Left:* $\chi(0) = 0.9$ *(top) and* $\chi(0) = 0.95$ *(below); Right:* $\chi(0) = 4.14$ *(below) and* $\chi(0) = 4.15$ *(top) .*

where for certain initial values the solutions are caught in resonance, but for other initial values the solutions leave the resonance zone (see Figure 5.3*).*

Intersection of the Two Resonance Zones. *Finally, we have to study the case of intersection of the two resonance zones. This happens in a neighborhood of* $x_1 = x_2 = 0$. *As we have seen above, the angle* ψ *is forced through the resonance zone near* $x_1 = 0$, *so we expect no new phenomena here.*

Discussion. *System* (5.1) *(or* (5.2)*) displays a simple example of passage through resonance and being caught in resonance. The numerical illustrations should be sup-*

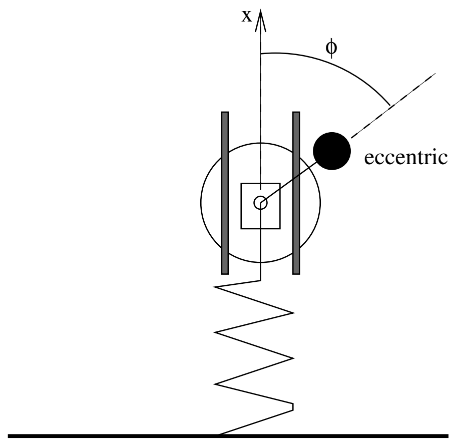

Fig. 5.4 *Eccentric flywheel, rotating on an elastic foundation.*

plemented by second order approximations, as was shown for the Mathieu equation; such approximations for the problem are presented in the next subsection.

The behavior of the solutions outside the resonance zones, as seen in Figure 5.3, *raises difficult questions: some solutions are unbounded, but in other cases* $x_2(t)$ *seems to converge to a definite value. Note that we can derive from system* (5.2) *the equation*

$$\ddot{\chi} + 2\varepsilon \cos\chi = \varepsilon - \varepsilon \cos\left(\psi(0) + x_1(0)t + \frac{1}{2}\varepsilon t^2\right).$$

This is a strongly nonlinear oscillator forced by a time-dependent term that is oscillating, but not periodically or even almost-periodically. As yet, we have not much understanding of such systems.

5.2. An Eccentric Flywheel. Many phenomena encountered in our toy Example 10 are found again in applications. In engineering mechanics, the imbalance of eccentric flywheels may cause resonance. We consider briefly a mechanical model, shown in Figure 5.4, where the phenomena of passage through resonance and being caught in resonance are both present.

Example 11. *An application in* [23, *Example* 12.11 *(with more references therein)*] *describes a slightly eccentric flywheel; see Figure* 5.4. *The vertical displacement* x *of a small mass on the flywheel and its rotation angle* ϕ *are given by*

$$\ddot{x} + x = \varepsilon(-x^3 - \dot{x} + \dot{\phi}^2 \cos\phi) + O(\varepsilon^2),$$
$$\ddot{\phi} = \varepsilon\left(\frac{1}{4}(2 - \dot{\phi}) + (1 - x)\sin\phi\right) + O(\varepsilon^2).$$

To analyze the system, we introduce

$$x = r\sin\phi_2, \dot{x} = r\cos\phi_2, \phi = \phi_1, \dot{\phi}_1 = \Omega,$$

with $r, \Omega > 0$. *We find to* $O(\varepsilon)$ *a system with two angles,* ϕ_1, ϕ_2, *and slowly varying* *variables r and Ω:*

$$\begin{aligned}
\dot{r} &= \varepsilon \cos\phi_2(-r^3 \sin^3\phi_2 - r\cos\phi_2 + \Omega^2 \cos\phi_1),\\
\dot{\Omega} &= \varepsilon\left(\frac{1}{4}(2-\Omega) + \sin\phi_1 - r\sin\phi_1 \sin_2\right),\\
\dot{\phi}_1 &= \Omega,\\
\dot{\phi}_2 &= 1 + \varepsilon\left(r^2\sin^4\phi_2 + \frac{1}{2}\sin 2\phi_2 - \frac{\Omega^2}{r}\cos\phi_1 \sin\phi_2\right).
\end{aligned}$$

Evaluating the trigonometric terms in the slowly varying equations for r and Ω, we find the angles $\phi_1, \phi_2, 2\phi_2, 4\phi_2$ and the combination angles $\phi_1 - \phi_2, \phi_1 + \phi_2$. The right-hand sides of the equations for the angles are positive, so the only resonance zone that can be active is when $\dot{\phi}_1 - \dot{\phi}_2 \approx 0$. As

$$\frac{d}{dt}(\phi_1 - \phi_2) = \Omega - 1 + O(\varepsilon),$$

this happens if Ω is near 1. Note that this analysis includes $O(\varepsilon)$ terms only; if we add higher order terms, more (but smaller) resonance zones may be found. Outside the resonance zone we average over the angles to find an approximation from

$$\begin{aligned}
\dot{r} &= -\frac{\varepsilon}{2}r,\\
\dot{\Omega} &= \frac{\varepsilon}{4}(2-\Omega).
\end{aligned}$$

Although simple looking, this result is already of interest. The deflection x of the flywheel will go exponentially to zero outside the resonance zone; $\Omega(t)$, the rotation speed, will tend to 2, but if $\Omega(0) < 1$, the flywheel will pass through the resonance zone. What happens there? As in the example above, we rescale locally in a neighborhood of $\Omega = 1$ and introduce the combination angle ψ:

$$\Omega = 1 + \delta(\varepsilon)\omega, \ \ \psi = \phi_1 - \phi_2.$$

We find

$$\begin{aligned}
\dot{r} &= O(\varepsilon),\\
\delta(\varepsilon)\dot{\omega} &= \varepsilon\left(\frac{1}{4} + \sin\phi_1 - \frac{1}{2}r\cos\psi + \frac{1}{2}r\cos(2\phi_1 - \psi)\right) + \cdots,\\
\dot{\phi}_1 &= 1 + \cdots,\\
\dot{\psi} &= \delta(\varepsilon)\omega + \cdots.
\end{aligned}$$

The dots represent higher order terms. The equations for ω and ψ have the same size terms if

$$\delta(\varepsilon) = \sqrt{\varepsilon},$$

which determines the size of the resonance zone. Averaging over the remaining angle

ϕ_1 and noting that $r(t)$ varies $O(\varepsilon)$ in the resonance zone, we find to $O(\sqrt{\varepsilon})$

$$\dot{\omega} = \sqrt{\varepsilon}\left(\frac{1}{4} - \frac{1}{2} r\cos\psi\right),$$
$$\dot{\psi} = \sqrt{\varepsilon}\omega,$$

where we have neglected higher order terms. Differentiating the equation for ψ, we find again a pendulum equation describing the dynamics in the resonance zone:

$$\ddot{\psi} + \frac{1}{2}\varepsilon r(0)\cos\psi = \frac{1}{4}\varepsilon.$$

It turns out that the resonance zone near $\Omega = 1$ is of size $O(\sqrt{\varepsilon})$, and the timescale of the dynamics is $\sqrt{\varepsilon}t$. The center stationary solution of the pendulum equation corresponds with a stable periodic solution, and the saddle with an unstable one. If we start with $0 < \Omega(0) < 1$, there exist initial values such that the solution is trapped in the resonance zone, resulting in periodic deflections of the flywheel.

Problems where averaging over angles (a torus) has to be used arise in many fields of application, for instance, in gyroscopic systems, and also in Hamiltonian mechanics. For an application to higher order resonance in two-degrees-of-freedom Hamiltonian systems, see [21]. Algebraic timescales of the form $\varepsilon^q t$ with q a rational number are natural in this context; see also [23] for the general theory and more examples.

6. Anticipation of Timescales. Considering the scientific literature, one observes that the use of asymptotic series to approximate solutions of differential equations takes all kind of different forms: averaging, multiple timing, renormalization, WKBJ, etc. The choice of a particular method seems to be a matter of taste and local customs. This adds to the widespread but incorrect opinion that applied mathematics is "a bag of tricks" and not an accomplished and respectable part of mathematics. In this respect it is very important to have such comparative and unifying studies as [17], [14], [15], [20], and [3], to name a few. A basic aspect of the discussion is of course that we have some freedom of expansion as Taylor series expansions, for instance, are unique, but asymptotic expansions are not.

The relationship between averaging, multiple timing, and the renormalization method was discussed in [2], [3], and [14]. In the seminal paper [17], the equivalence of the averaging method and multiple timing was established for standard equations like (2.3) on intervals of time of order $1/\varepsilon$. See also the extensive discussions in [15] and [20].

Often, a clear statement of equivalence of methods is lacking. Without these specifications, statements about equivalence of perturbation methods are too vague and sometimes not correct. Referring to the examples given above, we mention two classes of problems where multiple timing may be deficient while averaging and renormalization give the correct result.

- In bifurcation problems one encounters structural stability problems of matrices; this is the nature of such problems. In such cases unexpected algebraic timescales cannot be avoided.
- Problems with resonance manifolds may arise in systems of the form

  $$\dot{x} = \varepsilon X(x,\phi) + \varepsilon^2 + \cdots, \ \dot{\phi} = \Omega(x) + \varepsilon + \cdots,$$

  with x a Euclidean n-vector and ϕ an angle vector. Such problems arise in dissipative systems and in conservative systems. Higher order algebraic timescales and asymptotically small domains are natural here.

Normal form methods like averaging and renormalization have no need to anticipate the timescales that are relevant for the approximations. These timescales present themselves in the course of the analysis. Multiple timing, on the other hand, makes restricting choices of timescales but is safe to use if we confine the analysis to time intervals of order $1/\varepsilon$ and if we understand a priori the nature of the solutions. However, this is often not the case in research problems. Extension of validity of approximations beyond order $1/\varepsilon$ or to obtain higher order precision is usually not expedient for the multiple timescale method.

REFERENCES

[1] K. E. Avrachenkov, J. A. Filar, and P. G. Howlett, *Analytic Perturbation Theory and Its Applications*, SIAM, Philadelphia, 2014.
[2] L.-Y. Chen, N. Goldenfeld, and Y. Oono, *Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory*, Phys. Rev. E, 54 (1996), pp. 376–394.
[3] H. Chiba, *Extension and unification of singular perturbation methods for ODEs based on the renormalization group method*, SIAM J. Appl. Dynam. Syst., 8 (2009), pp. 1066–1115.
[4] J. A. Cochran, *Problems in Singular Perturbation Theory*, Ph.D. Thesis, Stanford University, 1962.
[5] W. M. Greenlee and R. E. Snow, *Two-timing on the half line for damped oscillation equations*, J. Math. Anal. Appl., 51 (1975), pp. 394–428.
[6] R. Haberman, *Applied Partial Differential Equations*, 5th ed., Pearson, Cranbury, NJ, 2012.
[7] J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York, 1969.
[8] M. Hénon and C. Heiles, *The applicability of the third integral of motion: Some numerical experiments*, Astron. J., 69 (1964), pp. 73–79.
[9] J. Kevorkian, *The Two Variable Expansion Procedure for the Approximate Solution of Certain Nonlinear Differential Equations*, Ph.D. Thesis, Calif. Institute Techn., Pasadena, 1961.
[10] J. Kevorkian, *Perturbation techniques for oscillatory systems with slowly varying coefficients*, SIAM Rev., 29 (1987), pp. 391–461.
[11] E. Kirkinis, *The renormalization group: A perturbation method for the graduate curriculum*, SIAM Rev., 54 (2012), pp. 374–388.
[12] N. Krylov and N. Bogoliubov, *Méthodes approchées de la mécanique nonlinéaire dans leur application a l'étude de la perturbation des mouvements périodiques et de divers phénomènes de résonance s'y rapportant*, Ac. Sciences Ukraine, 14 (1935).
[13] G. E. Kuzmak, *Asymptotic solutions of nonlinear second order differential equations with variable coefficients*, J. Appl. Math. Mech., 10 (1959), pp. 730–744.
[14] B. Mudavanhu and R. E. O'Malley, Jr., *A new renormalization method for the asymptotic solution of weakly nonlinear vector systems*, SIAM J. Appl. Math., 63 (2002), pp. 373–397.
[15] J. A. Murdock, *Perturbations: Theory and Methods*, Classics Appl. Math. 27, SIAM, Philadelphia, 1999 (republished with corrections from the 1991 John Wiley edition).
[16] A. H. Nayfeh, *Perturbation Methods*, Wiley-Interscience, New York, 1973.
[17] L. M. Perko, *Higher order averaging and related methods for perturbed periodic and quasi-periodic systems*, SIAM J. Appl. Math., 17 (1969), pp. 698–724.
[18] J. A. Sanders, *Are higher order resonances really interesting?*, Celestial Mech., 16 (1977), pp. 421–440.
[19] J. A. Sanders and F. Verhulst, *Approximations of higher order resonances with an application to Contopoulos' model problem*, in Asymptotic Analysis, from Theory to Application, F. Verhulst, ed., Lecture Notes in Math. 711, Springer, New York, 1979, pp. 209–228.
[20] J. A. Sanders, F. Verhulst, and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, 2nd ed., Springer, New York, 2007.
[21] J. M. Tuwankotta and F. Verhulst, *Symmetry and resonance in Hamiltonian systems*, SIAM J. Appl. Math., 61 (2000), pp. 1369–1385.
[22] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Springer, New York, 2000.
[23] F. Verhulst, *Methods and Applications of Singular Perturbations*, Springer, New York, 2005.
[24] F. Verhulst, *Timescales and error estimates in dynamical systems*, Chaotic Model. Simul., 4 (2013), pp. 473–494.