WISM100 Lecture 1

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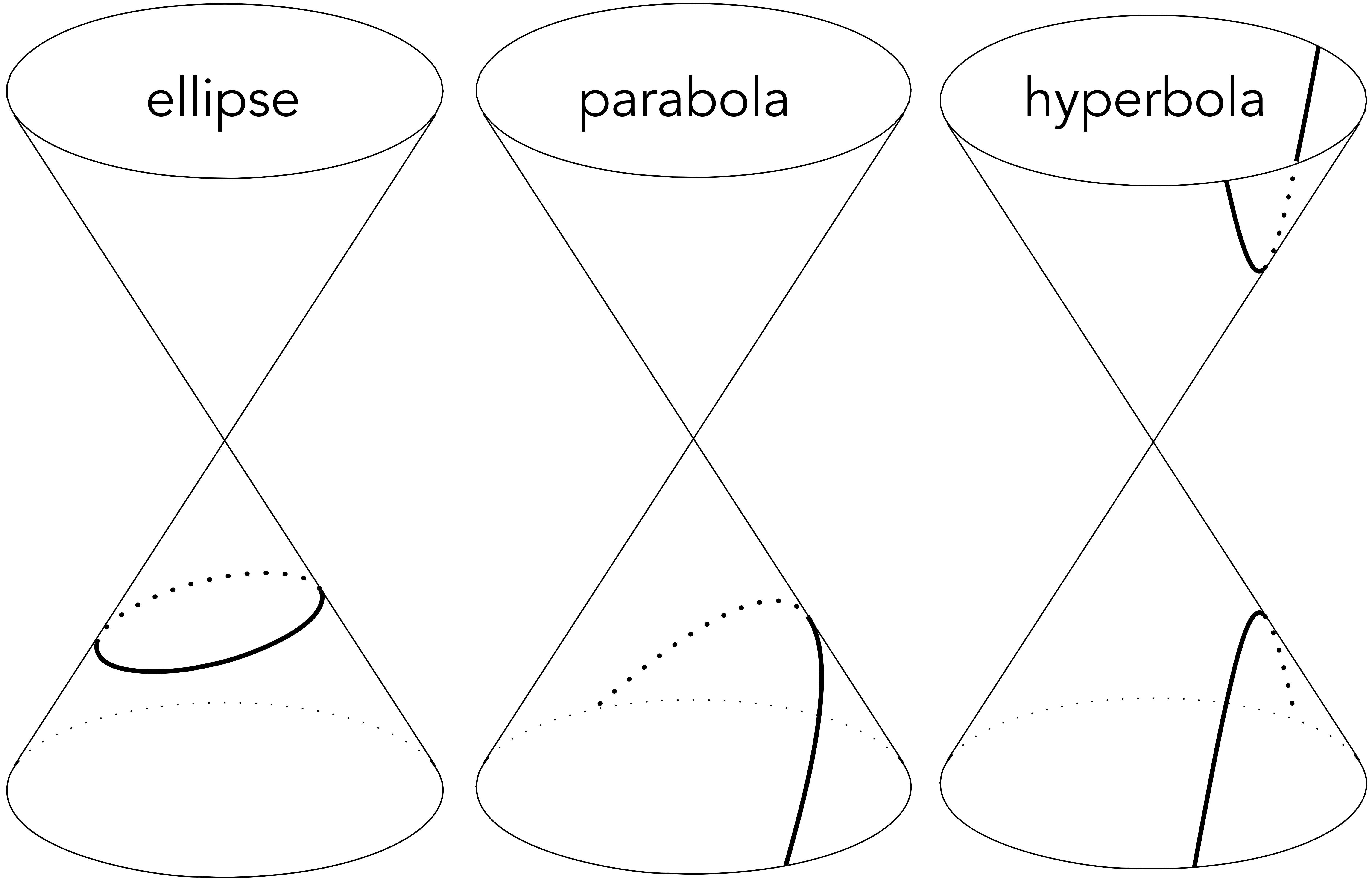
Conic sections



Why conic sections?

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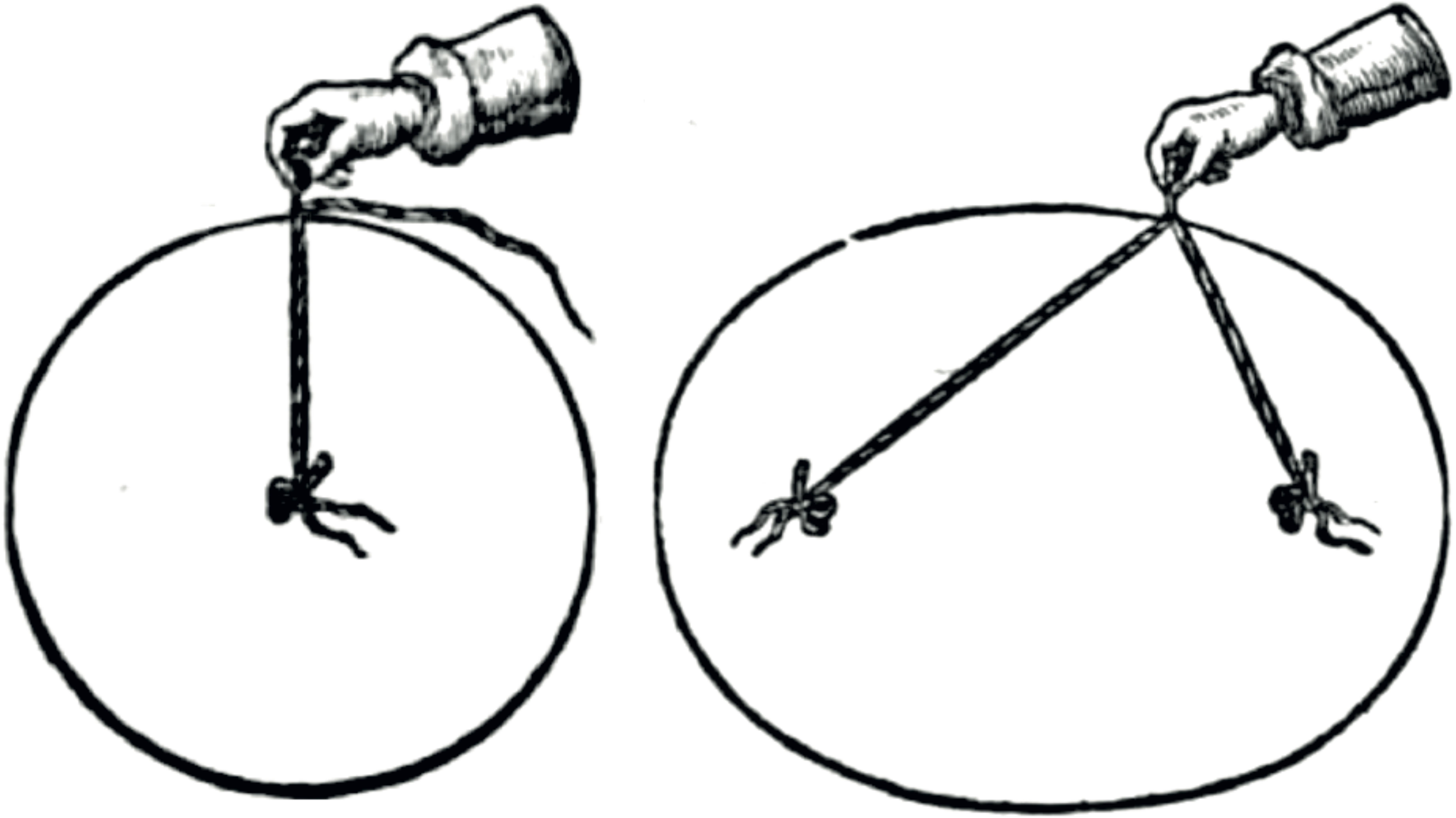
- Natural generalisations of circles in terms of mode of generation.



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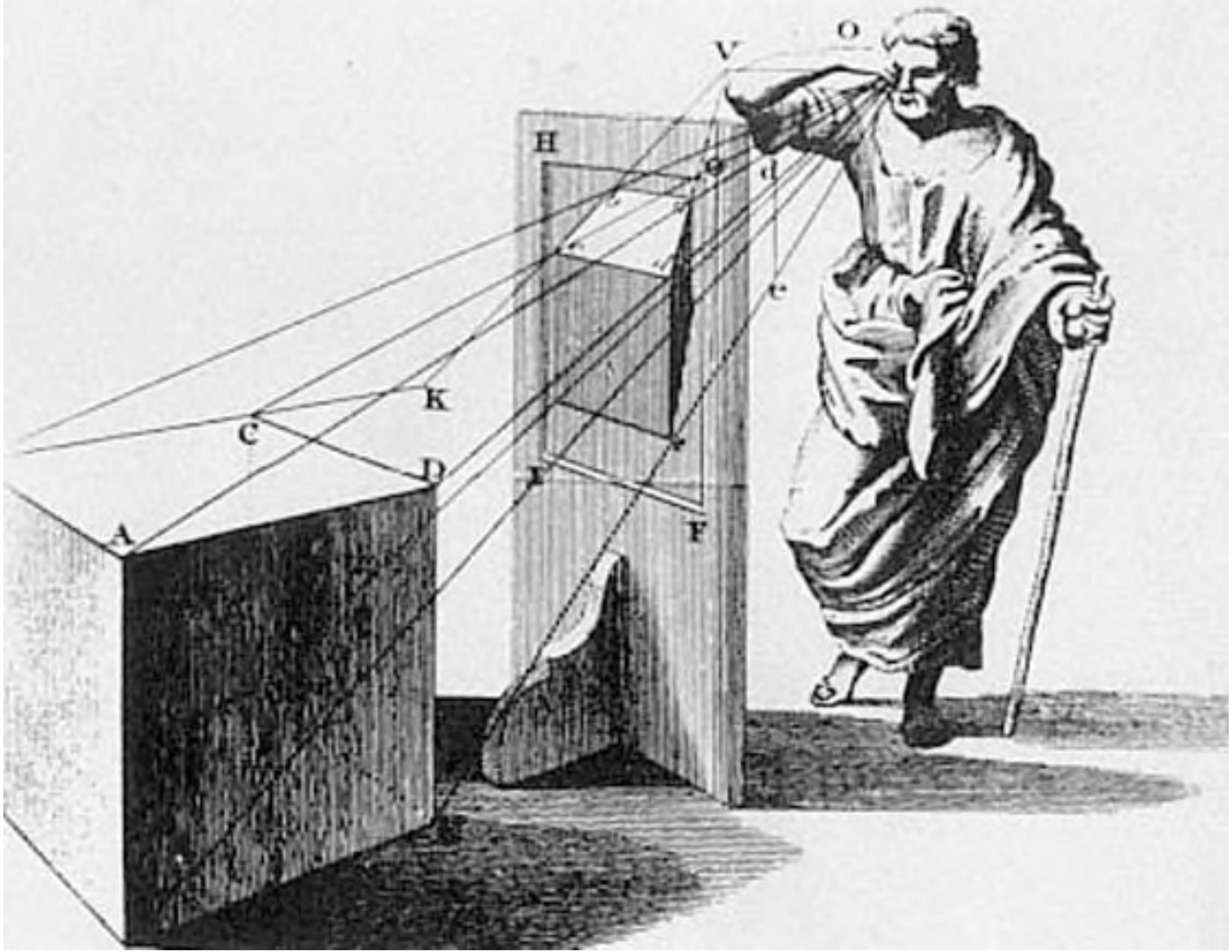
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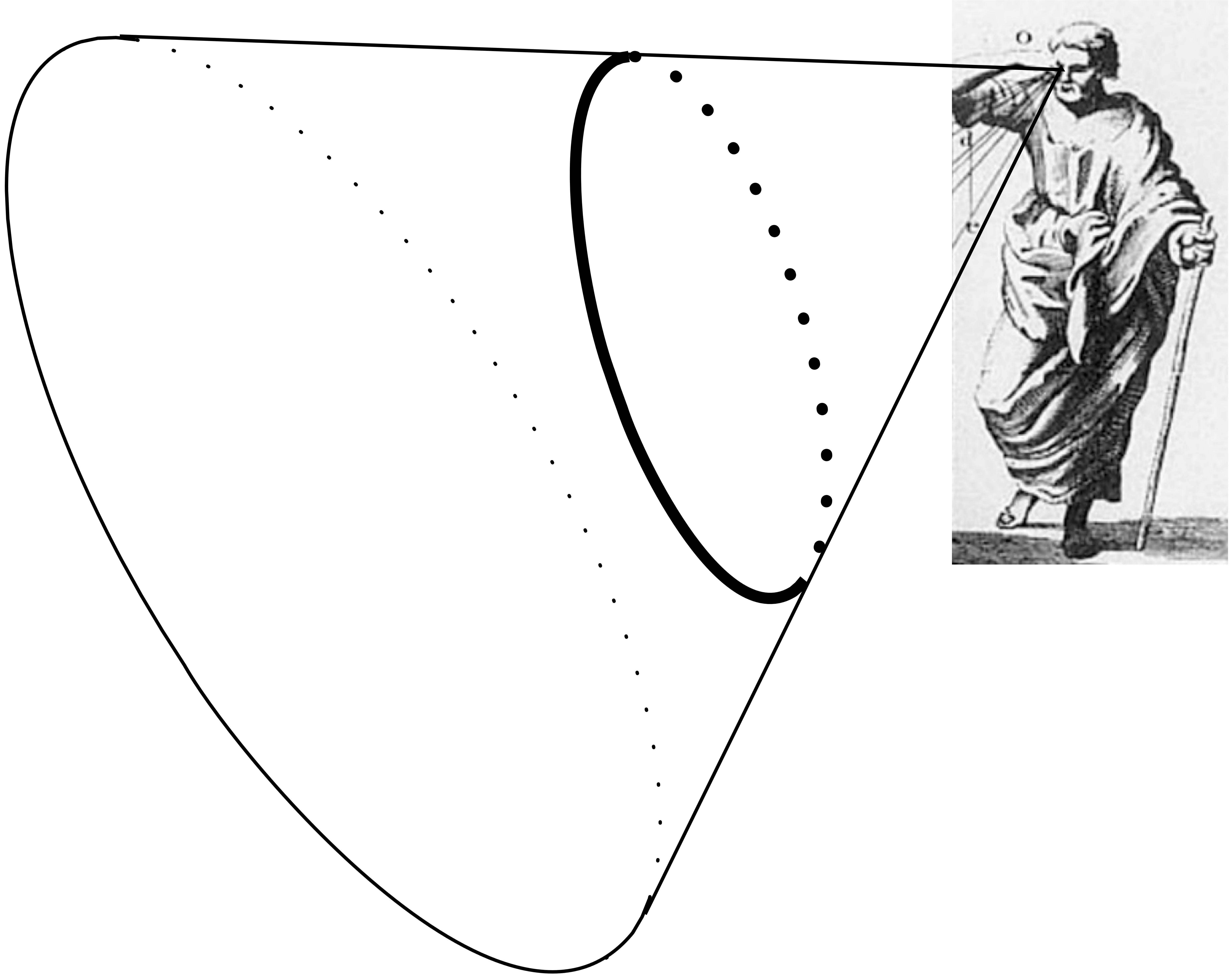

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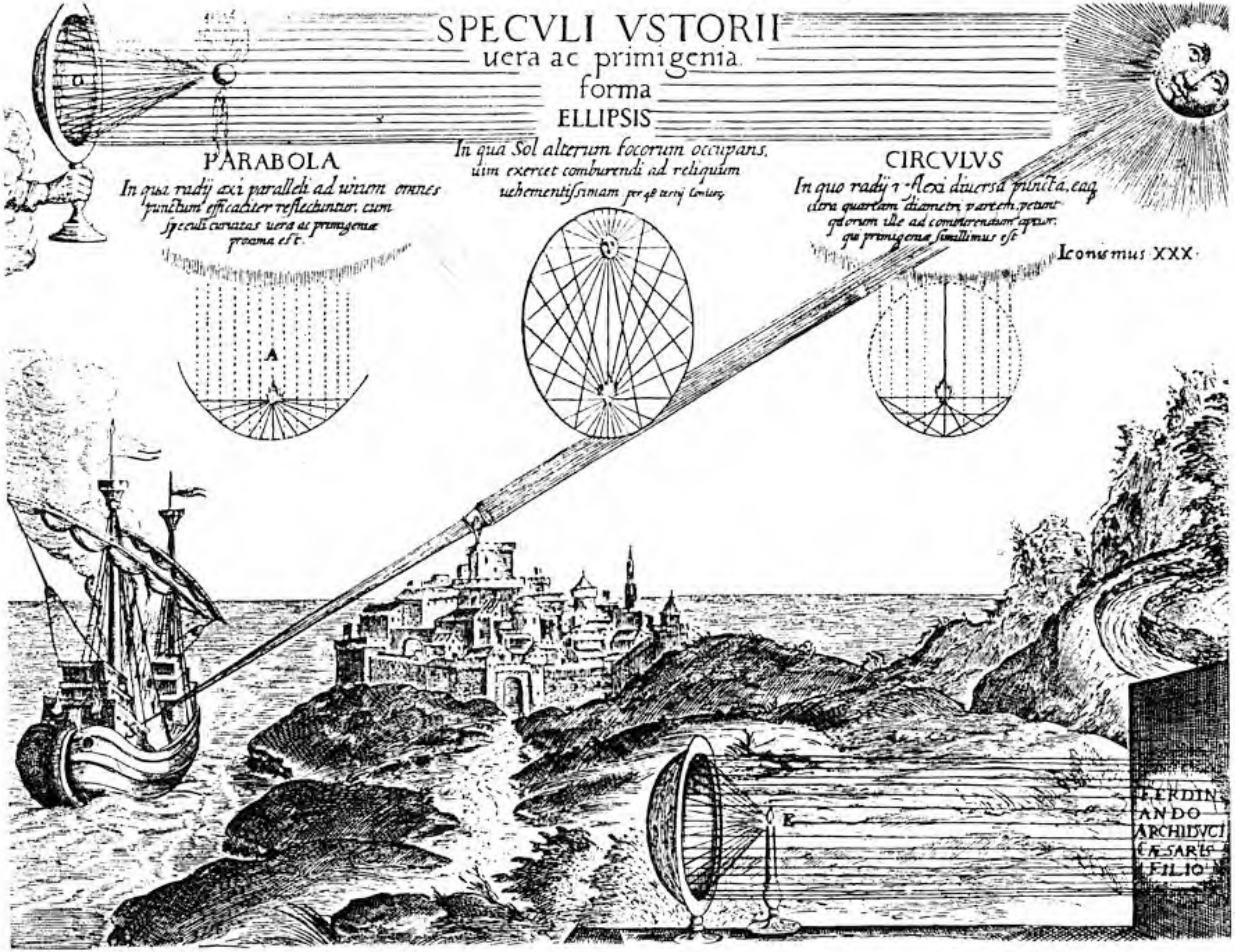
SPECVLI VSTORII
uera ac primigenia
forma
ELLIPSIS
In qua Sol alterum focorum occupans,
uim exercet comburendi ad reliquum
PARABOLA
CIRCVLVS
Iconismus XXX
A
E
FERDIN
ANDO
ARCHIDVCI
CÆSARIS
FILIO

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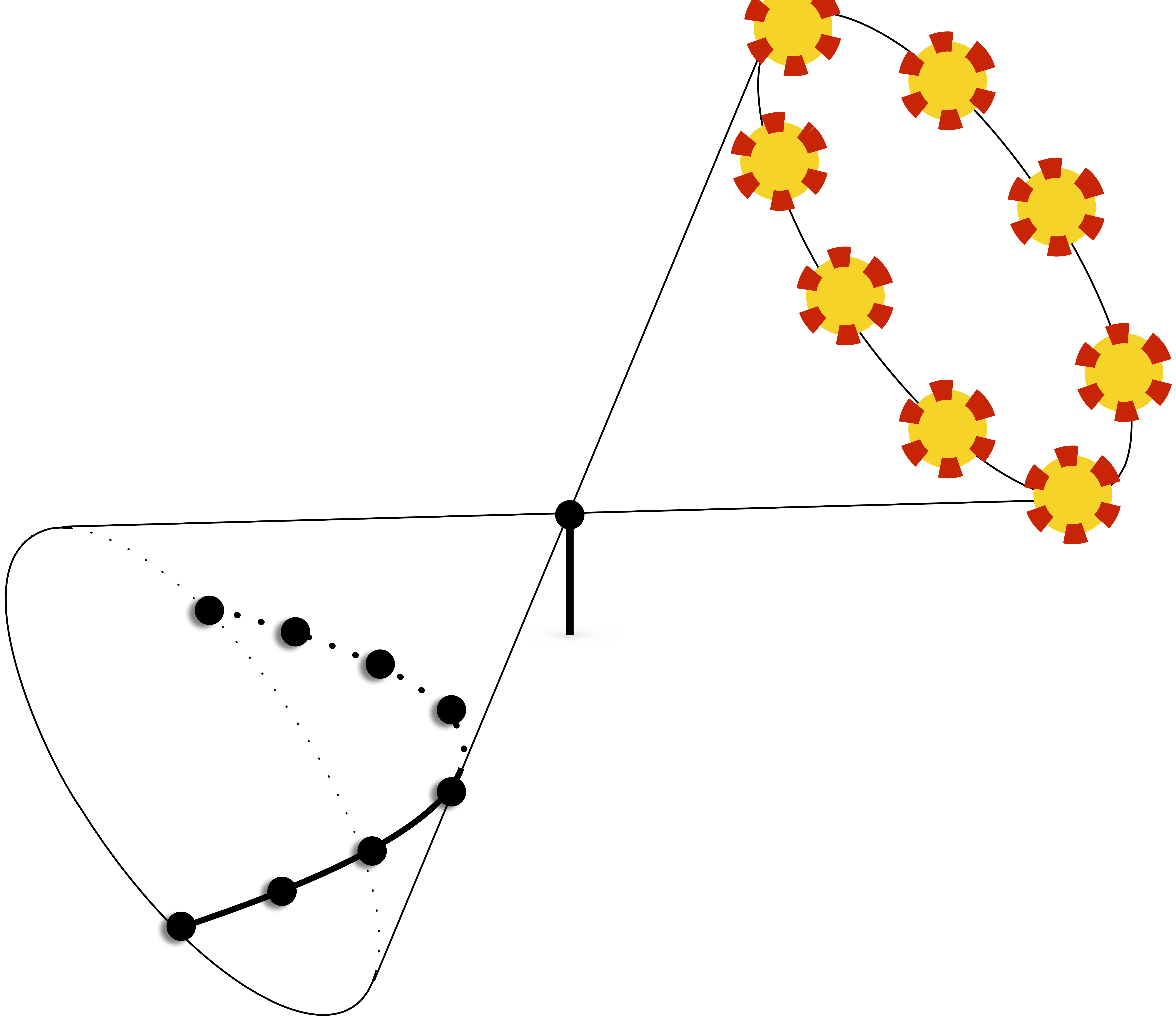


VI VII VIII IX X XI XII
XIII XIV
WMEZ
M.MEZ

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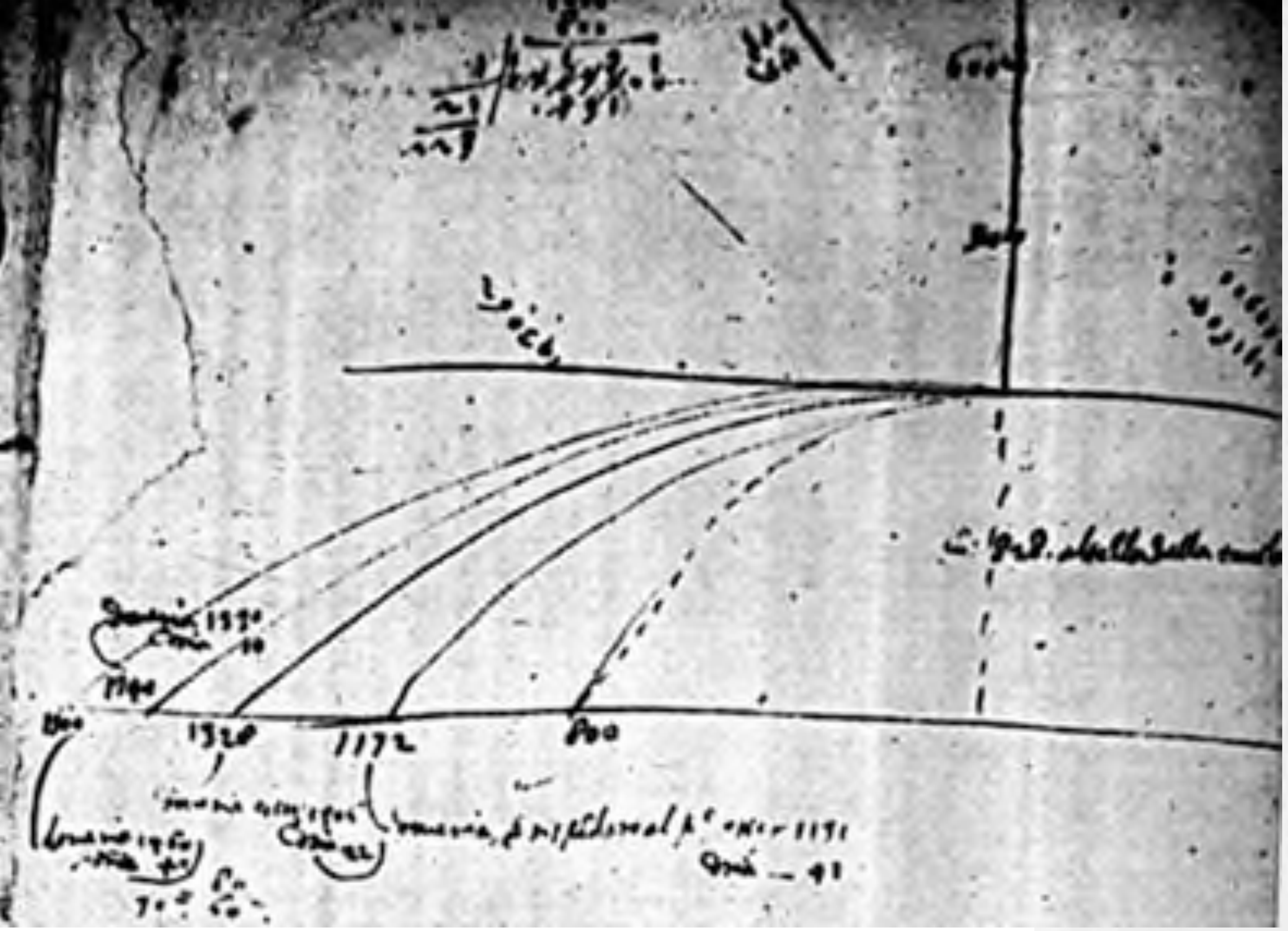
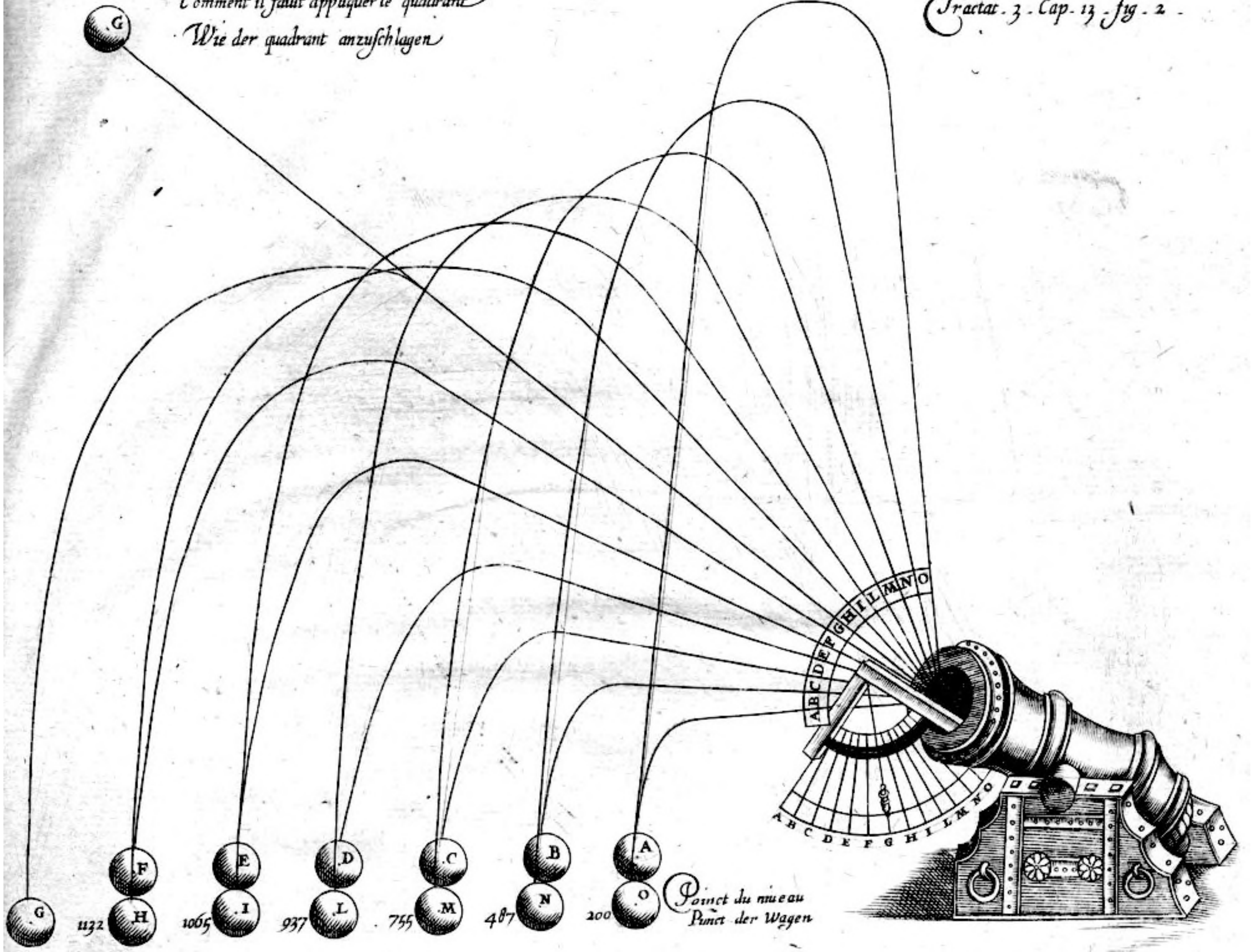
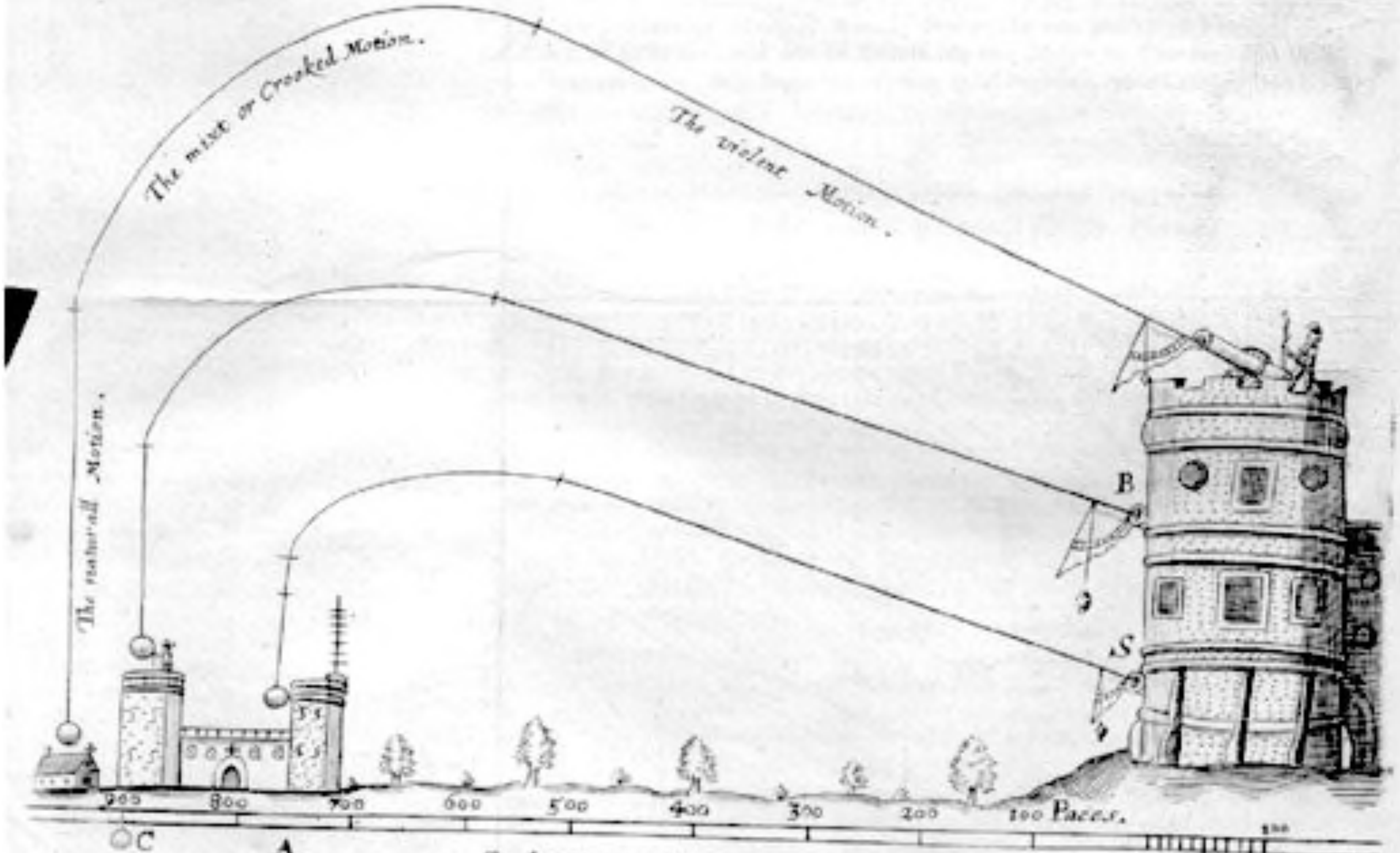

1172
800


Istituto
e Museo di Storia
della Scienza

Comment il fault appliquer le quadrant
Wie der quadrant anzuschlagen

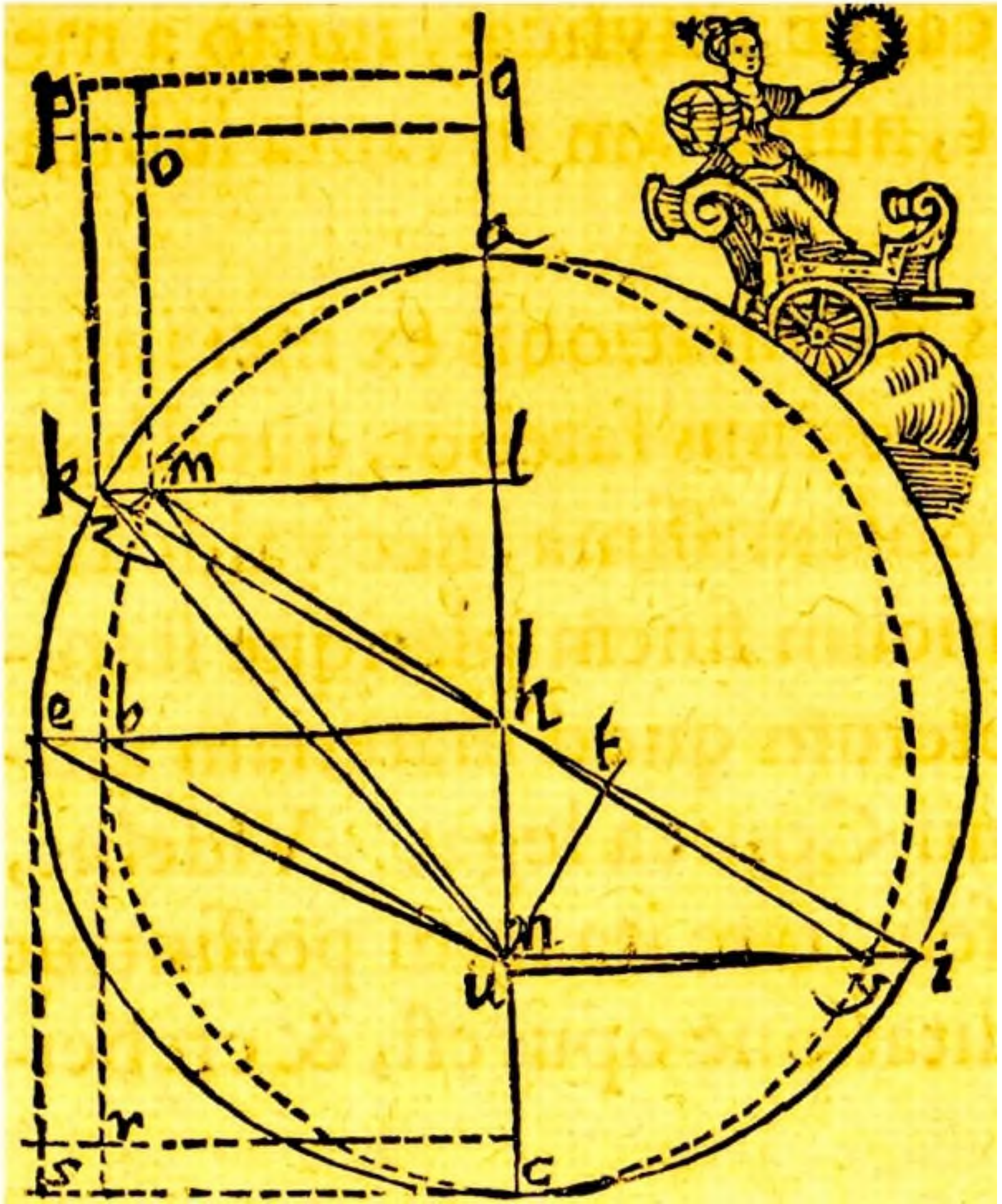




Book V. page 68 : 69.

PARS QVARTA.

	Tempus	Locus ☉	Solis a Terra distantia	Martis a Sole distantia
1582.	23 Nove. H. 16. 0	11.41 ♐	98345	158852
	26 Dece. H. 8.30	15. 4 ♑	98226	162104
	30 Dece. H. 8.10	19. 9 ♑	98252	162443
1583.	26 Janua. H. 6.15	16.33 ♒	98624	164421
1584.	21 Dece. H. 14. 0	10.16 ♑	98207	164907
1585.	24 Janua. H. 9. 0	14.53 ♒	98595	166210
	4 Febr. H. 6.40	26.10 ♒	98830	166400
	12 Mart. H. 10.30	2.16 ♈	99858	166170
1587.	25 Janua. H. 17. 0	16. 1 ♒	98611	166232
	4 Mart. H. 13.24	24. 0 ♓	99595	164737
	10 Mart. H. 11.30	29.52 ♓	99780	164382
	21 April. H. 9.30	10.48 ♉	101010	161027
1589.	8 Mart. H. 16.24	28.36 ♓	99736	161000
	13 April. H. 11.15	3.38 ♉	100810	157141
	15 April. H. 12. 5	5.36 ♉	100866	156900
	6 Maji. H. 11.20	25.49 ♉	101366	154326
1591.	13 Maji. H. 14. 0	2.10 ♊	101467	147891
	6 Junii H. 12.20	24.59 ♊	101769	144981
	10 Junii H. 11.50	28.47 ♊	101789	144526
	28 Junii H. 10.24	15.51 ♋	101770	142608
1593.	21 Julii H. 14. 0	8.26 ♌	101498	138376
	22 Aug. H. 12.20	9.11 ♍	100761	138463
	29 Aug. H. 10.20	11.54 ♍	100562	138682
	3 Octo. H. 8. 0	20.15 ♎	99500	140697
1595.	17 Sept. H. 16.45	4.18 ♎	99990	143222
	27 Octo. H. 12.20	13.59 ♏	98851	147890
	3 Nove. H. 12. 0	21. 2 ♏	98694	148773
	18 Dece. H. 8. 0	6.43 ♑	98200	154539



Planetary orbits are ellipses

Kepler, Astronomia Nova, 1609

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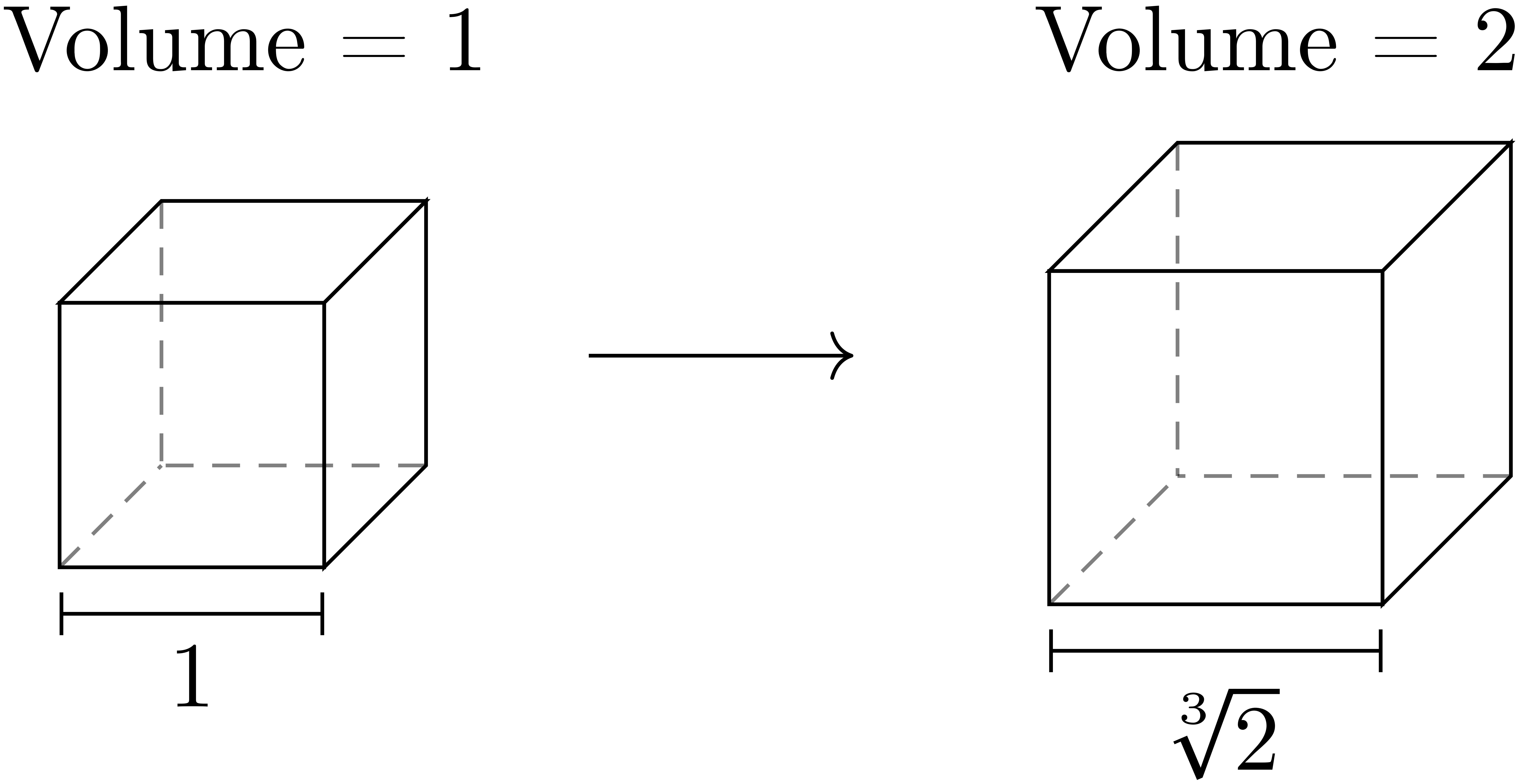
- Natural in astronomy and sundial-construction (path of shadow in the course of a day).
- Correspond to natural motion (projectiles and planets).
- Can be used to double the cube.


Volume = 1
1
Volume = 2
$\sqrt[3]{2}$

$y=x^2$
$y=\frac{2}{x}$
y
x
3
2.5
2
1.5
1
0.5
0
-0.5
-1.5
-1
0.5
1
1.5
2
2.5
3
3.5
4
4.5
$\sqrt[3]{2}$

Which reasons did the Greeks care about?

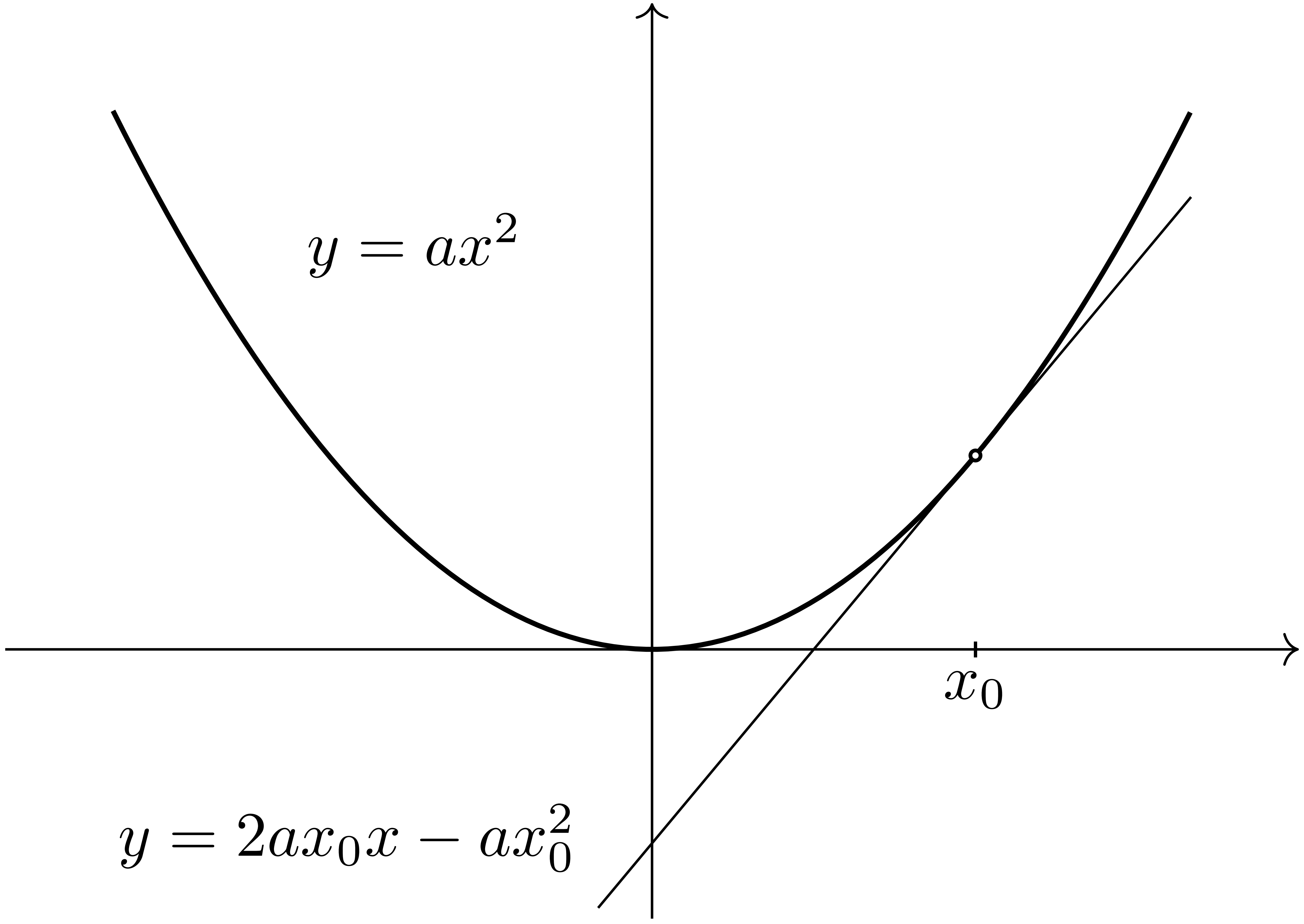
- Natural generalisations of circles in terms of mode of generation. ?
- Natural in optics and perspective painting (image of circle). ?
- Natural in optics (focal properties). ?
- Natural in astronomy and sundial-construction (path of shadow in the course of a day). ?
- Correspond to natural motion (projectiles and planets). No
- Can be used to double the cube. Yes

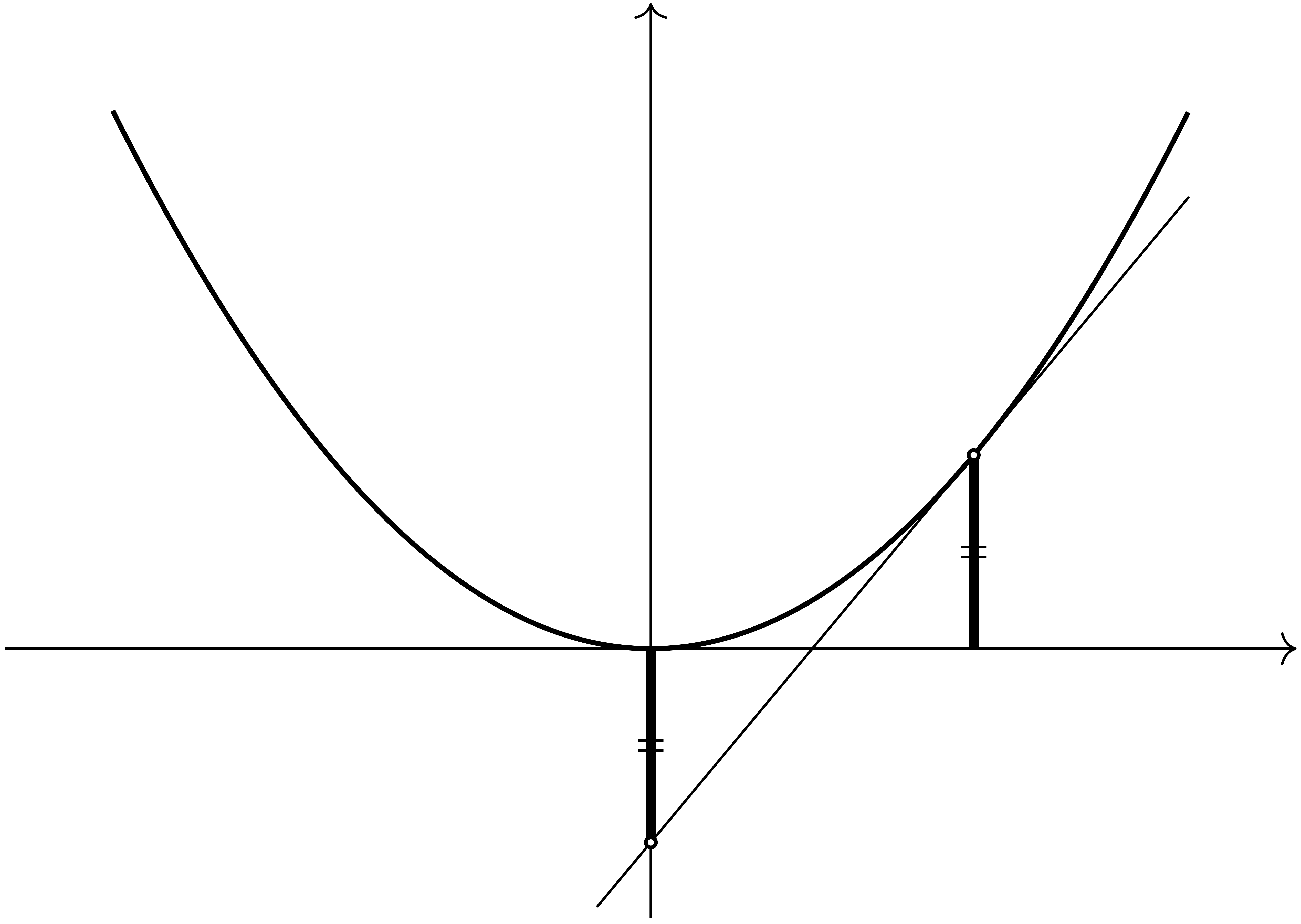
Ariſtippus Philoſophus Socraticus, naufragio cum ejectus ad Rhodienſium litus animadvertiſſet Geometrica ſchemata deſcripta, exclamaviſſe ad comites ita dicitur, Bene ſperemus, Hominum enim veſtigia video.

APOLLONII PERGÆI
CONICORUM
LIBRI OCTO,
ET
SERENI ANTISSENSIS
DE SECTIONE
CYLINDRI & CONI
LIBRI DUO.

OXONIÆ,
E THEATRO SHELDONIANO, An. Dom. MDCCX.

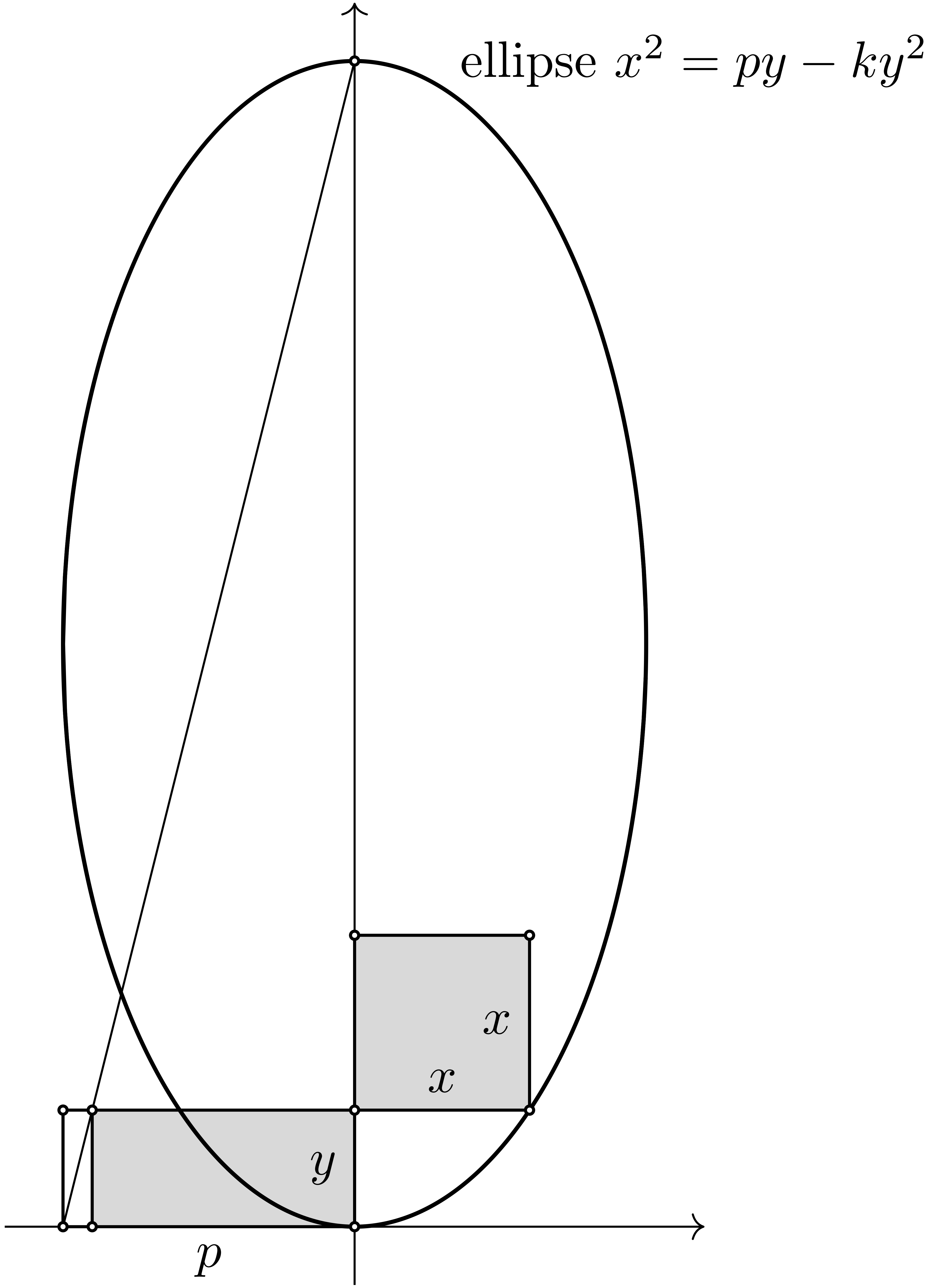
Apollonius, Κωνικά, I.33

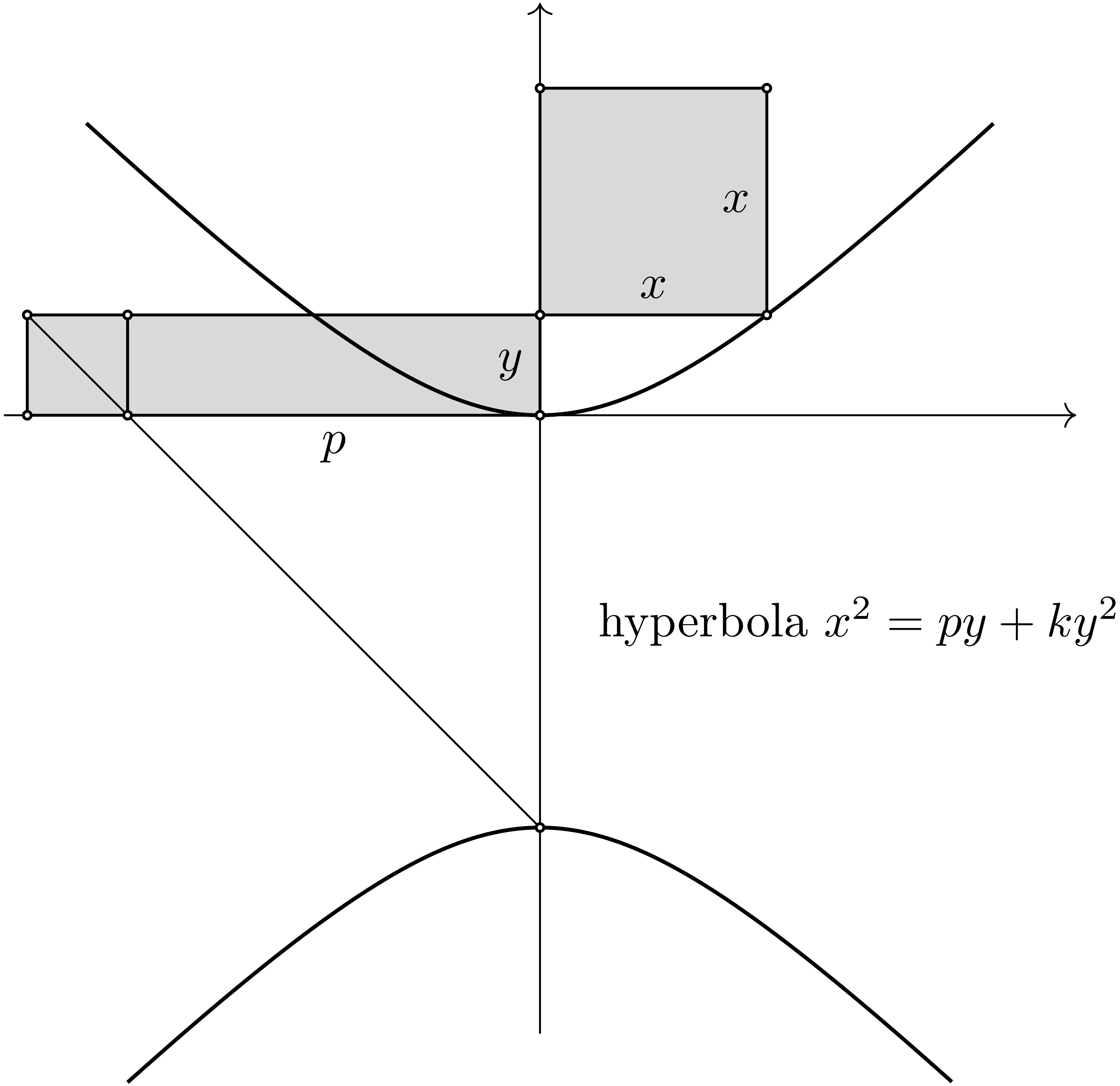




parabola $py = x^2$


parabola
p
F
ellipse
p
F
hyperbola
p
F


ellipse $x^2 = py - ky^2$
x
x
y
p


x
x
y
p
hyperbola $x^2 = py + ky^2$

hyperbole

exaggerated statements or claims not meant to be taken literally
("too much")

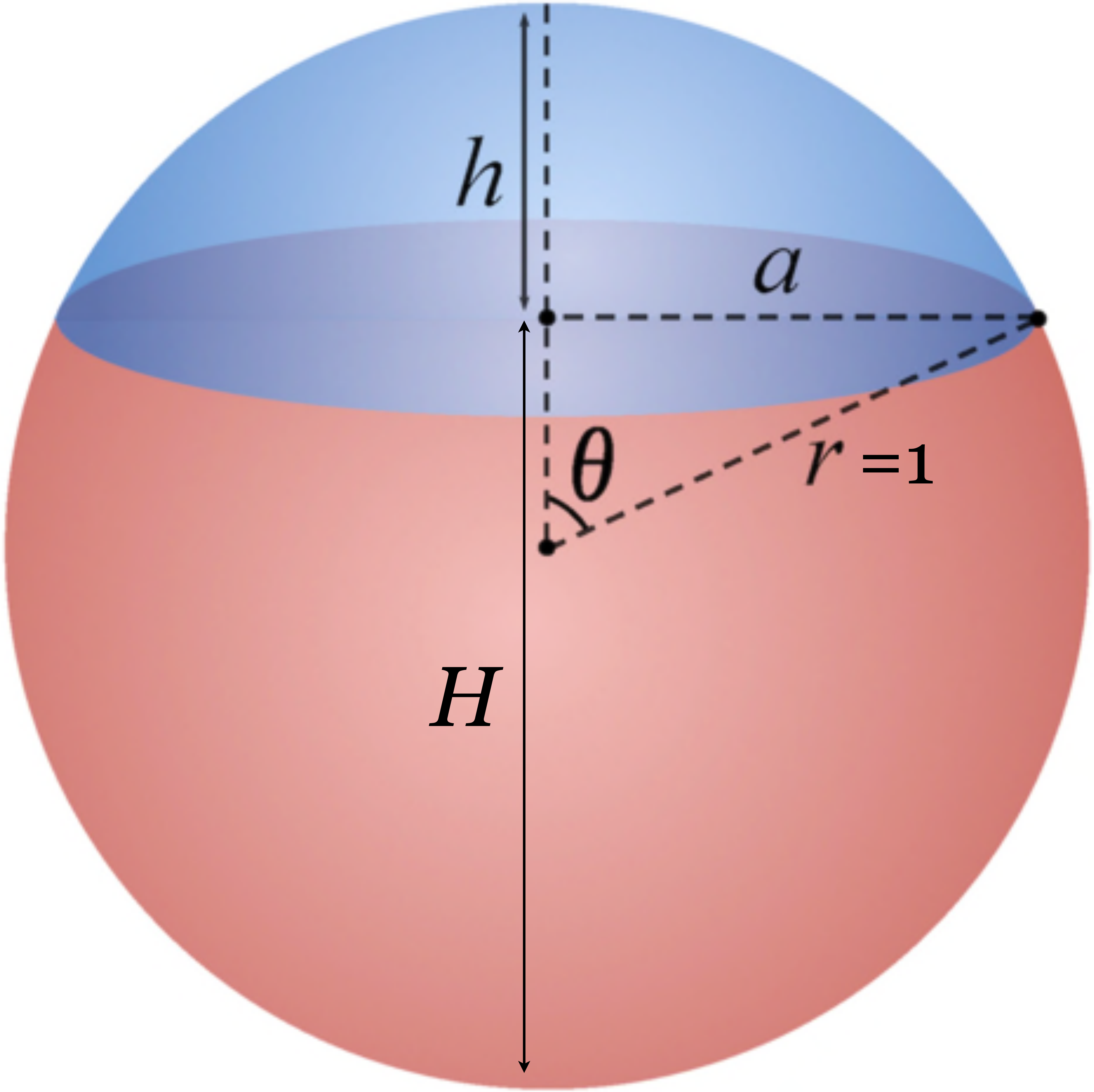
parable

a simple story used to illustrate a moral or spiritual lesson
("just right")

ellipsis (…)

the omission from speech or writing of words that are superfluous
("too little")

Archimedes, *Sphere and Cylinder*, II.4:
Find H such that $\dfrac{\text{red volume}}{\text{blue volume}} = k$

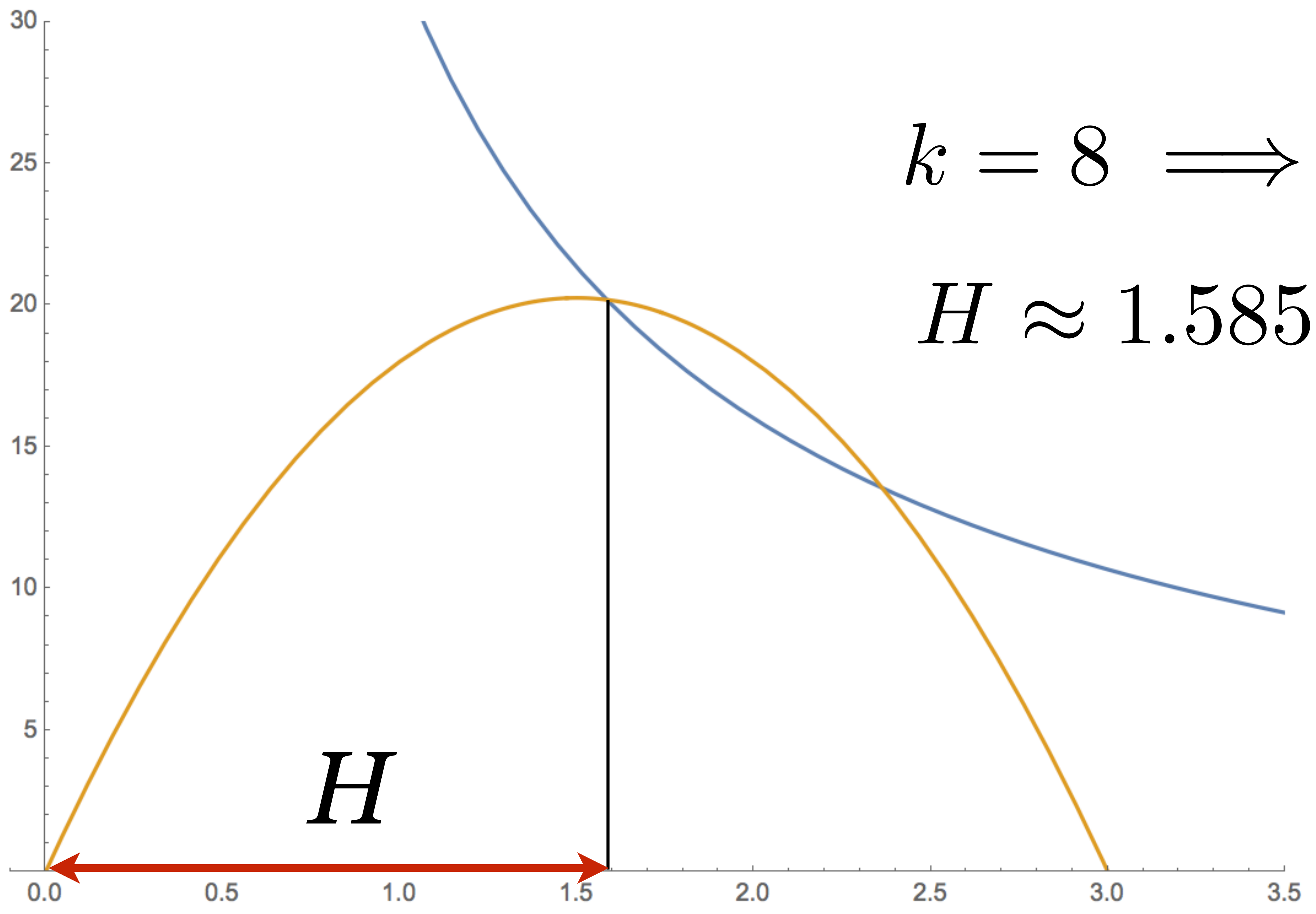


$$\Leftrightarrow \quad (4 - 3H^2 + H^3)k = (3 - H)H^2$$

$$\Leftrightarrow \quad \frac{4k}{H} = (3 + 3k)H - (1 + k)H^2$$

$\Leftrightarrow$ intersection of hyperbola $y = \dfrac{4k}{x}$

and parabola $y = (3 + 3k)x - (1 + k)x^2$



$k = 8 \implies$

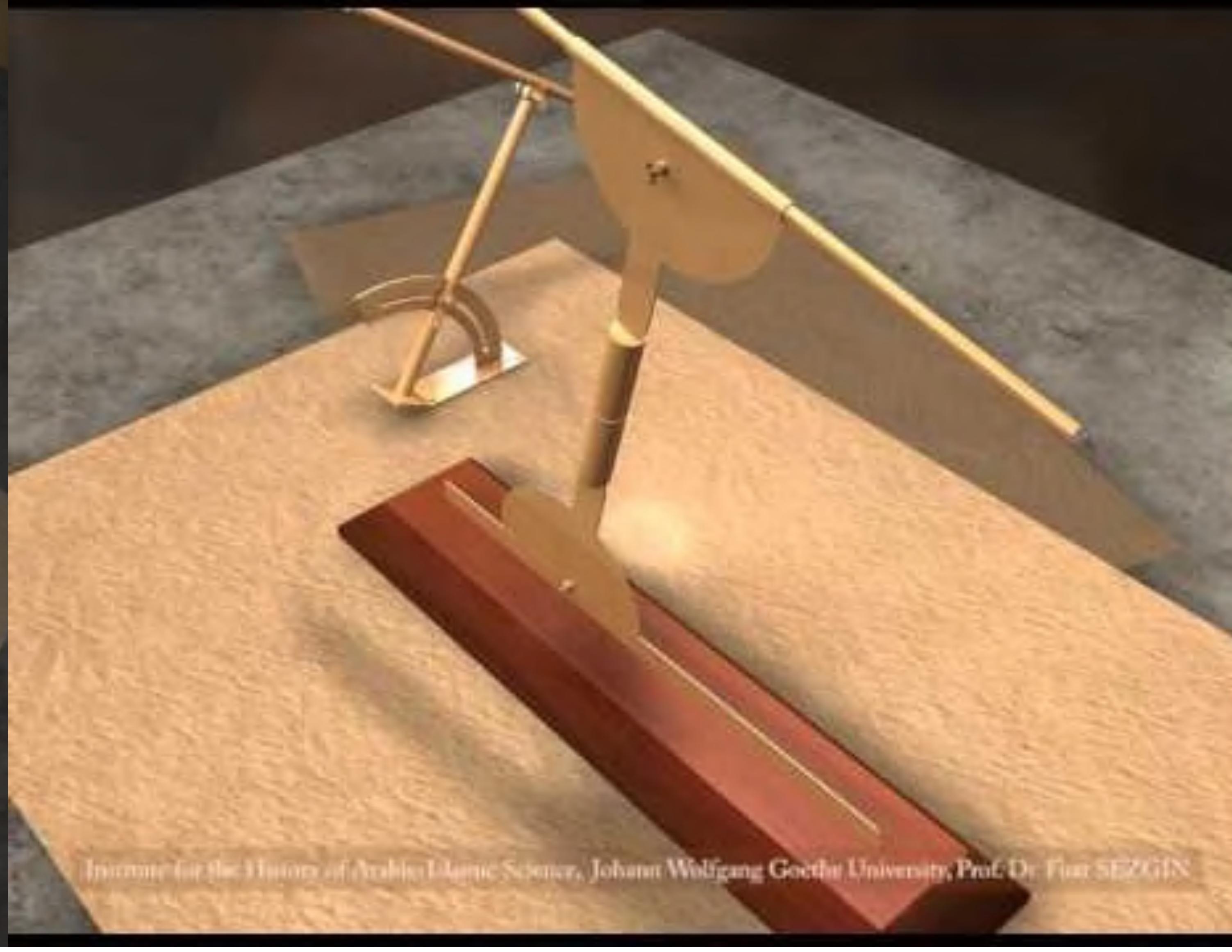
$H \approx 1.585$

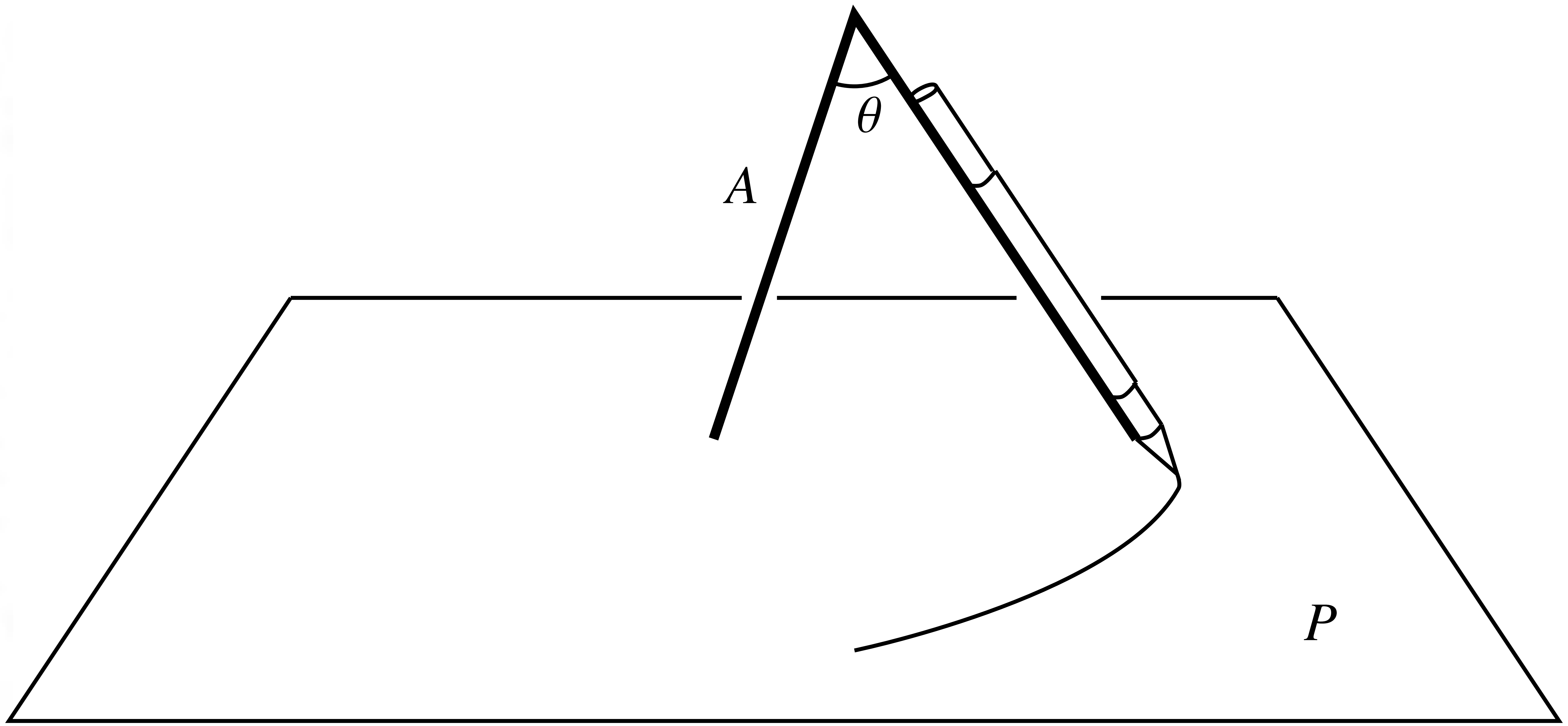
$$x^2 + x + 1 = 0 \quad \Longrightarrow \quad x = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$3^{x+2} = 7 \quad \Longrightarrow \quad x = \frac{\log 7}{\log 3} - 2$$

Kitāb al-Qūhī fī al-birkār al-tāmm, MS Istanbul, Raghib Pasha 569, fol. 235r.

al-Qūhī, c. 980





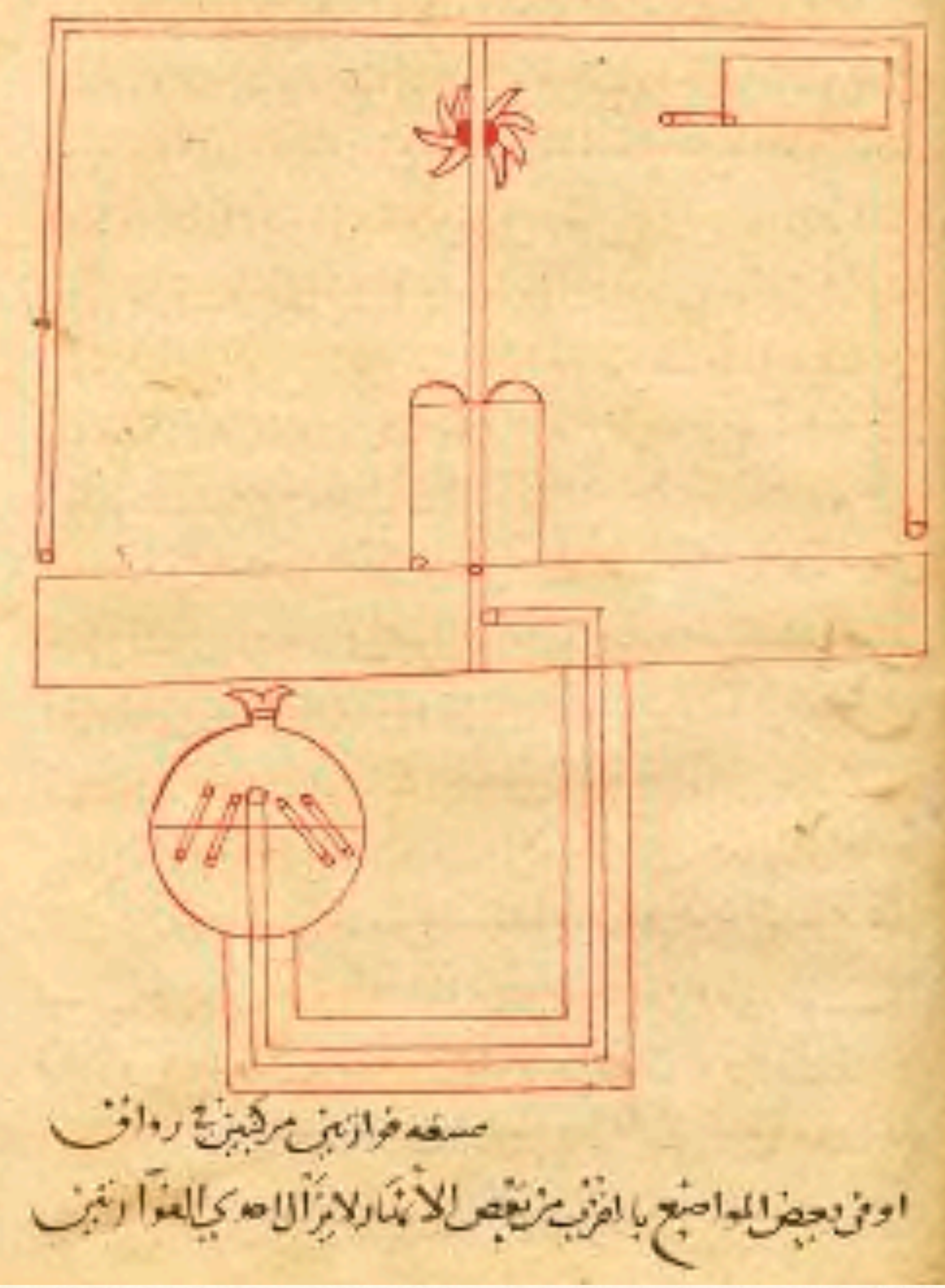


Plate 1. THE ARCHIMEDES AUTOMATIC WIND INSTRUMENTALIST.

British Museum MS., Or. Add. 23391.


Les automates
Jaquet Droz
a neuchatel

كبيرة فافهم ما وصفناه ولا قوة الا بالله ثم ما اردنا من

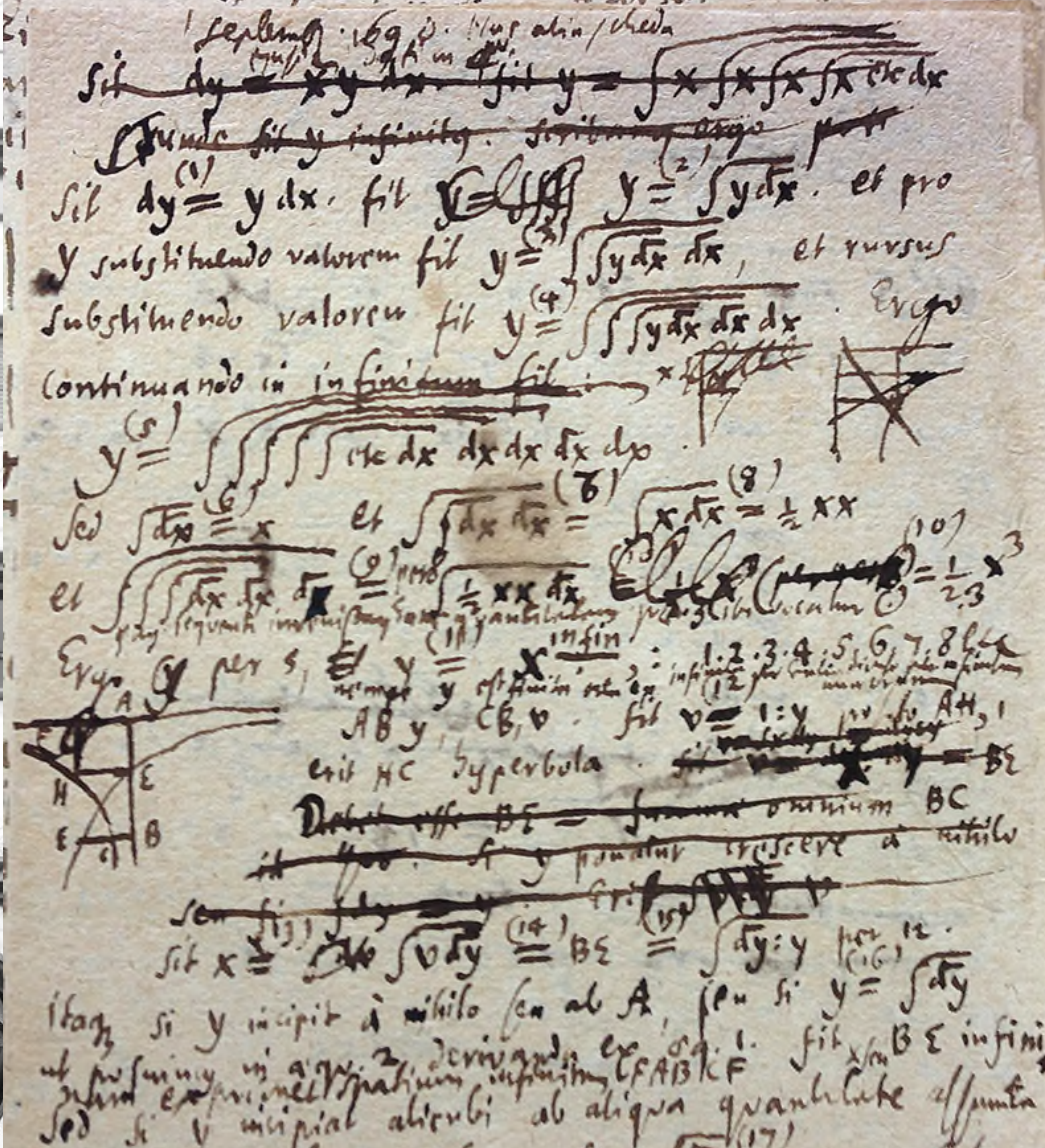
ذلك وبالله التوفيق صنعة الزامر لابلنيوس النجار

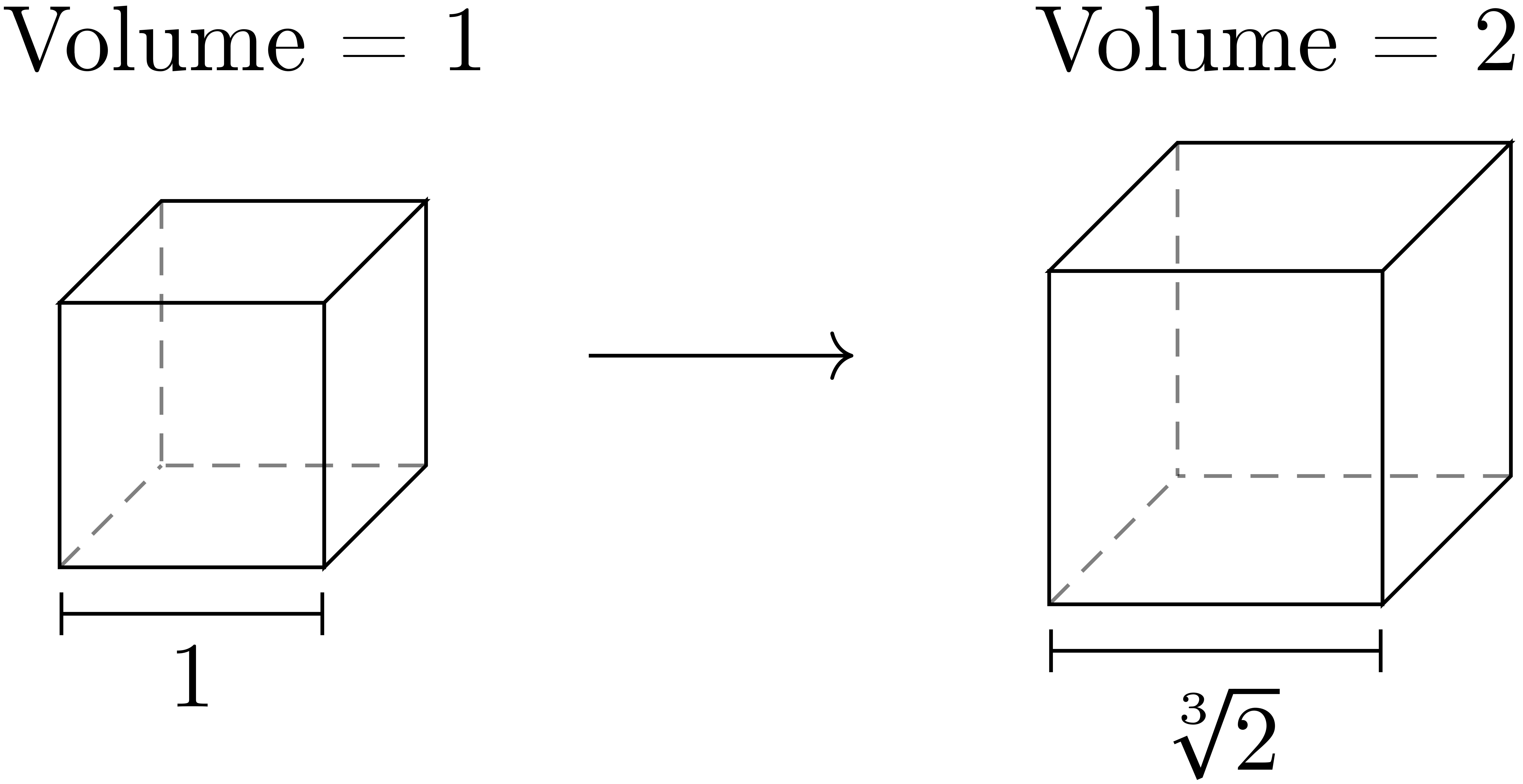
المهندس يتخذ صندوق مهندم مزفت او مرصص من

داخله كله اربعة اذرع في عرض اربعة اشبار في ارتفاع

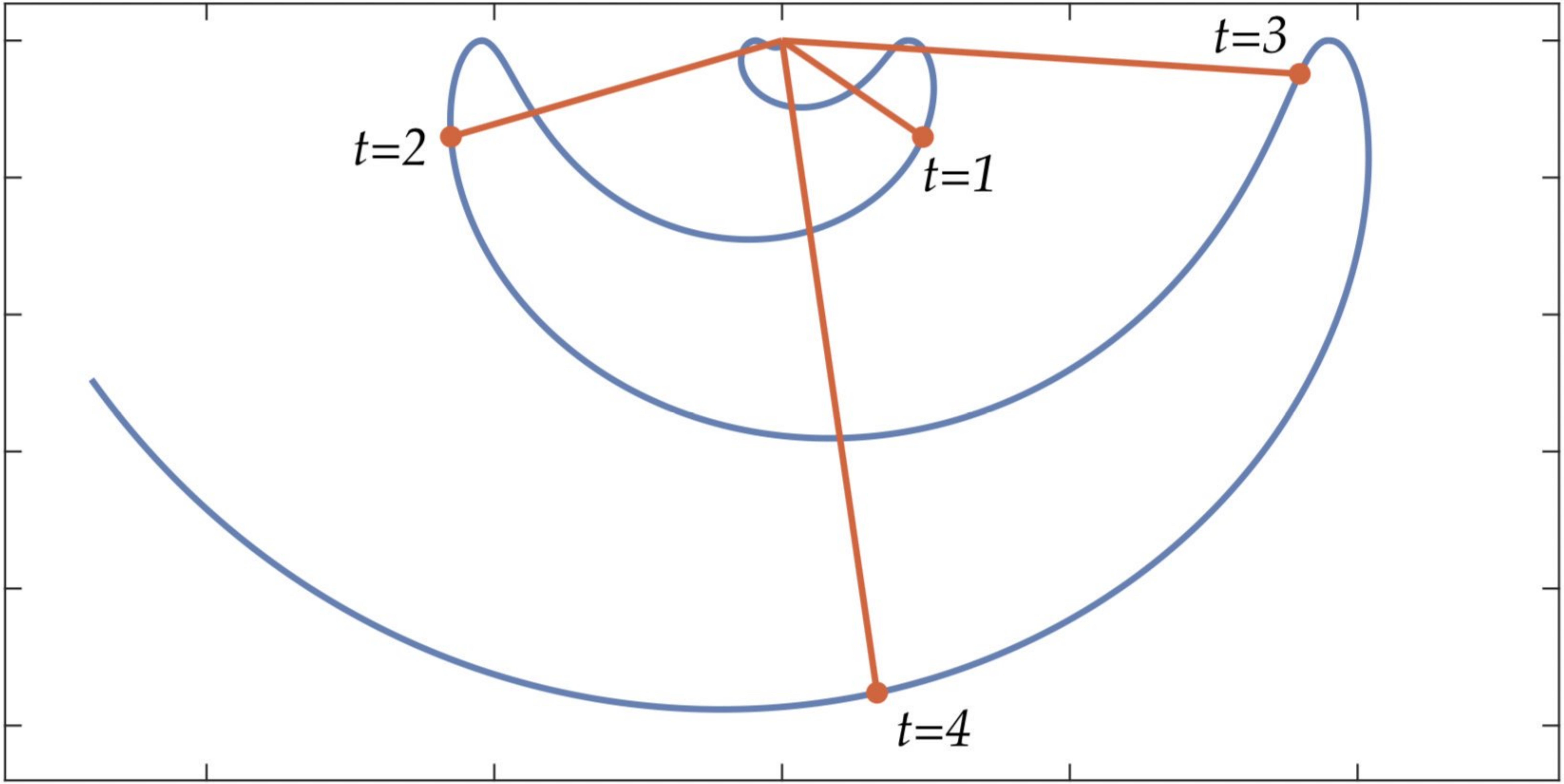
“Apollonius, the carpenter, the geometer”


GOTTFRIED WILHELM LEIBNIZ.


Sit dy = y dx. fit y = ∫ y dx. et pro
y substituendo valorem fit y = ∫∫ y dx dx, et rursus
substituendo valorem fit y = ∫∫∫ y dx dx dx. Ergo
continuando in infinitum fit
erit HC hyperbola


Volume = 1
Volume = 2
1
$\sqrt[3]{2}$

The Paracentric Isochrone $dr/dt = constant$



reduces to integrating $1/\sqrt{1-x^4}$


CHRISTIANUS HUGENIUS
natus 14 Aprilis 1629.
denatus 8 Junii 1695.


Young Leibniz studies with Huygens in Paris.
The student becomes the master.
Ageing Huygens studies Leibniz's new calculus.
Leibnitz
Hugenius
1670
1675
1680
1685
1690
1695
1700

Huygens' Hofwijck

Uw bezoek
prijzen & openingstijden

Wat is er te zien?
vast & actueel

Buitenplaats
huis & tuin

Verhuur
trouwen & zakelijk

The Hague

Utrecht

"I still do not understand anything about ddx, and I would like to know if you have encountered any important problems where they should be used, so that this gives me desire to study them."

"Je n'entens encore rien aux ddx, et je voudrois bien scavoir si vous avez rencontrè des problemes importants ou il faille les emploier, afin que cela me donne envie de les etudier."
(1693)

"As for the ddx, I have often needed them; they are to the dx, as the conatus to heaviness or the centrifugal solicitations are to the speed. Bernoulli, ... employed them for the lines of sails. And I had used them for the movement of the stars."

"Quant aux ddx, j'en ay eu sou vent besoin elles sont aux dx, comme les conatus de la pesanteur ou les solicitations centrifugues sont à la vitesse. M. Bernoulli marque dans les Actes de Leipzig de l'année passée p. 202 de les avoir employées pour les lignes des voiles. Et ie les avois deiâ employées pour le mouvement des astres dans les mêmes actes."

personality type	visionary ☞	maestro ♡	technocrat 🖩	pragmatist €
fundamental desire	beauty ♀		power ♂	
antiquity		Archimedes	Apollonius	Ptolemy
17th century	Descartes, Leibniz	Huygens, late Newton	Bernoulli, early Newton	
a problem is worth studying if it is	illuminating foundationally and methodologically	self-evidently important; anchored in tradition	formulable within existing technical frameworks	externally motivated
style of mathematics	sketchy; specifics included only to illustrate principles	elegant, definitive; self-contained mini-cosmos	exhaustive, repetitive, adaptable	back-of-an-envelope; the end justifies the means
reader is offered	global view of mathematics & methodology	aesthetic experience; display of brilliance	toolbox for doing more mathematics	toolbox for applying mathematics
attitude to technicalities	impatience	minimalism	pride	acceptance

Correspondence with other mathematical personality systems:

Freeman Dyson	birds (vision, unification)	frogs (detail, problem solving)
Gian Carlo Rota	Theorizers. Success in mathematics is not solving problems but trivializing them through conceptual insights.	Problem solvers. Care only about being the first to solve puzzles, in whatever way. Look at subsequent theorizing of this field with condescension and boredom.
Timothy Gowers	The point of solving problems is to understand mathematics.	The point of understanding mathematics is to solve problems.
Grothendieck	Place nut in environment that makes it open naturally.	Attack nut with hammer and chisel.

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fundamental desire	beauty ♀		power ♂	
antiquity		Archimedes	Apollonius	Ptolemy
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LA
GEOMETRIE.
LIVRE PREMIER.

Des probleſmes qu'on peut conſtruire ſans y employer que des cercles & des lignes droites.

TOus les Probleſmes de Geometrie ſe peuuent facilement reduire a tels termes, qu'il n'eſt beſoin par aprés que de connoiſtre la longeur de quelques lignes droites, pour les conſtruire.

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trouuer les autres. Et i'eſpere que nos neueux me ſçauront gré, non ſeulement des choſes que iay icy expliquées; mais auſſy de celles que iay omiſes volontairement, affin de leur laiſſer le plaiſir de les inuenter.

FIN.

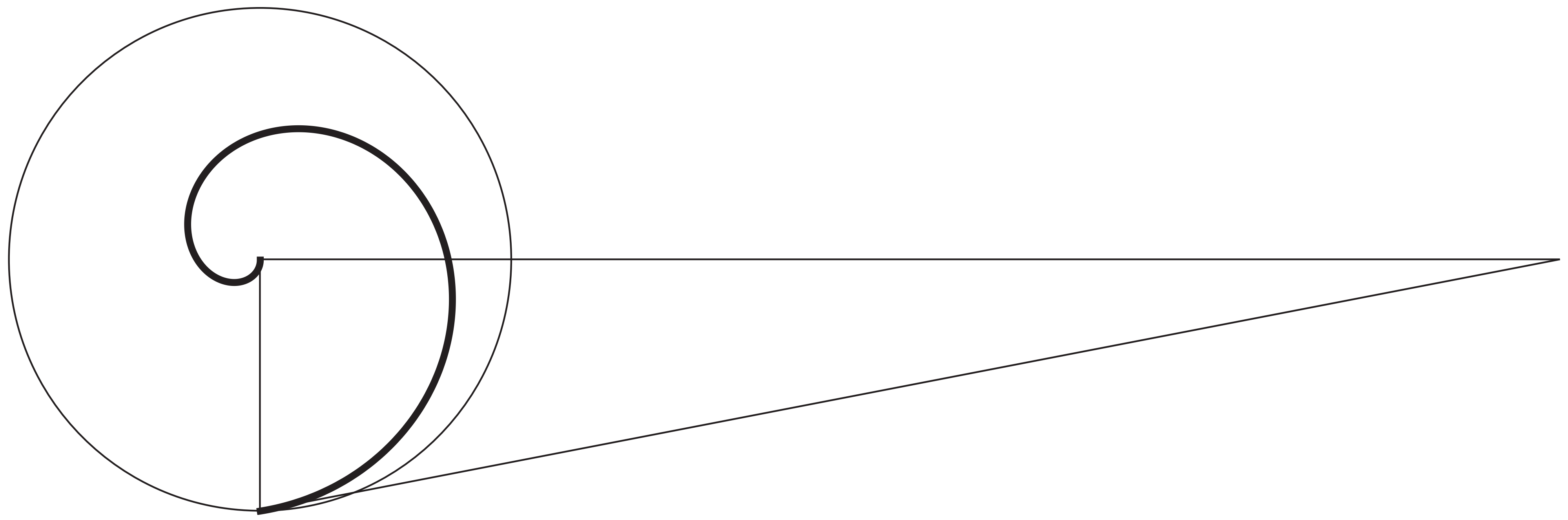
I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery.

visionary ☞ | maestro ♡ | technocrat | pragmatist €

beauty ♀ | power ♂

E.g. Archimedes:

volume of sphere = $\frac{2}{3}$ volume of cylinder



area of spiral = $\frac{1}{3}$ area of circle

area of triangle = area of circle

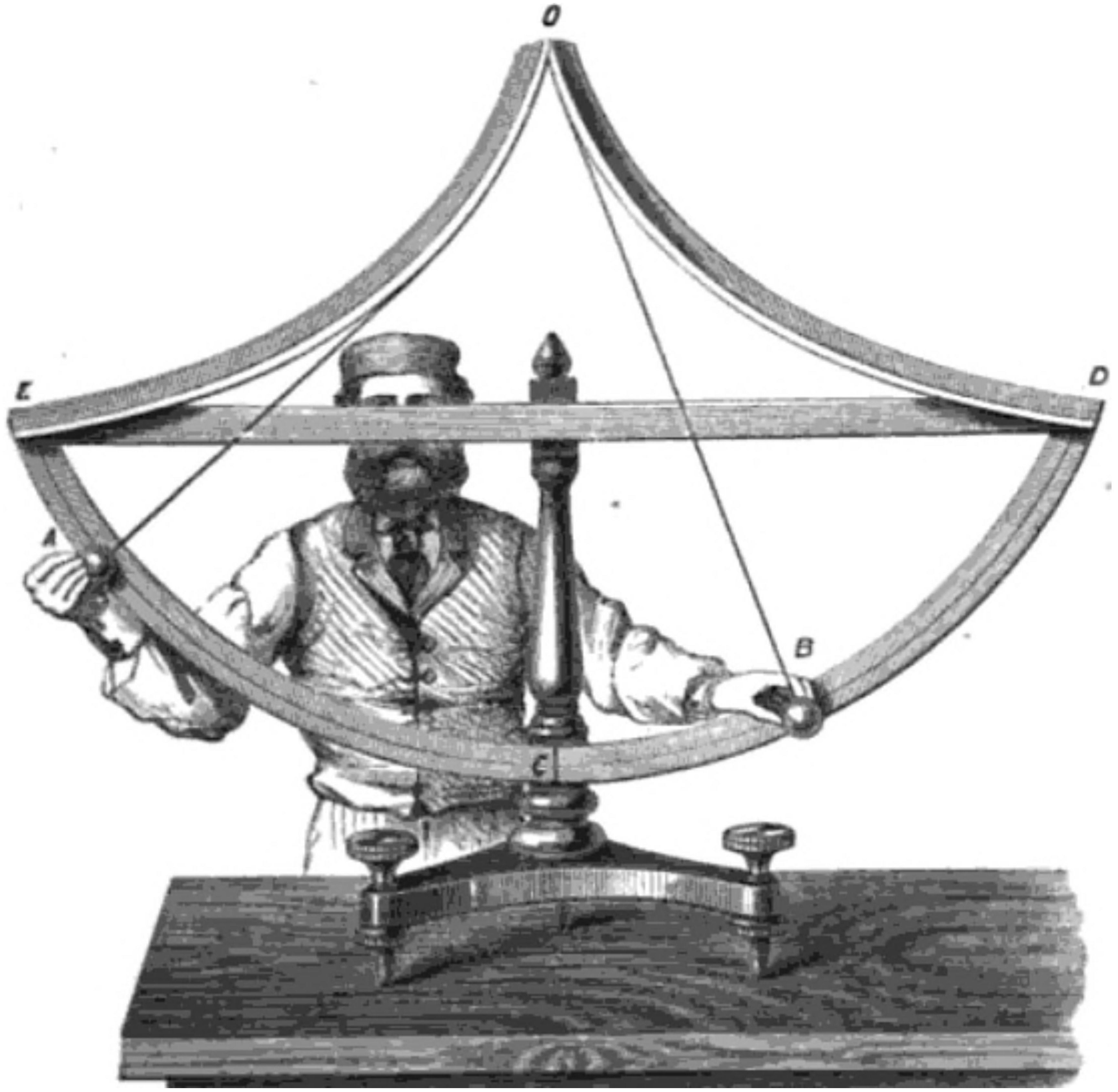
Technical calculus, polar coordinates, etc.

visionary ☞ | maestro ♡ | technocrat | pragmatist €

beauty ♀ | power ♂

Huygens:

tautochrone = cycloid = evolute of cycloid



Technical calculus, polar coordinates, etc.

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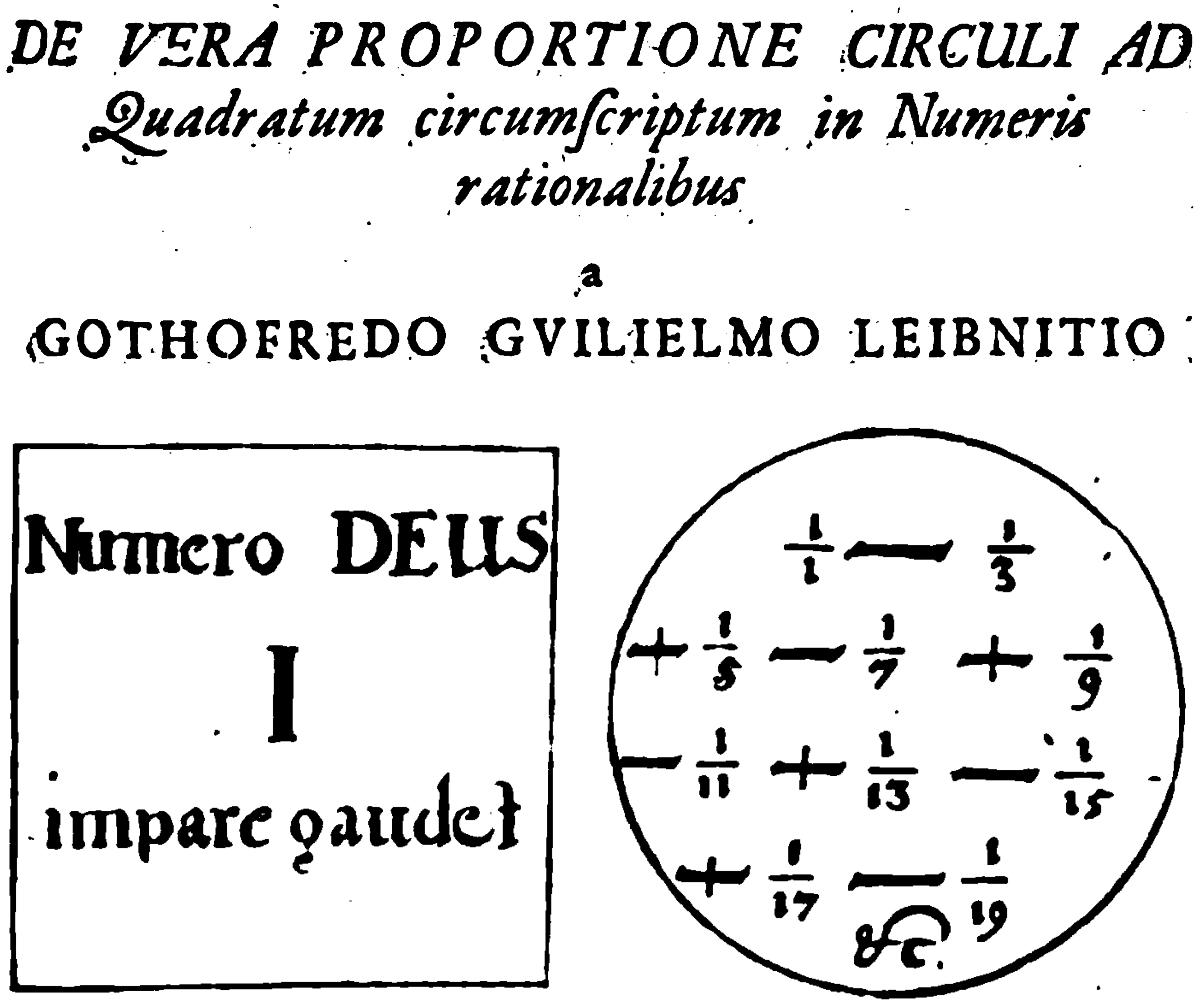
beauty ♀ | power ♂

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$


DE VERA PROPORTIONE CIRCULI AD Quadratum circumſcriptum in Numeris rationalibus

a

GOTHOFREDO GVILIELMO LEIBNITIO

Numero DEUS impare gaudet

$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}$ &c.

Johann Bernoulli:

$$\int_0^1 x^x \, dx = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

Not ♡ because not classically motivated and self-contained; presupposes "nerd" 🖩 interest in evaluating everything that can be symbolically formulated. Same with Euler's so-called beautiful (actually only 🖩) formula

$$e^{i\pi} + 1 = 0$$

visionary ☞ maestro ♡ technocrat 🖩 pragmatist €

$\underbrace{\text{visionary} \quad \text{maestro}}_{\text{beauty } ♀}$ $\underbrace{\text{technocrat} \quad \text{pragmatist}}_{\text{power } ♂}$

Huygens: I will learn calculus but only for ♡, not for 🖩. "I still do not understand anything about *ddx*, and I would like to know if you have encountered any important problems where they should be used, so that this gives me desire to study them." "[Natural] curves merit, in my opinion, that one selects them for study, but not those [curves] newly made up solely for using the geometrical calculus upon them."

Leibniz: Agree, 🖩 calculus worth little. "You are right, Sir, to not approve if one amuses oneself researching curves invented for pleasure." But the difference between ☞ and ♡ is that ☞ is more focussed on general methodological insights, which is why Leibniz adds: "I would however add a restriction: Except if it can serve to perfect the art of discovery."

L'Hôpital's Rule: typical 🖩 of the sort condemned here.

visionary ☞ | maestro ♡ | technocrat 🖩 | pragmatist €

beauty ♀ | power ♂

A typical ♡ versus 🖩 conflict/misunderstanding: Leibniz versus the English on power series in the 1670s.

Leibniz typical ♡, cares about singular, beautiful results: "I possess certain analytical methods, extremely general and far-reaching," but "exquisite" π series "especially is most wonderful."

English typical 🖩, care about plug-and-chug-ready formulas, criticise Leibniz for merely giving special cases. Collins: "infinite Series to be generally fitted to any equation proposed, so that an Algebraist being furnished with his Stock, will quickly fitt a Series." Newton: I gave "a general Method of doing in all Figures," whereas "Leibnitz never produced any other Series than numerical Series deduced from them in particular Cases."

But Leibniz has no interest in 🖩 that doesn't lead to ♡: "I too used this method [of series inversion] at one time, but after nothing elegant had resulted in the example which I had by chance taken up, I neglected it forthwith with my usual impatience."

$$\underbrace{\text{visionary ☞} \quad \text{maestro ♡}}_{\text{beauty ♀}} \qquad \underbrace{\text{technocrat 🖩} \quad \text{pragmatist €}}_{\text{power ♂}}$$

Later Newton turns from 🖩 to ♡, because more classical and elegant (and perhaps associated with a certain snobbery and sense of superiority): "He thought Huygens's stile and manner the most elegant of any mathematical writer of modern times, and the most just imitator of the antients. Of their taste, and form of demonstration, Sir Isaac always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret of his mistake at the beginning of his mathematical studies, in applying himself to the work of Des Cartes and other algebraic writers."

Euler disapproves, goes back to 🖩, values toolbox adaptability more than beauty: "I always have the same trouble, when I might chance to glance through Newton's *Principia*: Whenever the solutions of problems seem to be sufficiently well understood by me, yet by making only a small change, I might not be able to solve the new problem using this method."

visionary ☞ | maestro ♡ | technocrat 🖩 | pragmatist €

$\underbrace{\text{visionary} \quad \text{maestro}}_{\text{beauty } ♀} \qquad \underbrace{\text{technocrat} \quad \text{pragmatist}}_{\text{power } ♂}$

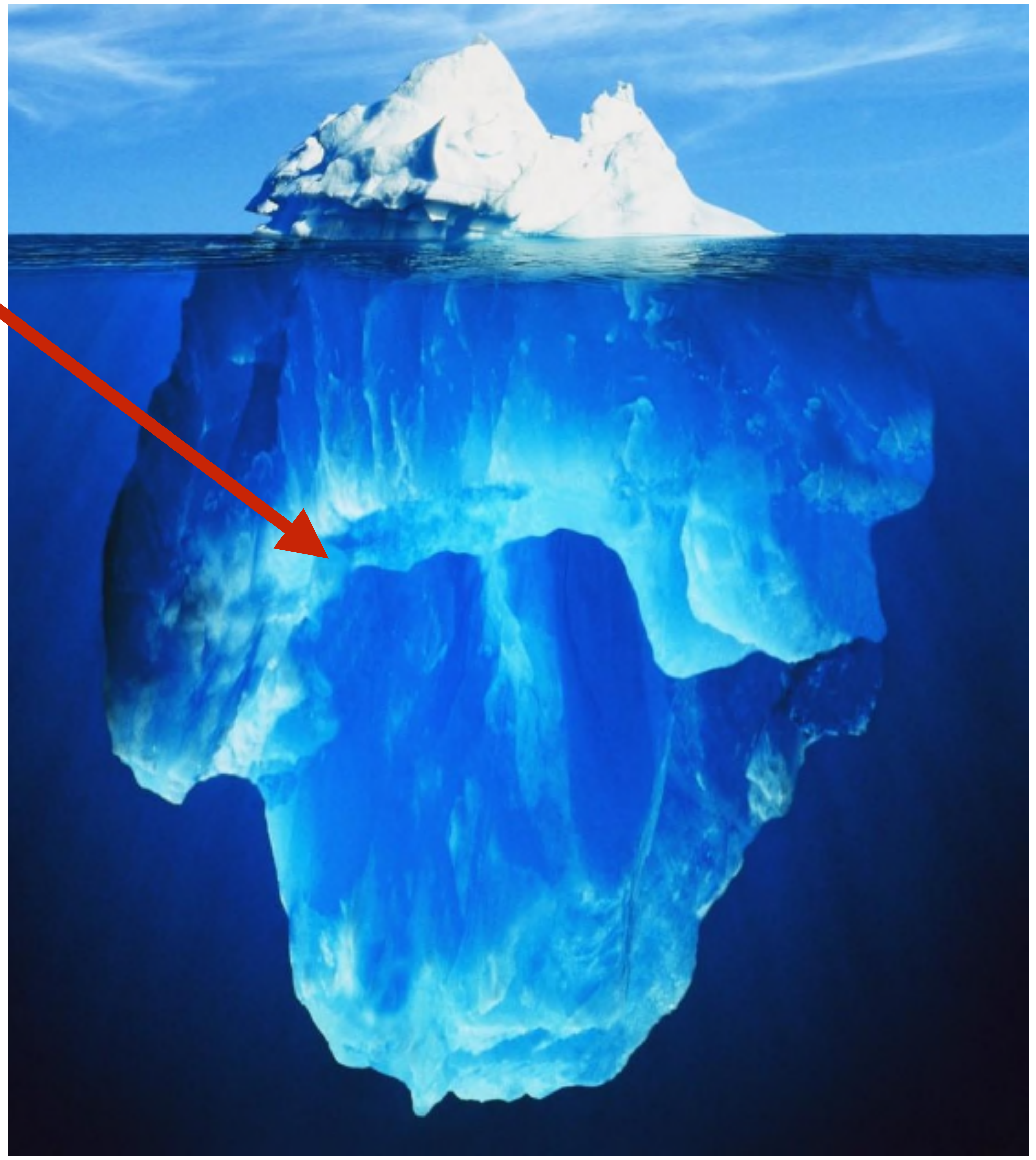
Leibniz is by nature a ☞. The ♡ tendencies in the π series episode are coloured by the influence of Huygens, who, in typical ♡ manner, praised the π series as “a discovery always to be remembered among mathematicians.”

Later Leibniz resisted ♡ and saw it as a distraction from his main task of ☞. This is why, for example, he fights not to get drawn into the brachistochrone problem (a true ♡ problem): “The problem draws me reluctantly and resistingly to it by its beauty, like the apple did Eve. For it is a grave and harmful temptation to me.”

visionary ☞ | maestro ♡ | technocrat 🖩 | pragmatist €

beauty ♀ | power ♂

☞ need 🖩 to spell out the details of their systems. E.g. Descartes: Van Schooten; Leibniz: l'Hôpital, Johann Bernoulli.



Leibniz: "I wish there were young people who would apply themselves to these calculations. With me it's like the tiger who lets run whatever he does not catch in one or two or three attempts."

Leibniz is no more than 5% 🖩 : "Had I 20 heads, or better yet 20 good friends, I would put one of them toward working out the theory of conics."

visionary ☞ | maestro ♡ | technocrat 🖩 | pragmatist €

beauty ♀ | power ♂

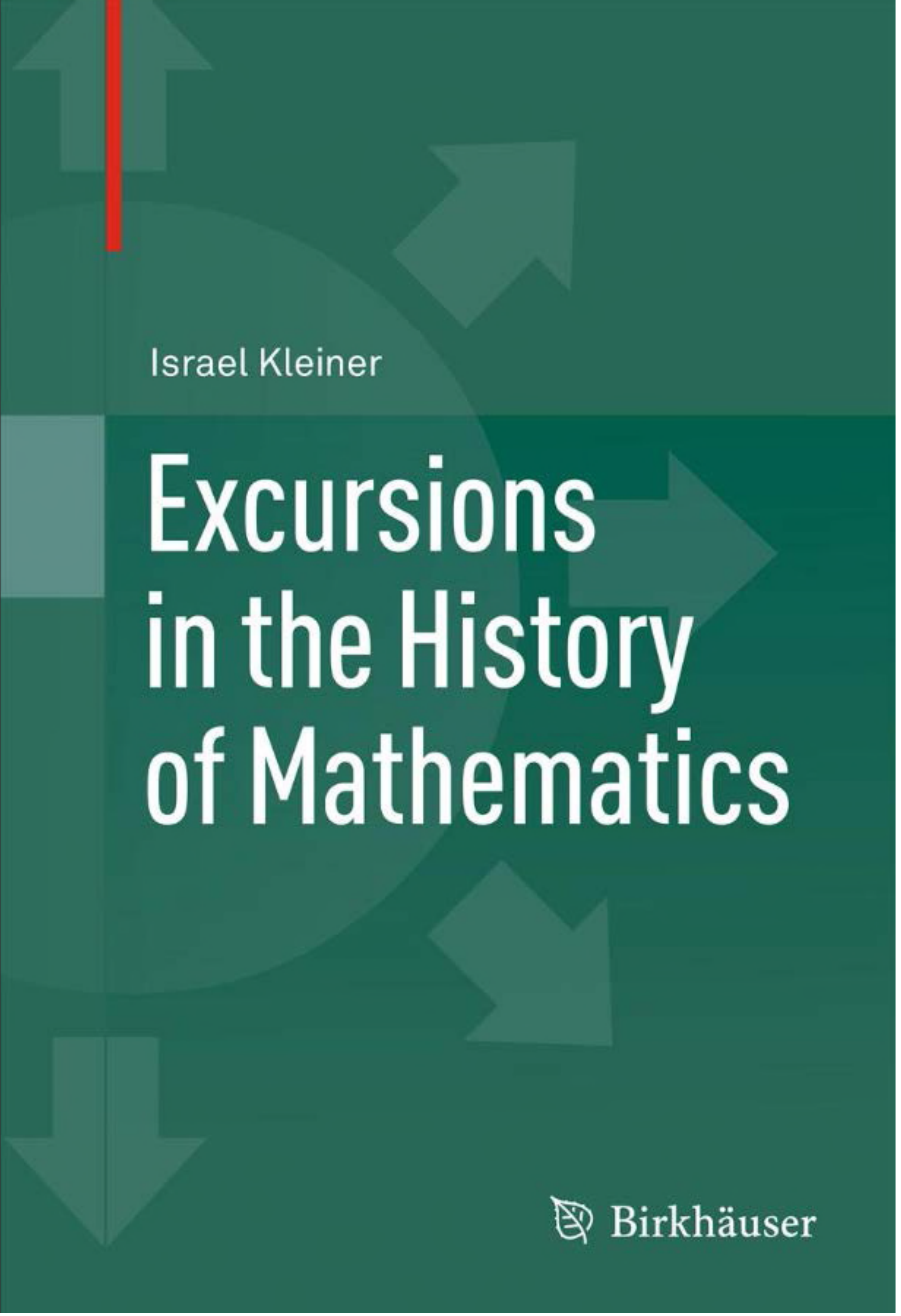
Systematic theory of integration by partial fractions: a 🖩 topic needed for ☞ , namely “a question of the greatest importance: whether all rational quadratures can be reduced to the quadrature of the hyperbola and the circle” (Leibniz). This forces Leibniz, reluctantly and contrary to his nature, to do some 🖩 work, with poor results (Leibniz erroneously believes that “$\int dx : (x^4 + a^4)$ can be reduced to neither the circle nor the hyperbola by [partial fractions], but establishes a new kind of its own”). A typical ☞ , Leibniz clearly has very little interest in actually evaluating integrals, and only cares about giving a big-picture methodological-foundational account of integration in general.

visionary ☞ maestro ♡ technocrat 🖩 pragmatist €

beauty ♀ power ♂

Myth: Early Leibnizian calculus driven by applications; lacks attention to rigour. Typical €.

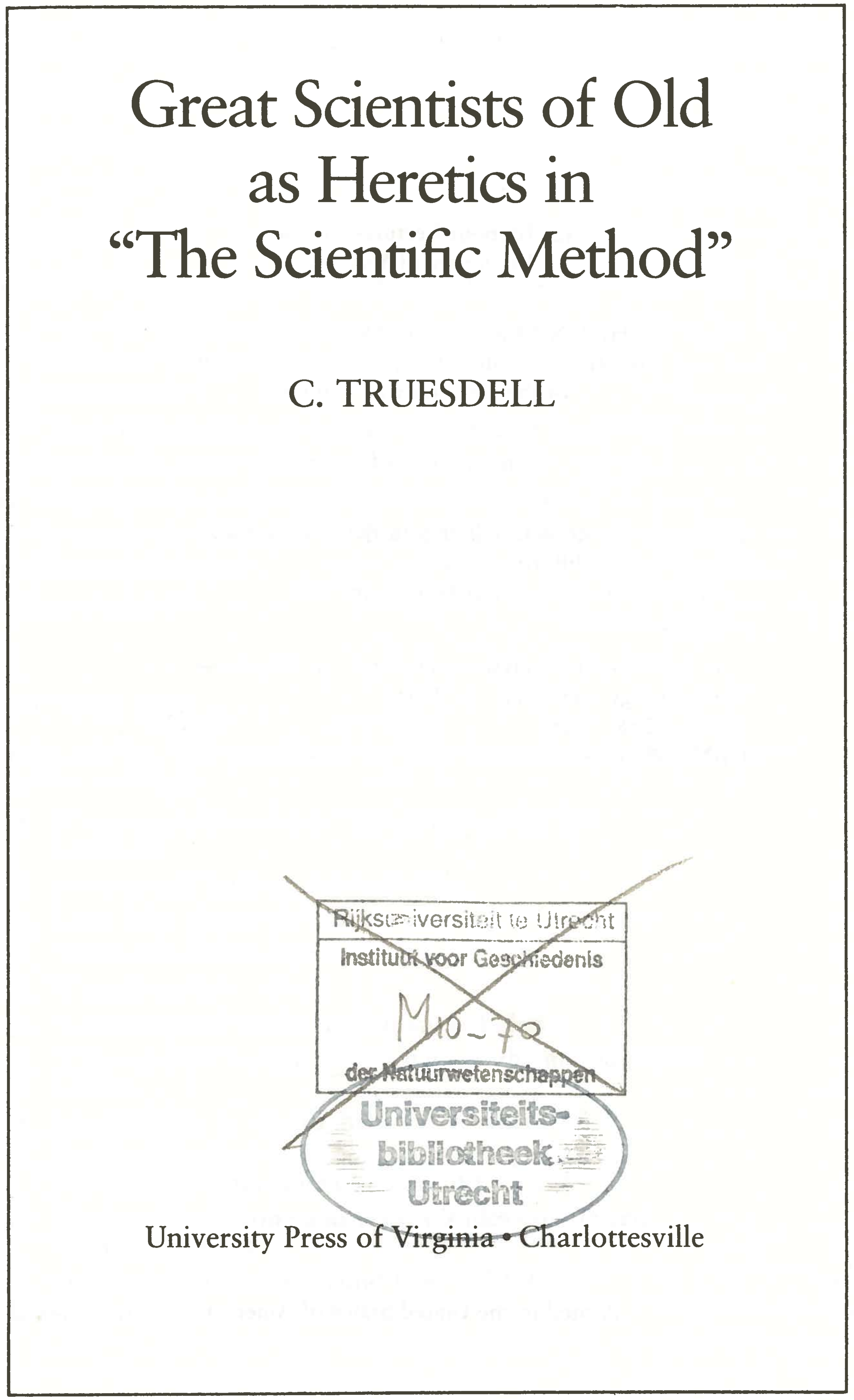
Reality: The exact opposite: Early Leibnizian calculus primarily concerned with ☞; indifferent to physics; consumed by rigour.



visionary ☞ maestro ♡ technocrat pragmatist €

beauty ♀ power ♂

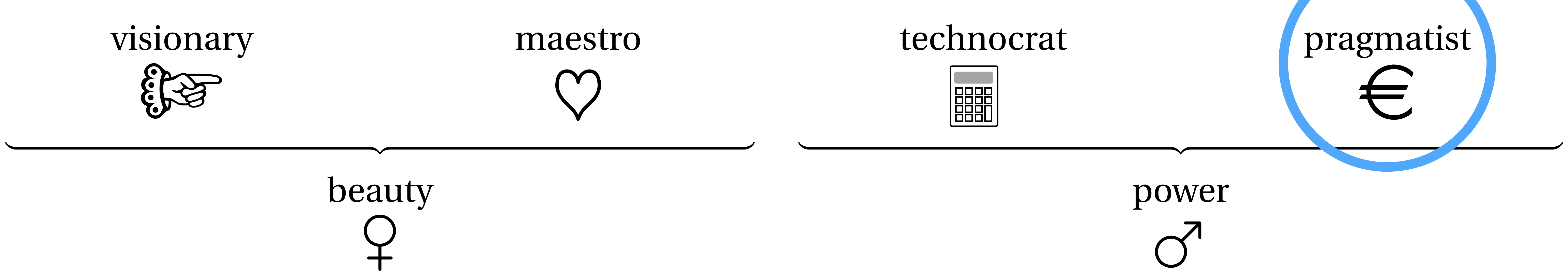
It was not uncommon for mathematicians of the seventeenth and eighteenth centuries to resort to mathematical techniques which were at best questionable, often inconsistent. They usually also recognized that their methods were unsatisfactory, but were willing to tolerate them because they yielded correct results. Justification of otherwise inexplicable notions on the grounds that they yield useful results has occurred frequently in the evolution of mathematics Of course, out of confusion



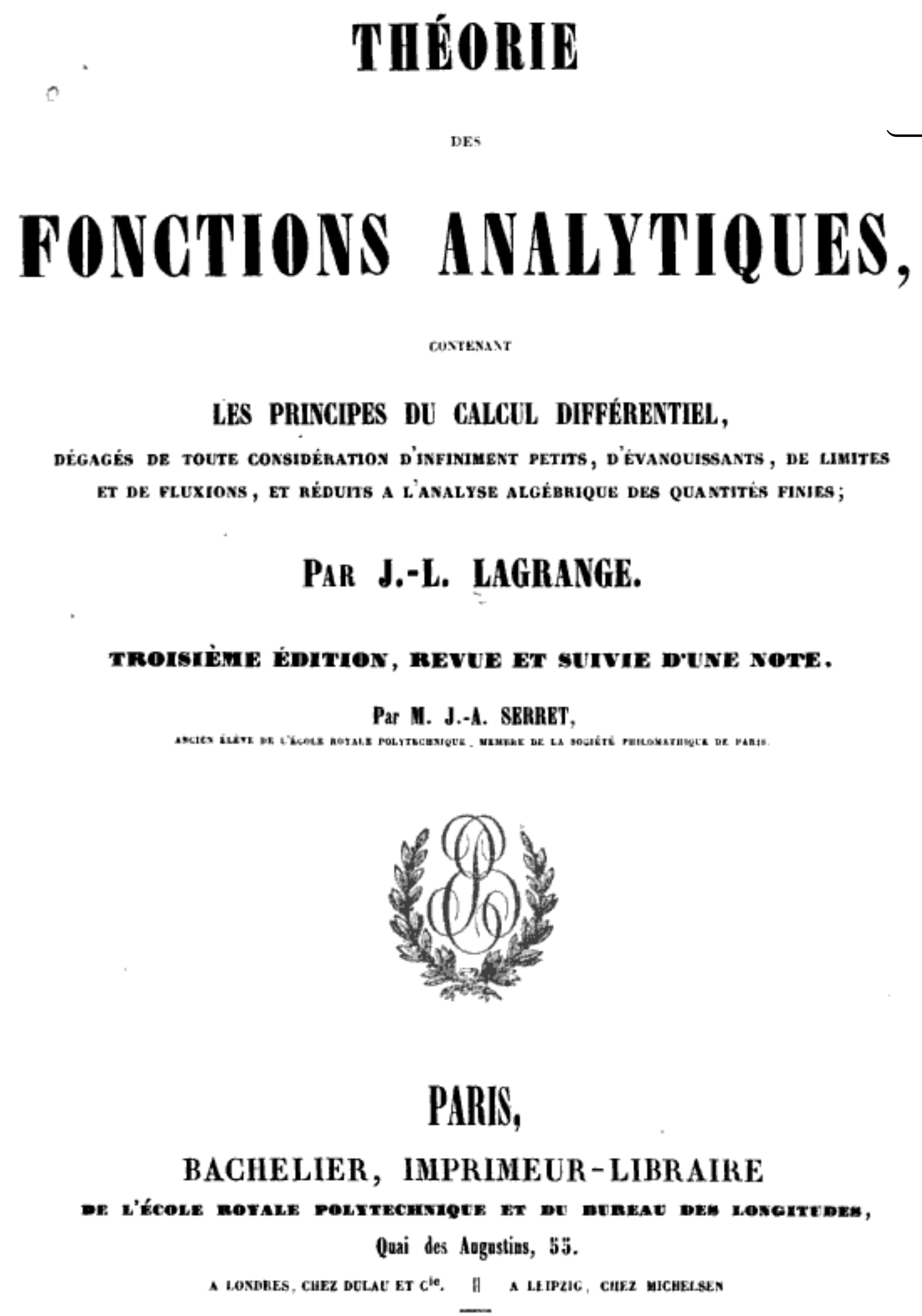
Great Scientists of Old as Heretics in “The Scientific Method”

C. TRUESDELL

University Press of Virginia • Charlottesville



I have written the story in articles and books published from the 1950s onward. In brief, the infinitesimal calculus and rational mechanics together, the former largely responding to conceptual problems set by the latter, were developed, organized into particular structures, and broadly expanded.



THÉORIE

DES

FONCTIONS ANALYTIQUES,

CONTENANT

LES PRINCIPES DU CALCUL DIFFÉRENTIEL,

DÉGAGÉS DE TOUTE CONSIDÉRATION D'INFINIMENT PETITS, D'ÉVANOUISSANTS, DE LIMITES ET DE FLUXIONS, ET RÉDUITS A L'ANALYSE ALGÉBRIQUE DES QUANTITÉS FINIES;

PAR J.-L. LAGRANGE.

TROISIÈME ÉDITION, REVUE ET SUIVIE D'UNE NOTE.

Par M. J.-A. SERRET,

ANCIEN ÉLÈVE DE L'ÉCOLE ROYALE POLYTECHNIQUE, MEMBRE DE LA SOCIÉTÉ PHILOMATHIQUE DE PARIS.

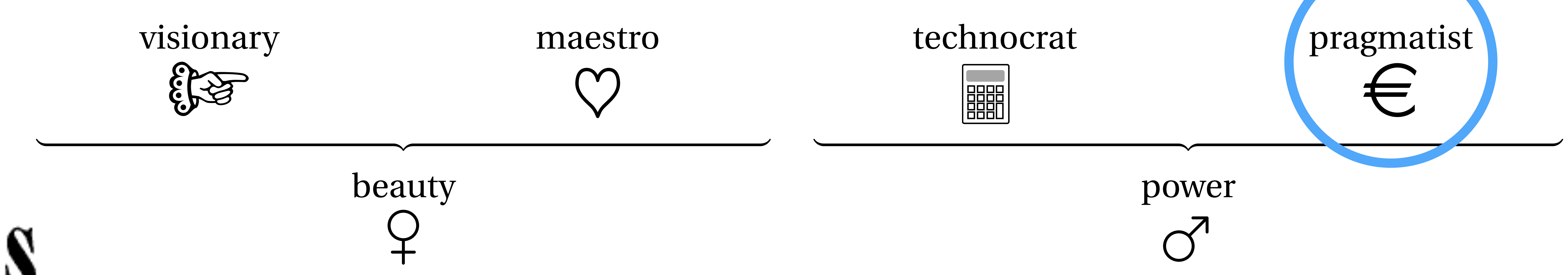
PARIS,

BACHELIER, IMPRIMEUR-LIBRAIRE

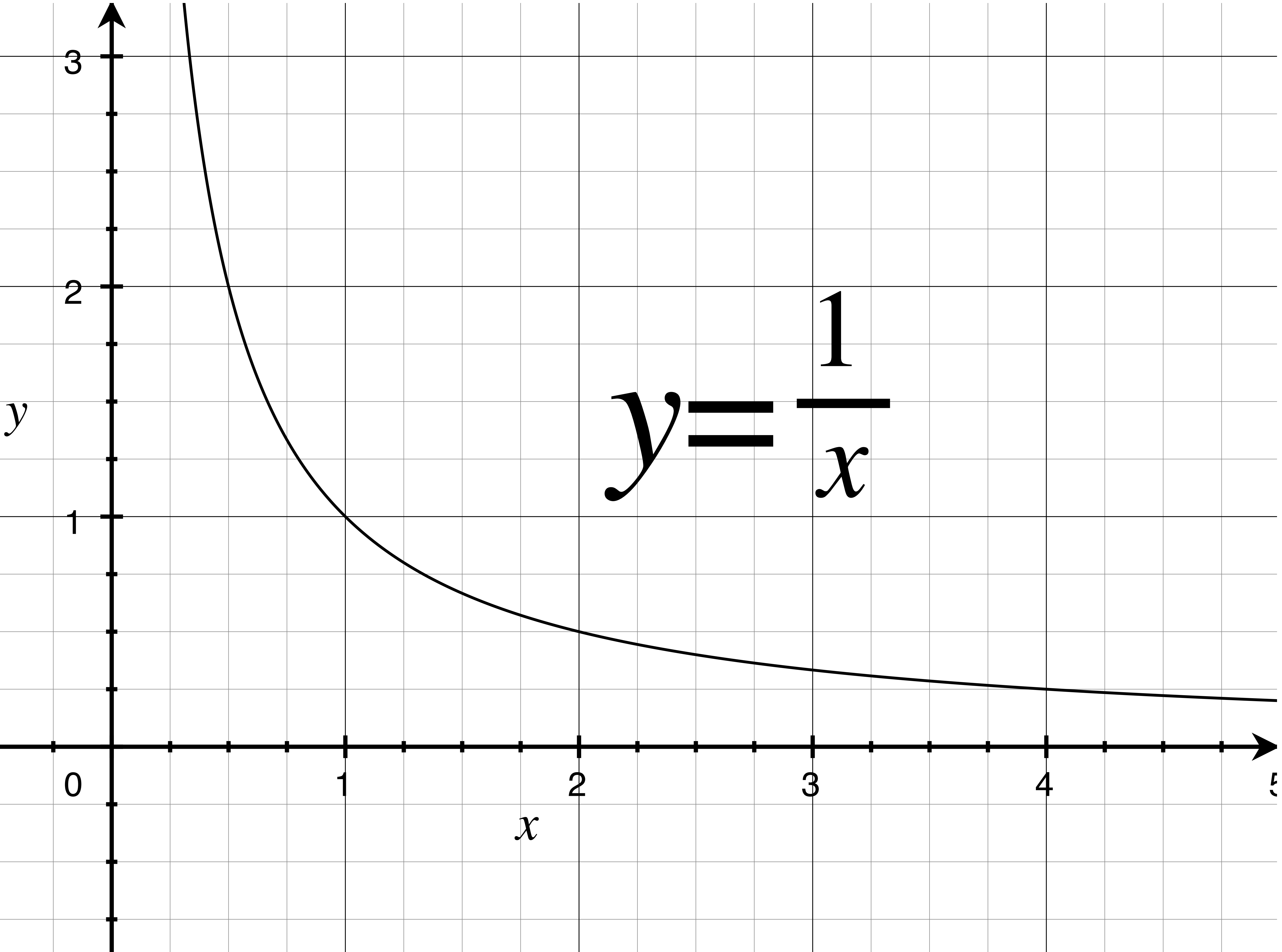
DE L'ÉCOLE ROYALE POLYTECHNIQUE ET DU BUREAU DES LONGITUDES,

Quai des Augustins, 55.

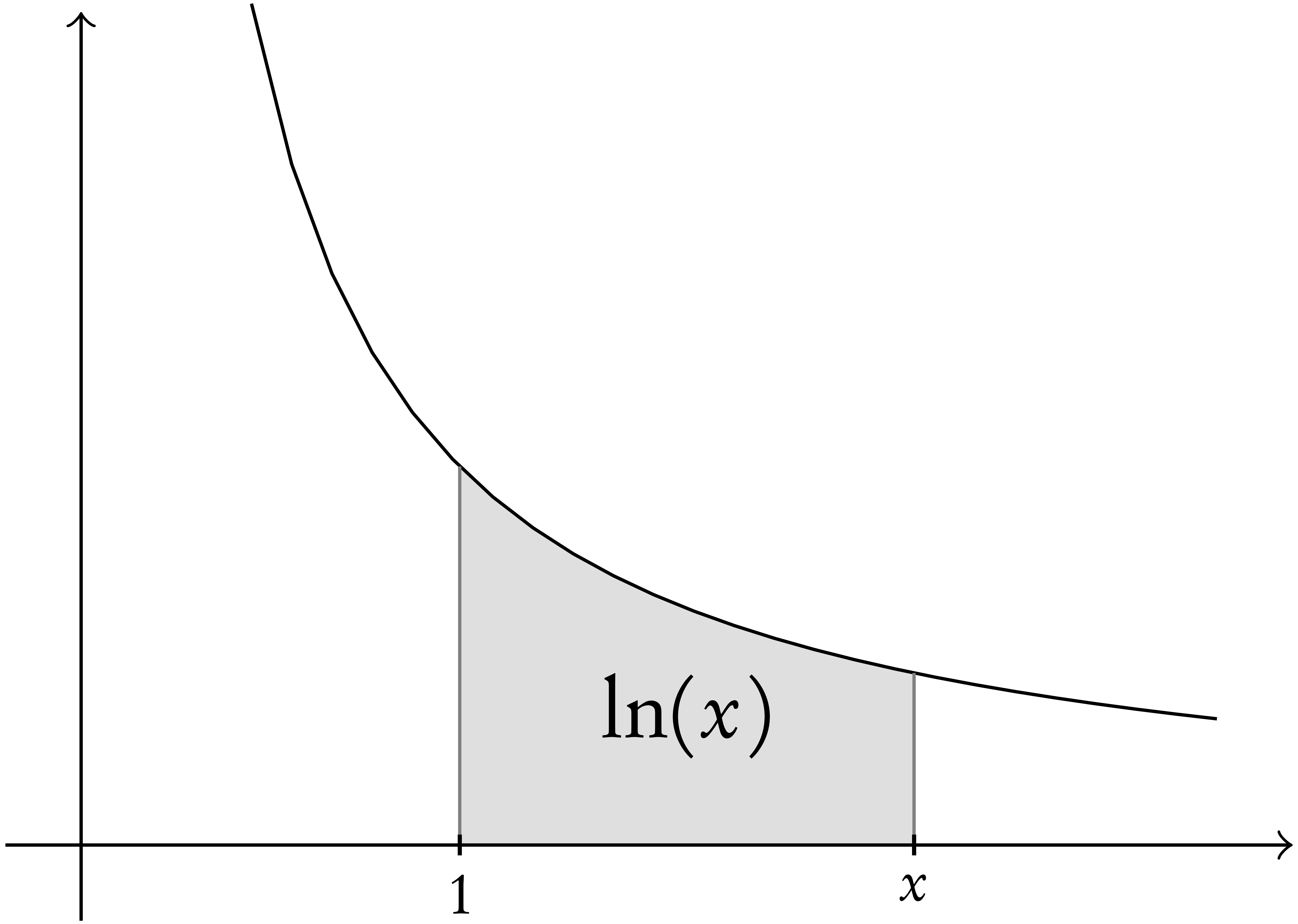
A LONDRES, CHEZ DULAU ET C^ie. || A LEIPZIG, CHEZ MICHELSEN



ceux qui admirent avec raison l'évidence et la rigueur des anciennes démonstrations regrettent de ne pas trouver ces avantages dans les principes de ces nouvelles méthodes.



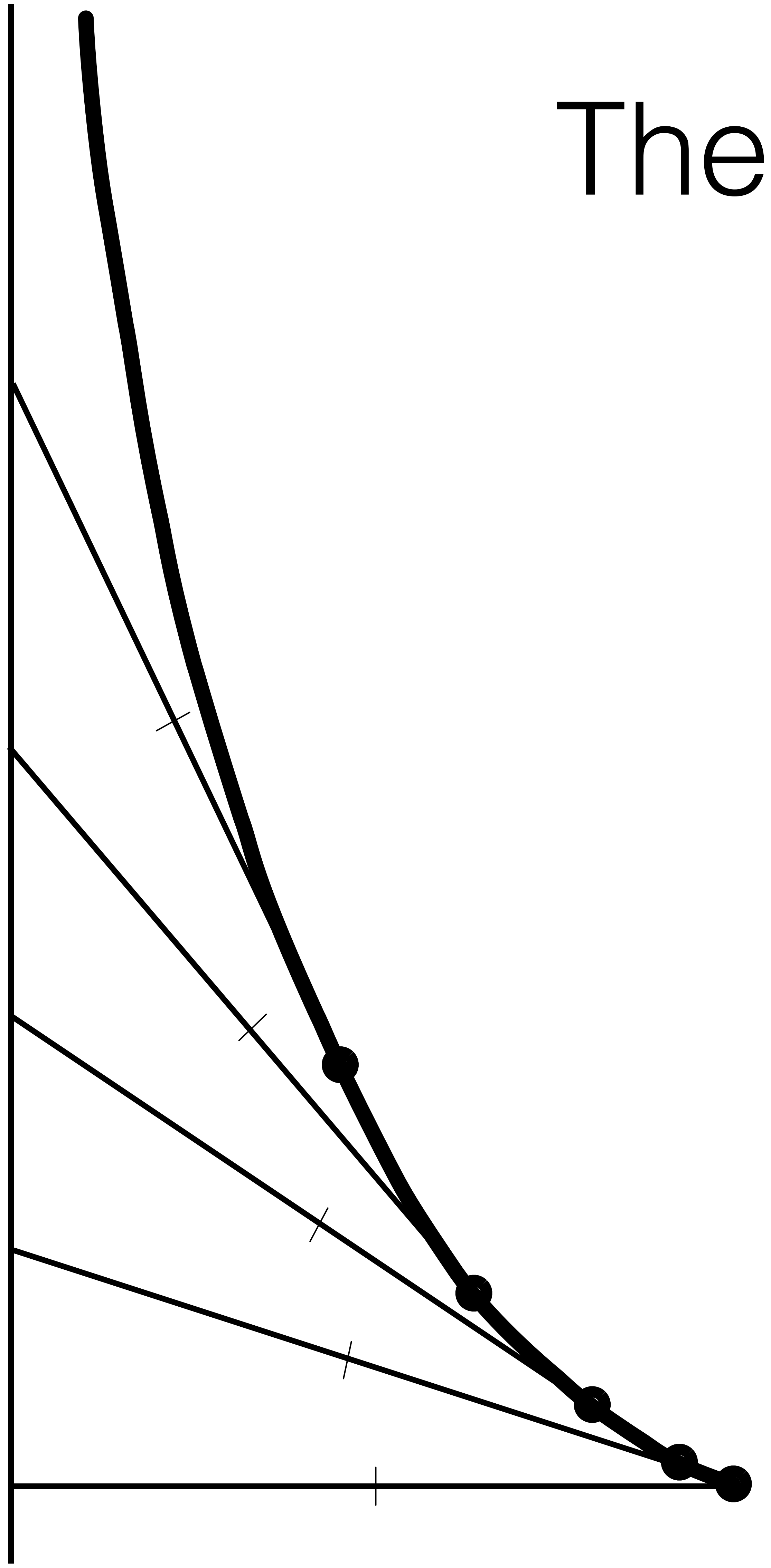
3
2
1
0
1
2
3
4
x
y
$y = \frac{1}{x}$

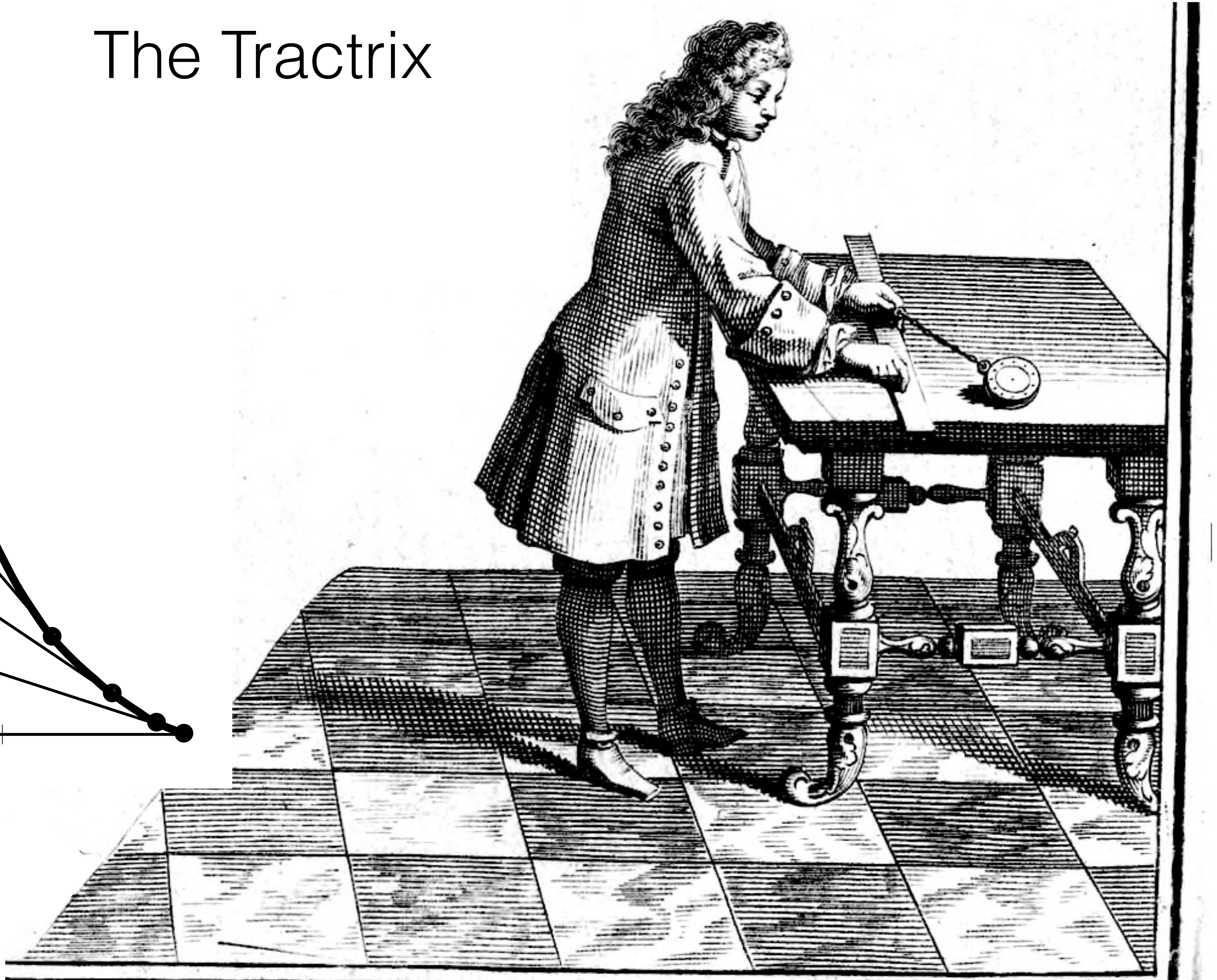


ln(x)
1
x

$y=x^2$
Volume = 1
Volume = 2
1
$\sqrt[3]{2}$
$y=\frac{2}{x}$
y
x
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
-0.5

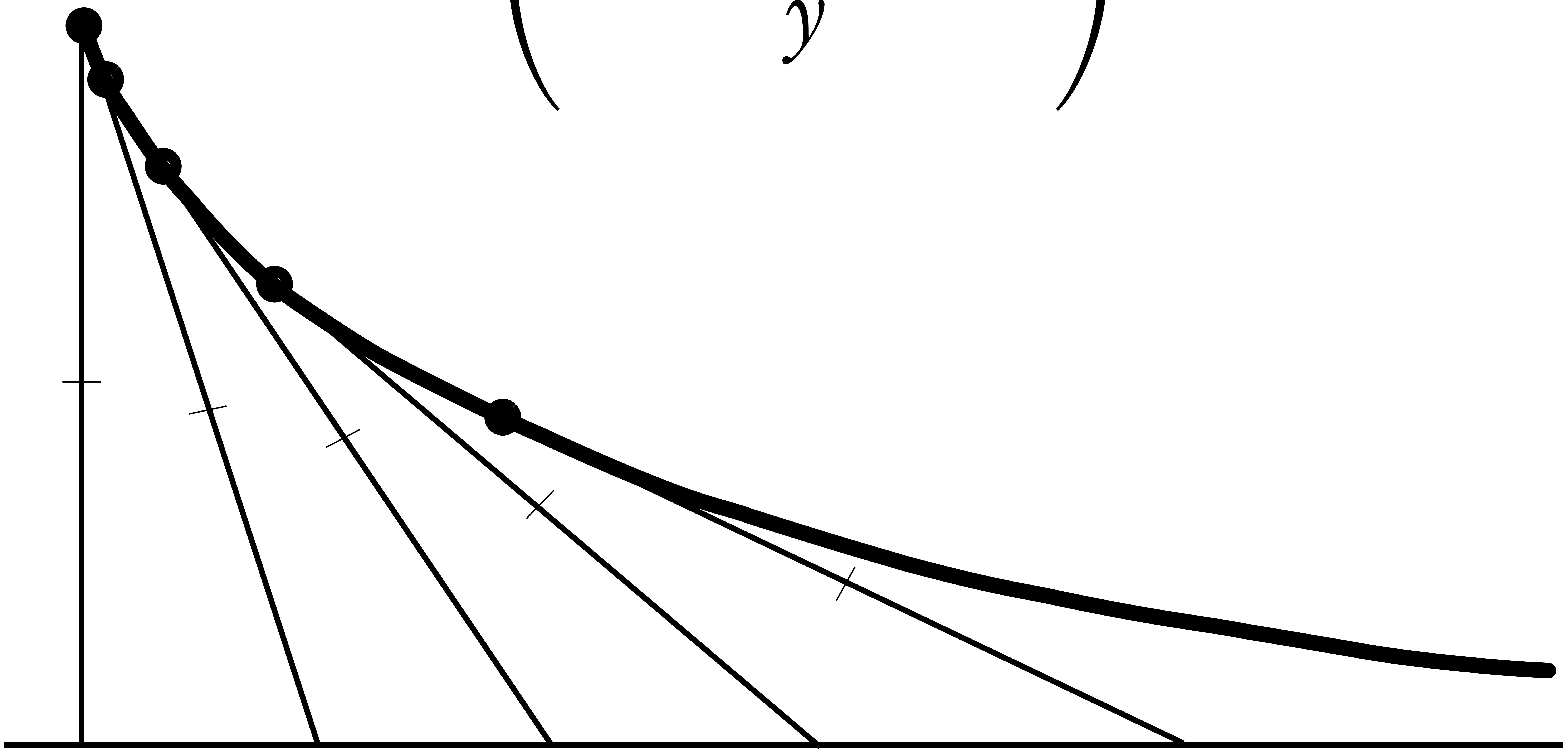
The Tractrix





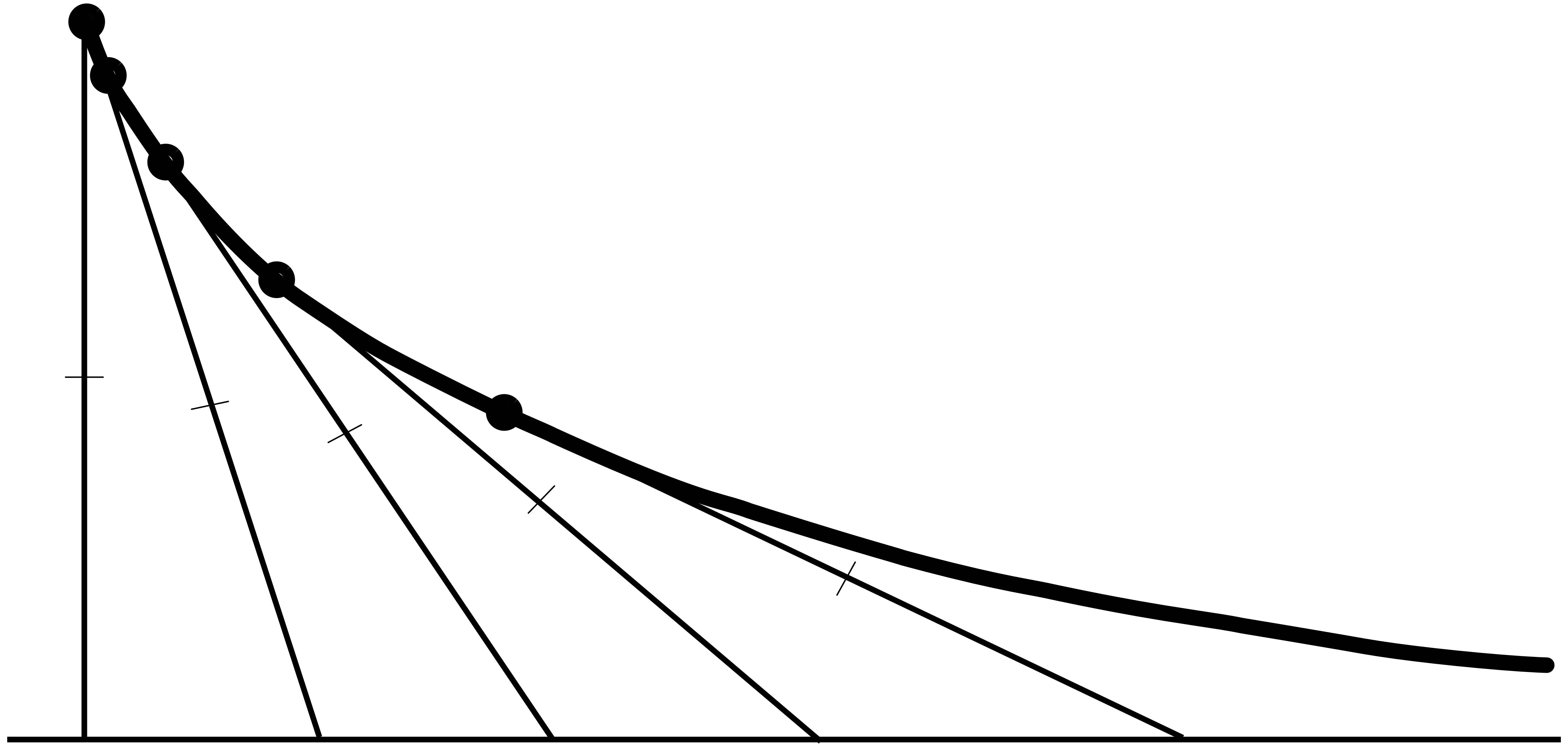
Giovanni Poleni, **Epistolarum mathematicarum fasciculus**, *1729*

$$x = \log\left(\frac{1+\sqrt{1-y^2}}{y}\right) - \sqrt{1-y^2}$$



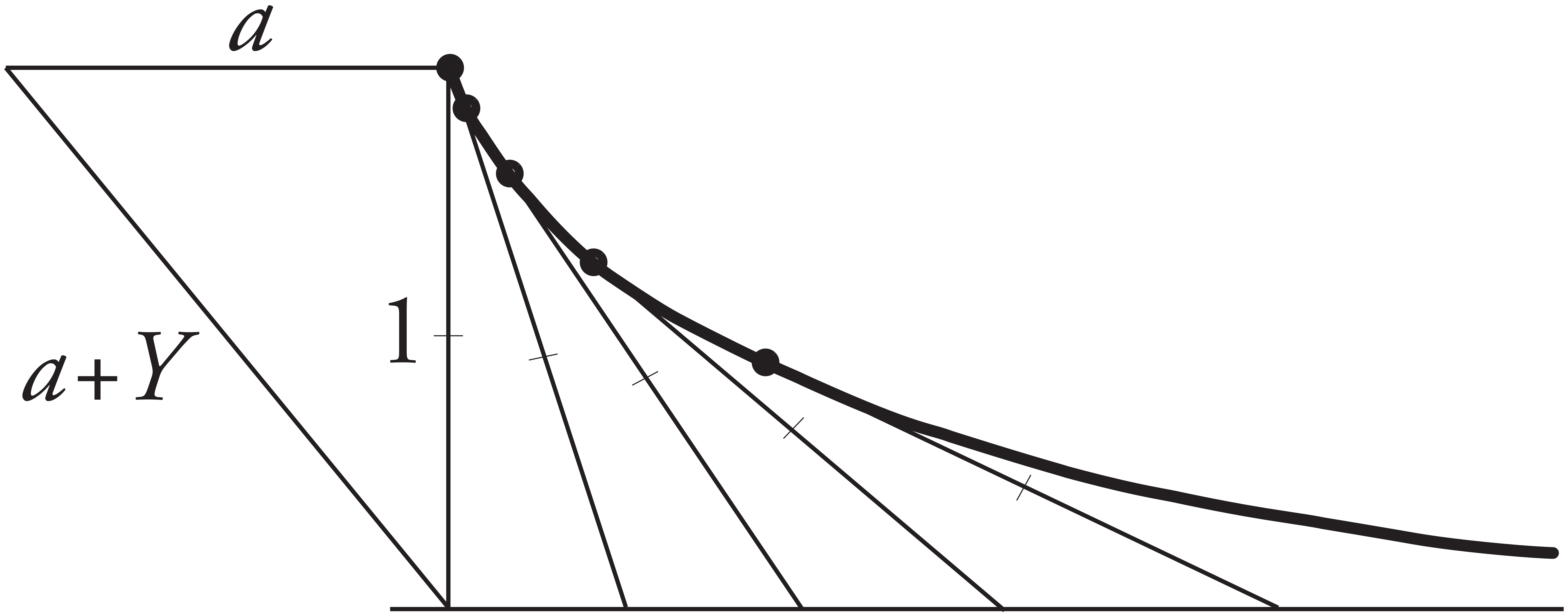
Given: tractrix

Sought: $\log(1/Y)$



Given: tractrix

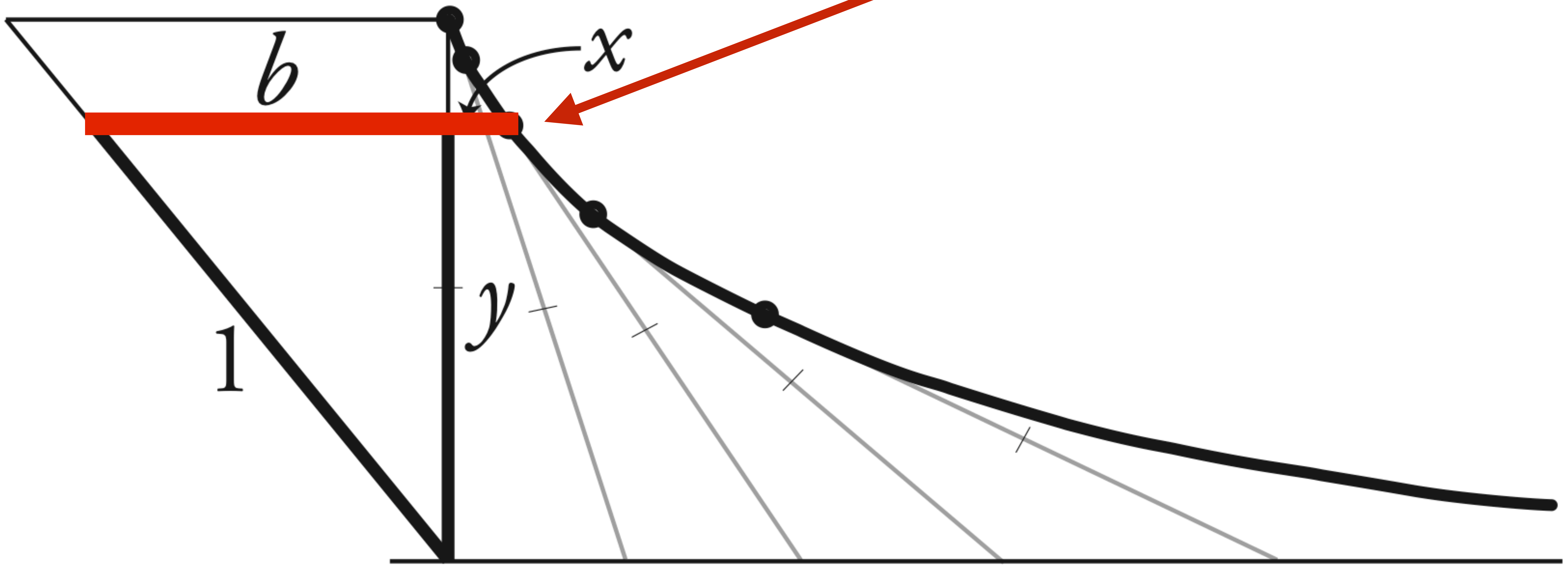
Sought: $\log(1/Y)$



$$(a+Y)^2 = a^2 + 1^2 \qquad \text{so that} \qquad a = \frac{1-Y^2}{2Y}$$

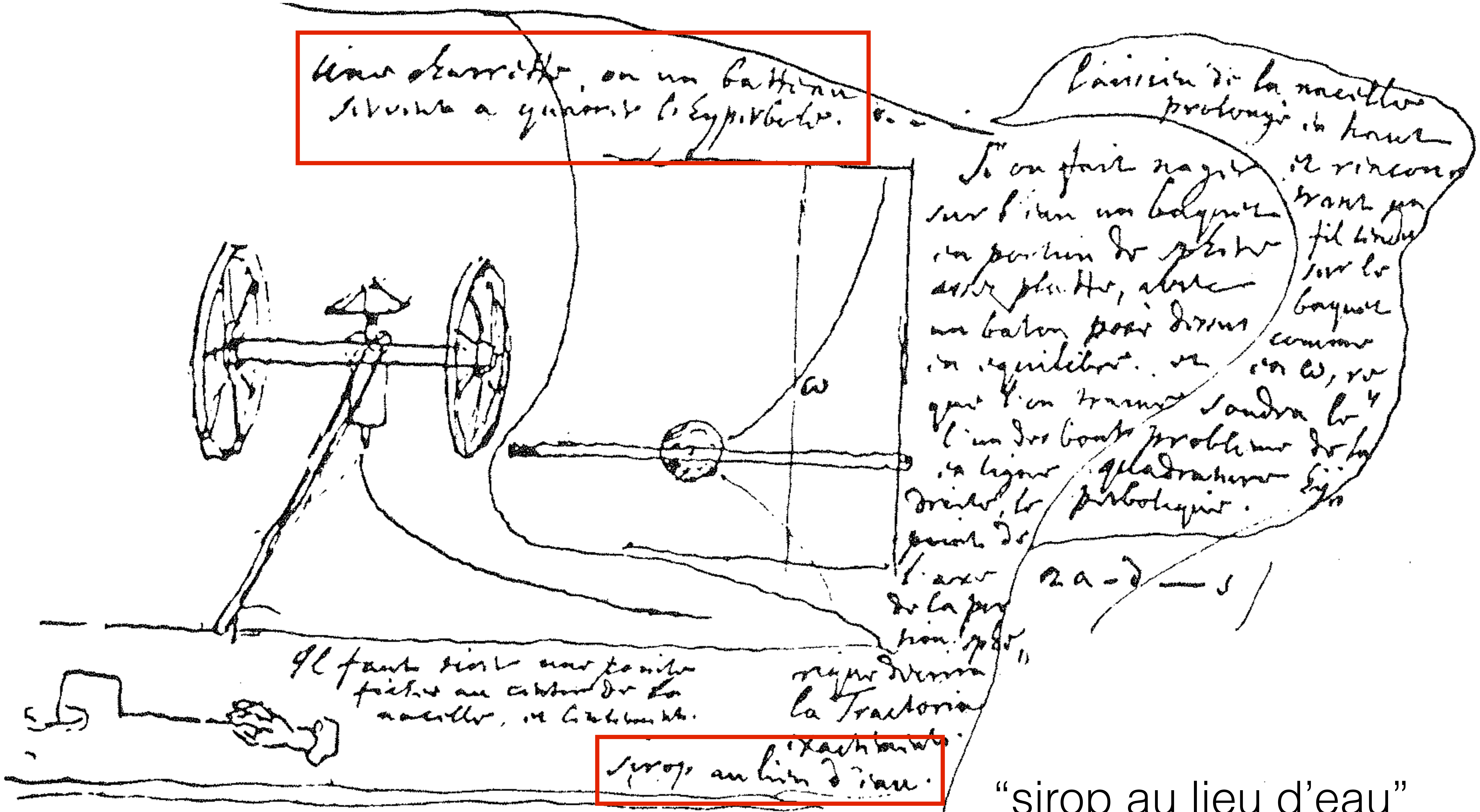
Given: tractrix

Sought: $\log(1/Y)$



$y=x^2$
$y=\frac{2}{x}$
x
y
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
2.5
2
1.5
1
0.5
-0.5
$\sqrt[3]{2}$

“Une charette, ou un batteau servira a quarrer l’hyperbole”
“a little cart or boat will serve to square the hyperbola”



“sirop au lieu d’eau”
“syrup instead of water”

Christiaan Huygens, 1692, Leiden University Library, Ms. Hug. 6, fol. 64r

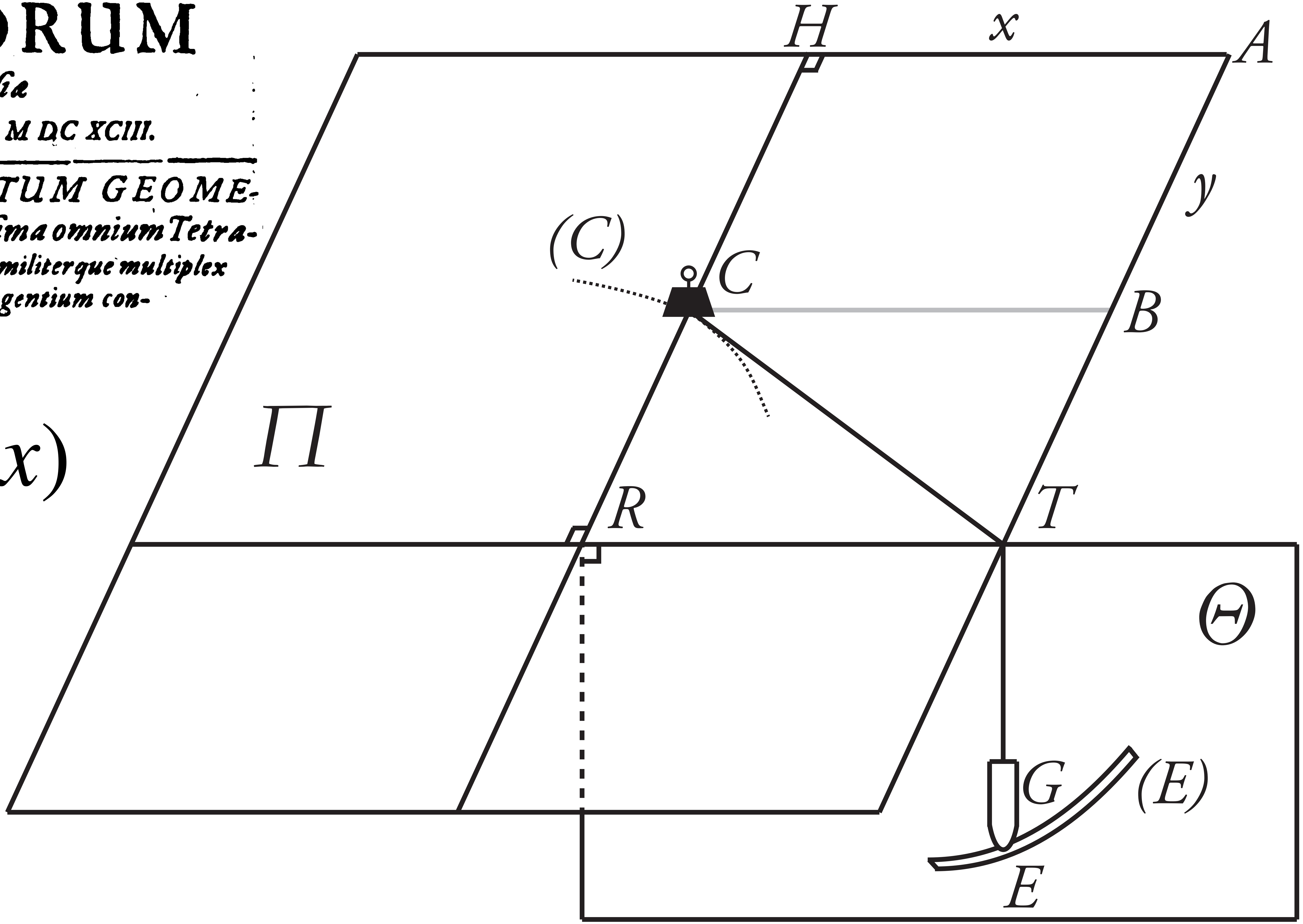
ACTA ERUDITORUM

publicata Lipsiæ

Calendis Septembris, Anno M DC XCIII.

G. G. L. SUPPLEMENTUM GEOMETRIÆ Dimensoriæ, seu generalissima omnium Tetragonismorum effectio per motum: Similiterque multiplex constructio lineæ ex data tangentium conditione.

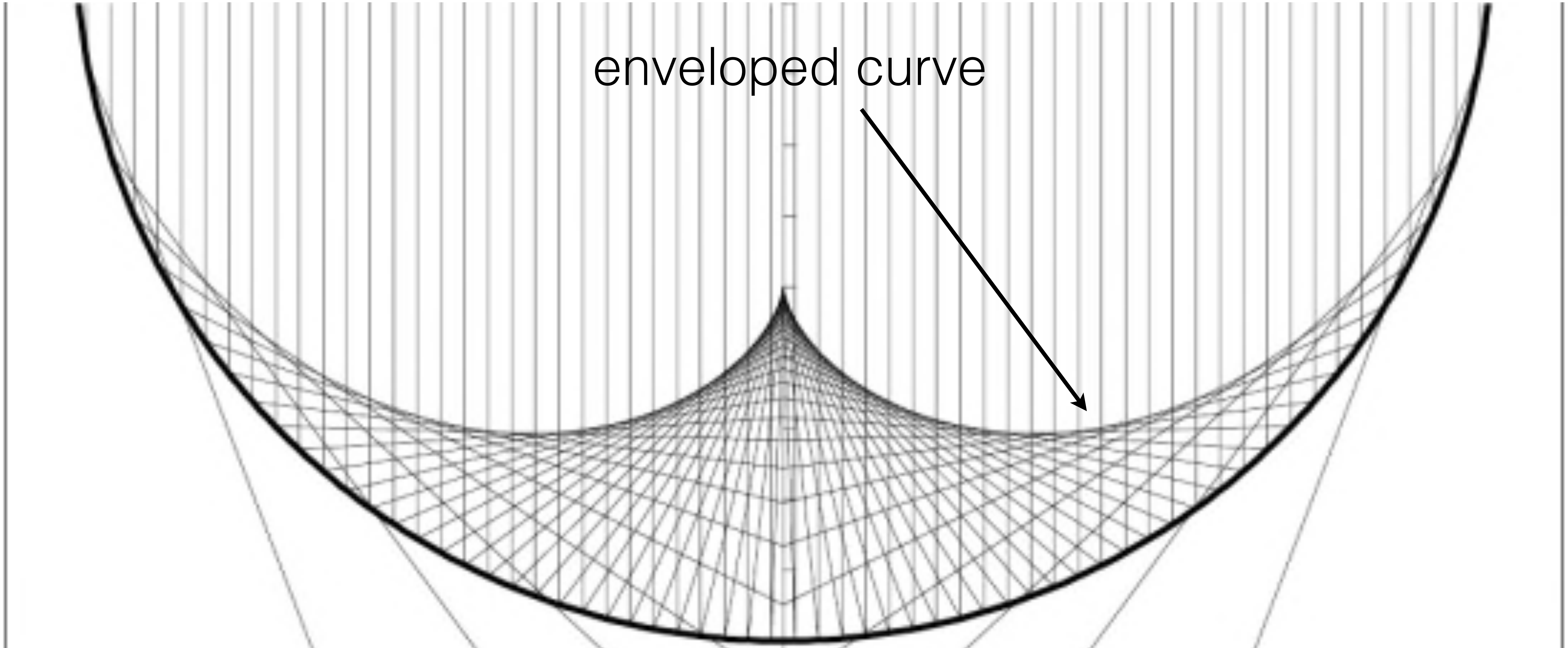
$$dy/dx = f(x)$$




enveloped curve

Leibniz's envelope rule

$$\left.\begin{aligned} f(x,y,a) &= 0 \\ \frac{\partial}{\partial a} f(x,y,a) &= 0 \end{aligned}\right\} \quad \text{eliminate } a$$



DE LINEA EX LINEIS NUMERO INFINITIS ordinatim ductis inter ſe concurrentibus formata, easque omnes tangente, ac de novo in ea re Analyſis infinitorum uſu.

Autore O. V. E.

ORdinatim applicatas vocare ſolent Geometræ rectas quotcunque inter ſe parallelas, quæ a curva ad rectam quandam (*directricem*) usque ducuntur, quæ cum ad directricem (tanquam *Axem*) ſunt normales, ſolent vocari *Ordinatæ κατ' ἐξοχήν*. *Deſarguеſius* rem prolatavit, & ſub Ordinatim applicatis etiam comprehendit rectas *convergentes* ad unum punctum commune, aut ab eo *divergentes*. Et ſane parallelæ ſub convergentibus aut divergentibus comprehendi poſſunt, fingendo punctum concurſus infinite abhinc diſtare. Verum quia multis aliis modis fieri poteſt, ut infinitæ duci intelligantur lineæ ſecundum legem quandam communem, quæ tamen non ſint parallelæ vel convergentes ad punctum omnibus commune, aut a puncto omnibus communi divergentes, ideo nos tales lineas generaliter vocabimus *Ordinatim ductas*, vel ordinatim (poſitione) datas. Exempli cauſa, ſi ſpeculum aliquod, vel potius ſectio ejus a plano per axem, cujuscunque figuræ poſitione datæ, radios Solares ſive immediate, ſive poſt aliam quandam reflexionem aut refractionem advenientes reflectat; iſti radii reflexi erunt infinitæ lineæ rectæ ordinatim ductæ, & dato quovis puncto ſpeculi (cæteris manentibus) dabitur radius reflexus ei *reſpondens*. Verum ego ſub ordinatim ductis non tantum rectas, ſed & curvas lineas qualescunque accipio, modo *lex* habeatur, ſecundum quam dato lineæ cujusdam datæ (tanquam *ordinatricis*) puncto, reſpondens ei puncto linea duci poſſit, quæ una erit ex ordinatim ducendis, ſeu ordinatim poſitione datis. *Ordine* enim percurrendo puncta ordinatricis (verbi gratia lineæ, cujus rotatione fit ſpeculum paulo ante dictum, ſeu ſectionis ejus per axem) ordine prodibunt lineæ illæ ordinatim datæ. Porro etſi eæ non concurrant omnes ad unum punctum commune, tamen regulariter duæ quævis tales lineæ *proximæ*, (id eſt *infiniteſime* differentes, ſeu infinite parvam habentes diſtantiam) concurrunt inter ſe, punctumque con-

curſus eſt aſſignabile, & his concurſibus ordinatim ſumtis nova prodit *linea concurſuum*, quæ eſt omnium concurſuum inter proximas locus communis, habetque hoc egregium, quod omnes ordinatim ductas, quarum concurſu formatur, tangit, quam proprietatem, cum meditantibus ſatis appareat, demonſtrare hic non eſt opus. Talis eſt *linea evolutione generans*, ea enim omnes rectas ad curvam evolutione generatam perpendiculares tangit, ex Hugeniano invento. Tales ſunt lineæ plures *coëvolutione generantes*, quas Dn. D. T. excogitavit, & *quaſi Foci* ab eodem introducti, cum concurſus radiorum non fiunt in puncto, ſed in ejus locum *Focus* eſt *linearis*, concurſu ſaltem duarum proximarum quarumcunque formatus. Sed cum hæc non niſi ad rectas pertineant, ſciendum eſt aliquid analogum & in curvis locum habere. Ita linea reflectens, quæ radios ſecundum quamcunque præſcriptam legem a lucido, vel ſpeculo aut lente (una pluribusve) datarum figurarum, venientes reddit iterum convergentes (divergentes aut parallelas) cujus conſtructionem in his Actis dedimus, formatur ex concurſu infinitarum ellipſium (hyperbolarum aut parabolarum.) Et hinc quoque Methodus haberi poterat, problema illud prima fronte tam difficile ſolvendi: nam infinitæ illæ ellipſes ſunt ordinatim poſitione datæ, adeoque & linea concurſuum data eſt, ſeu haberi poteſt. Et hæc Methodus ad multa alia præſtanda aditum præbet, quæ alias vix videbantur eſſe in poteſtate. Quæ etiam cauſa eſt, cur viam hanc novam Geometris aperire voluerim. Res autem pendet a noſtra *Analyſi indiviſibilium*, & calculus hujus Methodi tantum applicatio eſt noſtri calculi differentialis. Nempe conſtituta ſemel æquatione locali (ſeu ad curvam lineam, unam ex ordinatim datis,) ſed generali, (legem omnibus communem exhibente) hujus æquationis jam quæratur æquatio differentialis, modo mox dicendo, & ope harum æquationum habetur quæſitum. Et quidem cum linea alicujus curvæ ad punctum quodcunque in ea datum quæritur tangens, tunc etiam tantum opus eſt *æquationem ejus* curvæ *differentiare*, ſeu quærere æquationem, quæ ſit differentialis ad æquationem curvæ localem, ſed tunc *parametri* ſeu rectæ magnitudine *conſtantes*, lineæ conſtructionem; vel æquationis pro ipſa calculum ingredientes, quæ per *a*, *b*, &c. deſignari ſolent, cenſentur *unica* ſeu *indif-*

ferentiabiles, quemadmodum & ipſa recta tangens, vel aliæ nonnullæ *functiones* ab ea pendentes, verb. gr. perpendiculares ad tangentem ab axe ad curvam ductæ. Verum tam *ordinata* quam *abſciſſa*, quas per *x* & *y* deſignari mos eſt (quas & *coordinatas* appellare ſoleo, cum una ſit ordinata ad unum, altera ad alterum latus anguli, a duabus *condirectricibus* comprehenſi) eſt *gemina* ſeu *differentiabilis*. Hic vero in noſtro calculo præſenti cum non quæritur tangens quæcunque unius curvæ in quocunque ejus puncto, ſed tangens unica infinitarum curvarum ordinatim ductarum, unicuique in ſuo puncto reſpondenti occurrens, adeoque cum quæritur uni ex his curvis aſſumptæ reſpondens punctum contactus, tunc contrarium evenit, & tam x quam y (vel alia functio ad punctum illud determinandum æquivalens) eſt *unica*; ſed aliqua minimum parameter *a* vel *b* debet eſſe *gemina* ſeu differentiabilis, ea nimirum, qua variata etiam variantur curvæ ordinatim datæ. Et quidem, licet unius curvæ plures poſſint eſſe rectæ conſtantes ſeu parametri, (exempli cauſa ellipſis omnis, & hyperbolæ pleræque habent duas, cum parabola & circulus habeant tantum unicam,) tamen hic ſemper oportet ex datis eo rem tandem poſſe deduci, ut unica tantum ſuperſit *conſtans* (in eadem curva) *variabilis* (pro diverſis) alioqui modus ordinatim eas ducendi non ſatis eſt determinatus. Interim nihil impedit cum plures dantur æquationes determinantes, conſiderari plures parametros ut differentiabiles, cum etiam plures æquationes differentiales pro ipſis determinandis haberi poſſint. Et plerumque datur *conſtantiſſima* (una vel plures) ſeu parameter communis omnibus ordinatim ducendis; adeoque litera eam deſignans in calculo differentiali etiam manet indifferentiabilis. Hinc patet, eandem æquationem poſſe habere diverſas æquationes differentiales, ſeu variis modis eſſe differentiabilem, prout poſtulat ſcopus inquiſitionis. Imo fieri poſſe expertus ſum, ut plures modi differentiandi eandem æquationem jungantur inter ſe. Hæc omnia explicanda eſſent diſtinctius, atque exemplis illuſtranda, ſi inſtitutiones quasdam novæ noſtræ *Analyſeos infinitorum* tradere vellemus; ſed ea res nec hujus eſt loci, & nec temporis noſtri. Et qui priora noſtra intellexerint ac porro meditari volent, ad hæc quoque non difficulter pertingent, & eo quidem jucundius, quod in partem inventionis venire

ſibi videbuntur. *Vocabulis* utor ſubinde *novis*, ſed quæ ipſe contextus explicat, neque ego in verbis facile novare ſoleo, niſi cum evidens eſt fructus, non tantum ad brachylogiam, (alioqui enim vix licuiſſet hæc ſine multiplici calculo tradere) ſed & ad quandam, ut ita dicam, admonitionem atque excitationem mentis, atque univerſalia animo concipienda.

Leibniz's 1692 paper on envelope rule:

- Rule vaguely alluded to.
- No formulas, no examples, no figures.

DE LINEA EX LINEIS NUMERO INFINITIS ordinatim ductis inter se concurrentibus formata, easque omnes tangente, ac de novo in ea re Analysis infinitorum usu.

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cursus est assignabile, & his concursibus ordinatim sumtis nova prodit *linea concursuum*, quæ est omnium concursuum inter proximas locus communis, habetque hoc egregium, quod omnes ordinatim ductas, quarum concursu formatur, tangit, quam proprietatem, cum meditantibus satis appareat, demonstrare hic non est opus. Talis est *linea evolutione generans*, ea enim omnes rectas ad curvam evolutione generatam perpendiculares tangit, ex Hugeniano invento. Tales sunt lineæ plures *coëvolutione generantes*, quas Dn. D. T. excogitavit, & *quasi Foci* ab eodem introducti, cum concursus radiorum non fiunt in puncto, sed in ejus locum *Focus* est *linearis*, concursu saltem duarum proximarum quarumcunque formatus. Sed cum hæc non nisi ad rectas pertineant, sciendum est aliquid analogum & in curvis locum habere. Ita linea reflectens, quæ radios secundum quamcunque præscriptam legem a lucido, vel speculo aut lente (una pluribusve) datarum figurarum, venientes reddit iterum convergentes (divergentes aut parallelas) cujus constructionem in his Actis dedimus, formatur ex concursu infinitarum ellipsium (hyperbolarum aut parabolarum.) Et hinc quoque Methodus haberi poterat, problema illud prima fronte tam difficile solvendi: nam infinitæ illæ ellipses sunt ordinatim positione datæ, adeoque & linea concursuum data est, seu haberi potest. Et hæc Methodus ad multa alia præstanda aditum præbet, quæ alias vix videbantur esse in potestate. Quæ etiam causa est, cur viam hanc novam Geometris aperire voluerim. Res autem pendet a nostra *Analysi indivisibilium*, & calculus hujus Methodi tantum applicatio est nostri calculi differentialis. Nempe constituta semel æquatione locali (seu ad curvam lineam, unam ex ordinatim datis,) sed generali, (legem omnibus communem exhibente) hujus æquationis jam quæratur æquatio differentialis, modo mox dicendo, & ope harum æquationum habetur quæsitum. Et quidem cum linea alicujus curvæ ad punctum quodcunque in ea datum quæritur tangens, tunc etiam tantum opus est *æquationem ejus* curvæ *differentiare*, seu quærere æquationem, quæ fit differentialis ad æquationem curvæ localem, sed tunc *parametri* seu rectæ magnitudine *constantes*, lineæ constructionem, vel æquationis pro ipsa calculum ingredientes, quæ per *a*, *b*, &c. designari solent, censentur *unicæ* seu *indif-*

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Leibniz's 1692 paper on envelope rule:

- Rule vaguely alluded to.
- No formulas, no examples, no figures.
- Optics mentioned as application:

Exempli causa, si speculum ... radios Solares ... reflectat

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Autore O. V. E.

LEIBNIZ AN RUDOLF CHRISTIAN VON BODENHAUSEN

Hannover, 25. September (5. Oktober) 1692.

O. V. E. sind die literae secundae mei nominis post primas G. G. L.

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Autore O. V. E.

ORdinatim applicatas vocare ſolent Geometræ rectas quotcunque inter ſe parallelas, quæ a curva ad rectam quandam (*directricem*) usque ducuntur, quæ cum ad directricem (tanquam *Axem*) ſunt normales, ſolent vocari *Ordinatæ* κατ' ἐξοχήν. *Deſargueſius* rem prolatavit, & ſub Ordinatim applicatis etiam comprehendit rectas *convergentes* ad unum punctum commune, aut ab eo *divergentes*. Et ſane parallelæ ſub convergentibus aut divergentibus comprehendi poſſunt, fingendo punctum concurſus infinite abhinc diſtare. Verum quia multis aliis modis fieri poteſt, ut infinitæ duci intelligantur lineæ ſecundum legem quandam communem, quæ tamen non ſint parallelæ vel convergentes ad punctum omnibus commune, aut a puncto omnibus communi divergentes, ideo nos tales lineas generaliter vocabimus *Ordinatim ductas*, vel ordinatim (poſitione) datas. Exempli cauſa, ſi ſpeculum aliquod, vel potius ſectio ejus a plano per axem, cujuscunque figuræ poſitione datæ, radios Solares ſive immediate, ſive poſt aliam quandam reflexionem aut refractionem advenientes reflectat; iſti radii reflexi erunt infinitæ lineæ rectæ ordinatim ductæ, & dato quovis puncto ſpeculi (cæteris manentibus) dabitur radius reflexus ei *reſpondens*. Verum ego ſub ordinatim ductis non tantum rectas, ſed & curvas lineas qualescunque accipio, modo *lex* habeatur, ſecundum quam dato lineæ cujusdam datæ (tanquam *ordinatricis*) puncto, reſpondens ei puncto linea duci poſſit, quæ una erit ex ordinatim ducendis, ſeu ordinatim poſitione datis. *Ordine* enim percurrendo puncta ordinatricis (verbi gratia lineæ, cujus rotatione fit ſpeculum paulo ante dictum, ſeu ſectionis ejus per axem) ordine prodibunt lineæ illæ ordinatim datæ. Porro etſi eæ non concurrant omnes ad unum punctum commune, tamen regulariter duæ quævis tales lineæ *proximæ*, (id eſt *infiniteſime* differentes, ſeu infinite parvam habentes diſtantiam) concurrunt inter ſe, punctumque con-

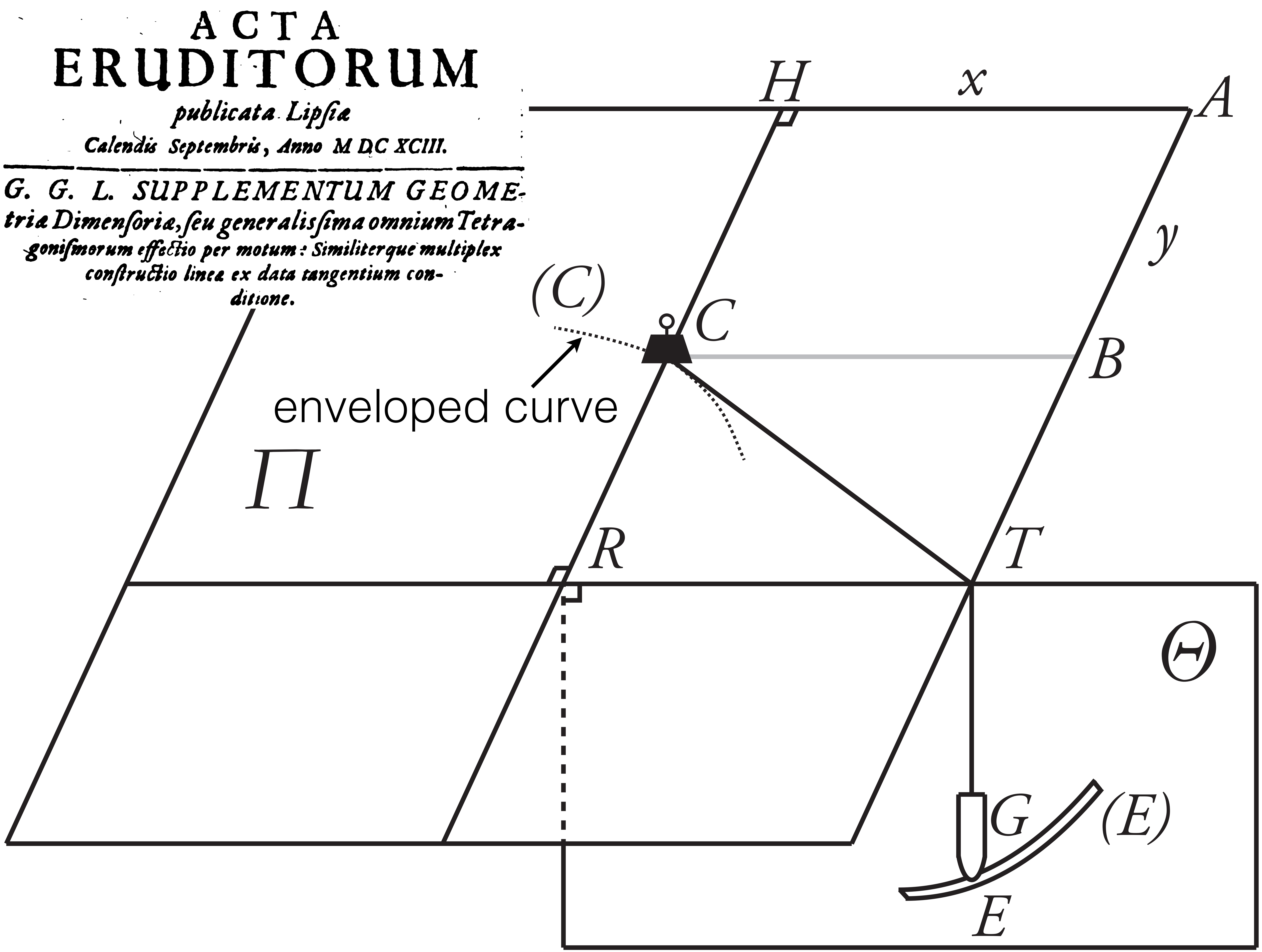
curſus eſt aſſignabile, & his concurſibus ordinatim ſumtis nova prodit *linea concurſuum*, quæ eſt omnium concurſuum inter proximas locus communis, habetque hoc egregium, quod omnes ordinatim ductas, quarum concurſu formatur, tangit, quam proprietatem, cum meditantibus ſatis appareat, demonſtrare hic non eſt opus. Talis eſt *linea evolutione generans*, ea enim omnes rectas ad curvam evolutione generatam perpendiculares tangit, ex Hugeniano invento. Tales ſunt lineæ plures *coëvolutione generantes*, quas Dn. D. T. excogitavit, & *quaſi Foci* ab eodem introducti, cum concurſus radiorum non fiunt in puncto, ſed in ejus locum *Focus* eſt *linearis*, concurſu ſaltem duarum proximarum quarumcunque formatus. Sed cum hæc non niſi ad rectas pertineant, ſciendum eſt aliquid analogum & in curvis locum habere. Ita linea reflectens, quæ radios ſecundum quamcunque præſcriptam legem a lucido, vel ſpeculo aut lente (una pluribusve) datarum figurarum, venientes reddit iterum convergentes (divergentes aut parallelas) cujus conſtructionem in his Actis dedimus, formatur ex concurſu infinitarum ellipſium (hyperbolarum aut parabolarum.) Et hinc quoque Methodus haberi poterat, problema illud prima fronte tam difficile ſolvendi: nam infinitæ illæ ellipſes ſunt ordinatim poſitione datæ, adeoque & linea concurſuum data eſt, ſeu haberi poteſt. Et hæc Methodus ad multa alia præſtanda aditum præbet, quæ alias vix videbantur eſſe in poteſtate. Quæ etiam cauſa eſt, cur viam hanc novam Geometris aperire voluerim. Res autem pendet a noſtra *Analyſi indiviſibilium*, & calculus hujus Methodi tantum applicatio eſt noſtri calculi differentialis. Nempe conſtituta ſemel æquatione locali (ſeu ad curvam lineam, unam ex ordinatim datis,) ſed generali, (legem omnibus communem exhibente) hujus æquationis jam quæratur æquatio differentialis, modo mox dicendo, & ope harum æquationum habetur quæſitum. Et quidem cum linea alicujus curvæ ad punctum quodcunque in ea datum quæritur tangens, tunc etiam tantum opus eſt *æquationem ejus* curvæ *differentiare*, ſeu quærere æquationem, quæ ſit differentialis ad æquationem curvæ localem, ſed tunc *parametri* ſeu rectæ magnitudine *conſtantes*, lineæ conſtructionem; vel æquationis pro ipſa calculum ingredientes, quæ per *a*, *b*, &c. deſignari ſolent, cenſentur *unica* ſeu *indif-*

ferentiabiles, quemadmodum & ipſa recta tangens, vel aliæ nonnullæ *functiones* ab ea pendentes, verb. gr. perpendiculares ad tangentem ab axe ad curvam ductæ. Verum tam *ordinata* quam *abſciſſa*, quas per x & y deſignari mos eſt (quas & *coordinatas* appellare ſoleo, cum una ſit ordinata ad unum, altera ad alterum latus anguli, a duabus *condirectricibus* comprehenſi) eſt *gemina* ſeu *differentiabilis*. Hic vero in noſtro calculo præſenti cum non quæritur tangens quæcunque unius curvæ in quocunque ejus puncto, ſed tangens unica infinitarum curvarum ordinatim ductarum, unicuique in ſuo puncto reſpondenti occurrens, adeoque cum quæritur uni ex his curvis aſſumptæ reſpondens punctum contactus, tunc contrarium evenit, & tam x quam y (vel alia functio ad punctum illud determinandum æquivalens) eſt *unica*; ſed aliqua minimum parameter *a* vel *b* debet eſſe *gemina* ſeu differentiabilis, ea nimirum, qua variata etiam variantur curvæ ordinatim datæ. Et quidem, licet unius curvæ plures poſſint eſſe rectæ conſtantes ſeu parametri, (exempli cauſa ellipſis omnis, & hyperbolæ pleræque habent duas, cum parabola & circulus habeant tantum unicam,) tamen hic ſemper oportet ex datis eo rem tandem poſſe deduci, ut unica tantum ſuperſit *conſtans* (in eadem curva) *variabilis* (pro diverſis) alioqui modus ordinatim eas ducendi non ſatis eſt determinatus. Interim nihil impedit cum plures dantur æquationes determinantes, conſiderari plures parametros ut differentiabiles, cum etiam plures æquationes differentiales pro ipſis determinandis haberi poſſint. Et plerumque datur *conſtantiſſima* (una vel plures) ſeu parameter communis omnibus ordinatim ducendis; adeoque litera eam deſignans in calculo differentiali etiam manet indifferentiabilis. Hinc patet, eandem æquationem poſſe habere diverſas æquationes differentiales, ſeu variis modis eſſe differentiabilem, prout poſtulat ſcopus inquiſitionis. Imo fieri poſſe expertus ſum, ut plures modi differentiandi eandem æquationem jungantur inter ſe. Hæc omnia explicanda eſſent diſtinctius, atque exemplis illuſtranda, ſi inſtitutiones quasdam novæ noſtræ *Analyſeos infinitorum* tradere vellemus; ſed ea res nec hujus eſt loci, & nec temporis noſtri. Et qui priora noſtra intellexerint ac porro meditari volent, ad hæc quoque non difficulter pertingent, & eo quidem jucundius, quod in partem inventionis venire

ſibi videbuntur. *Vocabulis* utor ſubinde *novis*, ſed quæ ipſe contextus explicat, neque ego in verbis facile novare ſoleo, niſi cum evidens eſt fructus, non tantum ad brachylogiam, (alioqui enim vix licuiſſet hæc ſine multiplici calculo tradere) ſed & ad quandam, ut ita dicam, admonitionem atque excitationem mentis, atque univerſalia animo concipienda.

First use of term “function” in print:

functiones ab ea pendentes


ACTA
ERUDITORUM
publicata Lipsiæ
Calendis Septembris, Anno M DC XCIII.
G. G. L. SUPPLEMENTUM GEOMETRIÆ Dimensoriæ, seu generalissima omnium Tetragonismorum effectio per motum: Similiterque multiplex constructio lineæ ex data tangentium conditione.
H
x
A
y
(C)
C
B
enveloped curve
Π
R
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Θ
G
(E)
E

G. G. L. NOVA CALCULI DIFFERENTIALIS Applicatio & usus, ad multiplicem linearum constructionem, ex data tangentium conditione.

Memini jam a me insinuatum in his Actis, ut rectarum ordinatim sumptarum concursu hactenus noto, ita & concursu curvarum lineas formari. Sed placet rem non parvi ad Geometriam augendam momenti exponere distinctius; nam ne in rectis quidem concurrentibus tota ejus vis fuit perspecta. In genere igitur hoc problema ad communis Geometriæ leges revocare hic docebo: *Lineis (rectis vel curvis) propositam tangentibus, positione ordinatim datis, invenire propositam*, vel quod eodem redit: *invenire lineam, quæ infinitas lineas ordinatim positione datas tangit*. Cujus usus cum latissime pateat, calculum in eam rem peculiarem jamdudum excogitavi, vel potius huc peculiari ratione applicui nostrum Differentialem compendio non contemnendo. Scilicet quemadmodum *Cartesius* loca veterum calculo exprimens æquationes adhibuit, quæ cuivis curvæ puncto conveniunt, ita nos æquationes hic adhibemus infinities ampliores, quæ cuilibet puncto cujuslibet curvæ *in serie ordinatim sumtarum curvarum* comprehensæ, accommodantur. Itaque x & y abscissa quidem & ordinata, seu *coordinatæ* esse intelliguntur cujusvis ex dictis curvis, sed speciatim tamen accipiuntur de curva ex ipsarum concursu *formata* seu ipsas tangente; utili quodam *æquivocationis characteristicæ* genere. *Coefficientes* a, b, c, in æquatione cum ipsis x & y usurpatæ, significant quantitates in eadem curva *constantes*, alias quidem *insitas* (nempe *parametros*) alias vero *extraneas*, quæ situm curvæ (adeoque verticis axisque) definiunt. Sed comparando curvas seriei inter se, seu transitum de curva in curvam considerando, aliæ coefficientes sunt *constantissimæ* seu *permanentes*, (quæ manent non tantum in una, sed & in omnibus seriei curvis,) aliæ sunt *variabiles*. Et quidem ut *seriei curvarum lex* data sit, necesse est unicam tantum in coefficientibus superesse variabilitatem, adeoque si in *primaria* pro omnibus curvis *æquatione* naturam earum communem explicante plures extent variabiles, necesse est dari alias *æquationes accessorias*, coefficientium variabilium dependentiam inter se exprimentes, qua-

rum ope omnes variabiles ex æquatione primaria tolli possint præter unam. Cæterum pro concursu duarum linearum proximarum, sua intersectione punctum curvæ quæsitæ (quam & tangere intelliguntur,) determinantium, manifestum est, concurrentes quidem, adeoque lineam ex concursu *formatam* tangentes, esse geminas; intersectionis autem seu concursus punctum esse unicum, adeoque & ordinatam ei respondentem unicam esse; cum alioqui in investigatione solita linearum propositam tangentium, rectarum vel curvarum (velut circulorum, parabolarum &c.) ex datæ curvæ ordinatis quærendarum, ordinatæ *geminæ*, tangentes *unicæ* concipiantur. Itaque quoad præsentem calculum, quo ipsæ ex tangentibus rectis vel curvis positione datis investigantur ordinatæ (contra quam in communi) manent coordinatæ x & y in hoc transitu (a proximo ad proximum) invariatæ, adeoque sunt indifferentiabiles; at coefficientes, (quæ in communi calculo indifferentiabiles censentur, quia constantes,) quatenus hic variabiles sunt, differentiantur. Notabile est autem, *si omnes insitæ coefficientes sint permanentes*, curvæque adeo ordinatim concurrentes sint congruæ inter se; perinde fore, ac si intelligantur *esse vestigia ejusdem lineæ motæ*; curvaque earum concursu formata lineam motam perpetuo durante motu tanget. Unde in hoc casu oritur connexio quædam cum generatione *trochoëidum*; nam & basis, super quâ volvitur generatrix trochoëidis, generatricem durante motu tangit.

Calculus autem ita instituetur: assumatur aliquis angulus rectus fixus, cujus crura utcunque producta constituere intelligantur duos axes relationis curvarum, seu axem cum axe conjugato; in quos demissæ normales ex puncto curvæ quocunque erunt ordinata, x, & ordinata conjugata seu abscissa, y; uno verbo: *coordinatæ*, x & y; quarum relationem ex datis quærendo habebitur *æquatio*, (1) quam paulo ante appellavimus *primariam*, cum fit cuilibet cujuslibet curvarum ordinatim sumtarum puncto communis. Quod si æquationi 1, insunt plures coefficientes variabiles, ut b, c, dabitur earum dependentia per *secundariam* æquationem, (2) unam vel plures; atque ita ex æq. 1 tollendo coefficiientes variabiles, præter unam b prodibit æq. (3) Hanc æquationem differentiando, ut prodeat æq. (4) cum in ea sola affutura sit differentialis ipsius b, evanescet differentialitas,

adeoque habemus æq. 4 ordinariam, cujus ope ex æq. 3 tollendo variabilem residuam b, habebitur æquatio, (5) in qua præter x & y tantum supererunt coefficientes invariabiles (ut a) quæ erit *æquatio* ad curvam quæsitam concursu seriei linearum formatam, adeoque *ad seriei linearum tangentem communem*. Sed & aliter institui potest calculus, prout facilitas invitabit, non tollendo statim variabiles, sed servando. Nempe datis æq. (1) primaria, & æq. (2) secundaria (una vel pluribus pro explicanda dependentia coefficientium variabilium inservituris) differentientur æq. 1, ut prodeat (3), & æq. 2, ut prodeat (4) (una vel plures, si pro æq. 2 affuerint plures.) Ita habebimus plures quantitates differentiales, sed tamen habebimus & æquationes sufficientes ad eas tollendas; & quidem modo tolli possint differentiales quantitates usque ad unam; etiam residua ista evanescet per se, & sic prodibit æq. (5) ordinaria, seu carens quantitate differentiali; quam conjungendo cum æq. 1 & 2 tolli poterunt variabiles omnes, & prodibit æq. (6) naturam exprimens curvæ quæsitæ, linearum concursu formatæ, quæ erit eadem cum æq. 5 calculi prioris.

Hac jam methodo solvi possunt innumera problemata sublimioris Geometriæ, hactenus non habita in potestate, pertinentiaque ad tangentium conversam; ex quibus nonnulla in specimen indicabo, magnæ utique generalitatis. Veluti: data relatione inter AT & AΘ resegmenta axium per curvæ tangentem CT facta, invenire curvam CC; nam rectæ curvam tangentes ordinatim positione dantur, adeoque & curva quæsita, quippe quæ earum concursu formatur. Vel si dato puncto axis T, detur lineæ datæ EE punctum E, sic ut juncta TE, si opus producta, quæsitam curvam CC tangat, patet ex dictis curvam CC præscripta hic methodo haberi. Similiter data relatione inter AP & AΠ, resegmenta axium facta per curvæ perpendicularem PC, licet invenire curvam CC: nam ob rectas PΠ ordinatim positione datas, etiam datur linea FF formata per earum concursum; hujus vero evolutione describetur curva CC quæsita. Unde hic quidem infinitæ curvæ satisfacientes dari possunt, omnes scilicet *parallelæ* inter se, quæ ejusdem lineæ evolutione *condescribuntur*. & data relatione inter AP & AΠ dari potest *curva quæsita* non tantum satisfaciens, sed & *transiens per punctum datum*. Interim hoc casu curva CC non semper est ordinaria, quoniam scilicet non ipsamet,

sed ipsius per evolutionem generatrix rectarum positione datarum concursu formatur. Certe cum ipsa curva formatur concursu, habetur *determinata*, nec in arbitrio est punctum, per quod transeat; quæ distinctio utilis est in hac doctrina.

Sed exemplum calculi dabimus in problemate itidem generali, ad aliquam tamen specialem lineam applicato: *data relatione perpendicularis* PC *ad proprium ab axe resegmentum* AP, *invenire lineam* CC. Patet enim datis positione punctis C, nempe centris circulorum, & radiis PC datis magnitudine (ob datam relationem ad AP) *dari ordinatim circulos lineam* CC *tangentes*, adeoque lineam ipsam circulorum concursu formatam haberi posse, id quod jam verbulo indicaveramus olim in *Actis* 1686 *mense Junio* p.300, sub schediasmatis finem. Itaq; centro P, radio PC magnitudine dato, describatur circulus CF. Ut ergo methodum paulo ante positam huc applicemus: ex puncto circuli quocunque F agantur normales ad crura anguli recti PAH, seu coordinatæ FG, y, & FH, x (quæ in casu concursus duorum circulorum incidunt in CB, CL) sit AP, b, & PC, c; fiet ex natura circuli, $xx + yy + bb \overset{(1)}{=} 2bx + cc$ æquatio primaria omnibus nostris circulis & cuique cujusque puncto communis. Quoniam autem datur relatio inter AP & PC, dabitur curva EE, cujus ordinata PE æquetur ipsi PC; hæc curva ponatur (exempli gratia) esse parabola, cujus parameter, a, & fiat $ab \overset{(2)}{=} cc$, quæ æquatio secundaria exhibet relationem seu dependentiam inter c & b. Hujus ope tollendo c, ex æq. 1, fiet $xx + yy + bb \overset{(3)}{=} 2bx + ab$; patet autem in æq. 1. præter coordinatas x & y, adesse coefficientes c, b, a; ex quibus c & b sunt in uno circulo constantes, & c quidem est circulo insita, cum ejus radium designet; b est extranea, quippe situm centri designans; ambæ variatis circulis sunt variabiles, sed a est constantissima sive permanens, cum non unius tantum circuli omnibus punctis, sed & pro omnibus circulis nostris in æquatione maneat eadem. Reducta jam æquatio 3 ad unam coefficientem variabilem b, differentietur, secundum b (solam in ea differentiabilem) & fiet $2bdb = xdb + adb$, seu (evanescente db) fit $b \overset{(4)}{=} x + a : 2$ (qui calcu-

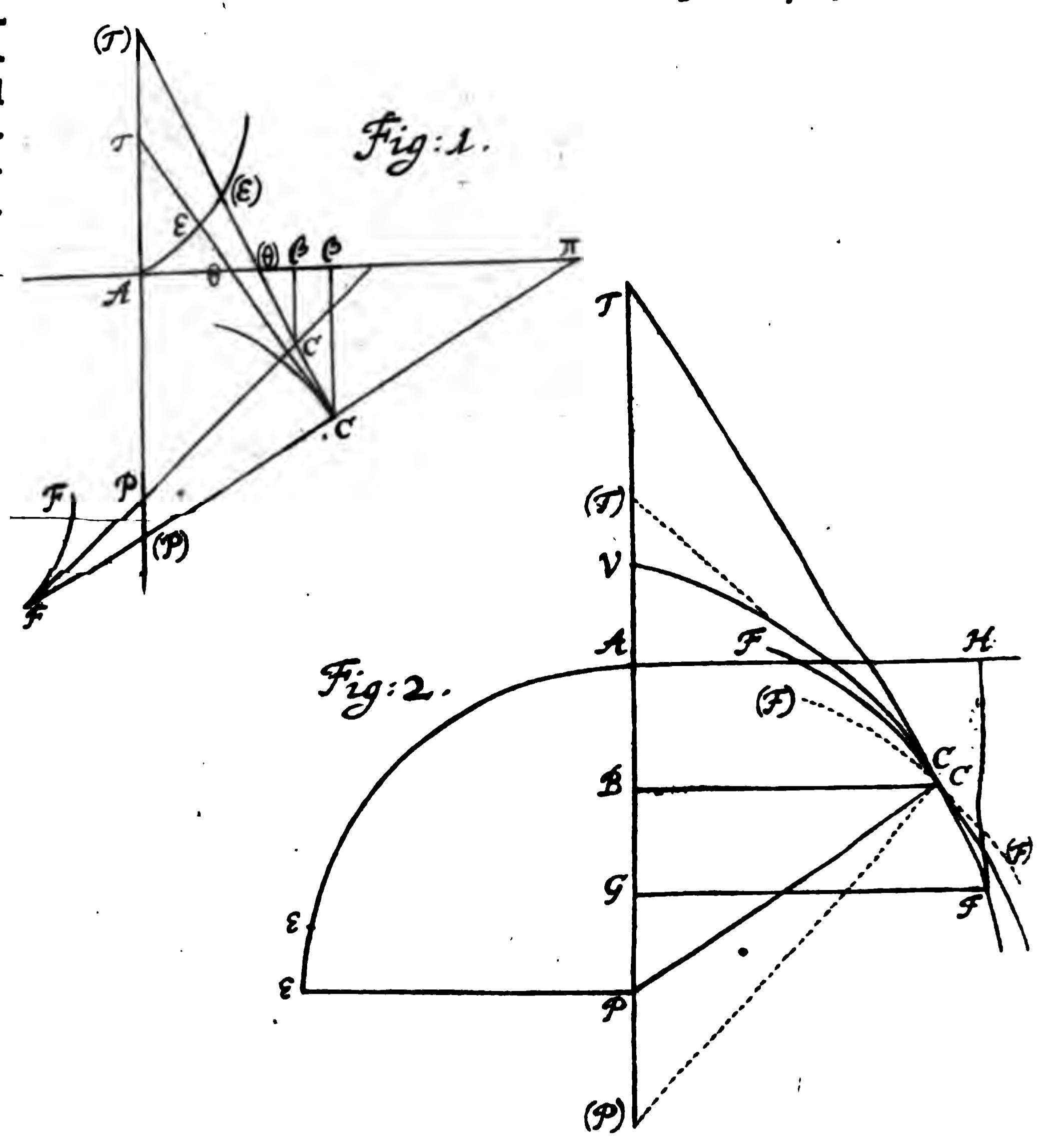
lus in casu unius differentiabilis in effectu coincidit cū methodo vetere de maximis & minimis a *Fermatio* proposita, ab *Huddenio* promota, sed quæ tantum est corollarium nostræ.) Jam ope æq. 4. tollendo residuam coefficienté variabilem b, ex æq. 3 fiet $ax + aa : 4 \overset{(5)}{=} yy$, quæ est æquatio ad curvam cc quæsitam. Idque indicio est eam esse parabolam, ipsi datæ AE congruentem, sed paulo tantum aliter sitam; continuata enim CC vertice suo V incidet in axem AP, sed supra datæ AE verticem A, ita ut distantia verticum AV sit communis lateris recti pars quarta. Si alteram calculandi rationem malis, per plures differentiales; resumtæ æquationes 1 & 2 differentientur, & ex 1 fiet $bdb \overset{(3)}{=} + db + cdc$, sed ex 2 fiet $adb \overset{(4)}{=} 2cdc$, quarum (3 & 4.) ope tollendo dc, evanescet simul & db, & fiet $b \overset{(5)}{=} x + a : 2$ ut paulo ante. Unde jam per 1, 2, 5 tollendo c & b coefficientes variabiles; prodibit $ax + aa : 4 \overset{(6)}{=} yy$ pro æquatione lineæ quæsitæ, ut ante.

Atque ita docuimus data relatione perpendicularis PC ad proprium ex axe resegmentum AP exhibere lineam CC, quia ordinatim dantur circuli lineam tangentes. Sed data relatione rectæ tangentis TC ad proprium ex axe resegmentum AT (seu circulis normalibus ad lineam ordinatim datis) invenire lineam, CC; alterius est methodi, & *constructione tractoria* talis linea haberi potest, a nobis in his *Actis Sept. anni superioris mense* explicata. Hujus autem præsentis methodi nostræ maximus præterea est usus ad complura alia problemata Geometriæ superioris, aut etiam ad mechanica vel physica applicatæ. Cum enim id agitur, ut figura formetur, in quovis puncto dato suæ lineæ terminantis præstans aliquid desideratum, persæpe consequimur quæsitam formando ipsam concursu linearum, quarum quævis in aliquo puncto satisfacit, ipsomet sc. puncto concursus. Hac ratione jam olim in schediasmate *de Lineis Opticis* inveni modum lineas exhibendi, quæ radios ordinatim positione datos, seu a datæ figuræ speculo venientes, reddant convergentes, aut divergentes, aut parallelos. Formatur enim talis linea ellipsium concursu, si radii debeant fieri convergentes; eademque methodus valet, si reddendæ sint parallelæ aut divergentes.

P. S.

Solutionem suam Problematis Bernoulliani mense nupero Majo una cum objectione Anonymi Actis Eruditorum insertam, Dn. Marchio Hospitalius Autor defendere non distulit, ostenditque, ut intellexi, Anonymum, si calculum suum ad finem perduxisset, ipsummet solutionis datæ successum fuisse deprehensurum. Cæterum Anonymus ille aliam solutionem non dedit, neque id secundum Analysin vulgarem facile præstari potest. Nostra autem nova, adeoque & Dn. Marchionis ac Dominorum Bernoulliorum Methodus, non hoc tantum, sed &, quemadmodum jam mense Julio p. 313. in Actis anni superioris est admonitum, innumera similia solvit, sive absolute pro re nata, sive per quadraturas. Et generale Problema sic concipi potest: *Data ratione inter duas Functiones invenire Lineam.* Data ratio intelligitur, quæ est inter duas datas, velut m & n. *Functionem* voco portionem rectæ, quæ ductis ope sola puncti fixi & puncti curvæ cum curvedine sua dati rectis, abscinditur. Tales sunt: Abscissa AB vel Aβ; ordinata BC vel βC; tangens CT vel Cθ; perpendicularis Cp vel Cπ; subtangentialis BT vel βθ; subperpendicularis BP vel βπ; per tangentem resecta AT vel Aθ; per perpendicularem resecta AP vel Aπ; corresecta PT vel πθ; radius osculi seu curvedinis CF; & aliæ innumeræ.

TAB. VII. ad A. 1694. pag. 311.



Leibniz’s 1694 paper on envelope rule:

- Rule thoroughly explained with formulas, figures and examples.

6. G. G. L. NOVA CALCULI DIFFERENTIALIS Applicatio & usus, ad multiplicem linearum constructionem, ex data tangentium conditione.

Memini jam a me insinuatum in his Actis, ut rectarum ordinatim sumptarum concursu hactenus noto, ita & concursu curvarum lineas formari. Sed placet rem non parvi ad Geometriam augendam momenti exponere distinctius; nam ne in rectis quidem concurrentibus tota ejus vis fuit perspecta. In genere igitur hoc problema ad communis Geometriæ leges revocare hic docebo: *Lineis (rectis vel curvis) propositam tangentibus, positione ordinatim datis, invenire propositam*, vel quod eodem redit: *invenire lineam, quæ infinitas lineas ordinatim positione datas tangit*. Cujus usus cum latissime pateat, calculum in eam rem peculiarem jamdudum excogitavi, vel potius huc peculiari ratione applicui nostrum Differentialem compendio non contemnendo. Scilicet quemadmodum *Cartesius* loca veterum calculo exprimens æquationes adhibuit, quæ cuivis curvæ puncto conveniunt, ita nos æquationes hic adhibemus infinities ampliores, quæ cuilibet puncto cujuslibet curvæ *in serie ordinatim sumtarum curvarum* comprehensæ, accommodantur. Itaque x & y abscissa quidem & ordinata, seu *coordinatæ* esse intelliguntur cujusvis ex dictis curvis, sed speciatim tamen accipiuntur de curva ex ipsarum concursu formata seu ipsas tangente; utili quodam *æquivocationis characteristicæ* genere. *Coefficientes* a, b, c, in æquatione cum ipsis x & y usurpatæ, significant quantitates in eadem curva *constantes*, alias quidem *infitas* (nempe *parametros*) alias vero *extraneas*, quæ situm curvæ (adeoque verticis axisque) definiunt. Sed comparando curvas seriei inter se, seu transitum de curva in curvam considerando, aliæ coefficientes sunt *constantissimæ* seu *permanentes*, (quæ manent non tantum in una, sed & in omnibus seriei curvis,) aliæ sunt *variabiles*. Et quidem ut *seriei curvarum lex* data sit, necesse est unicam tantum in coefficientibus superesse variabilitatem, adeoque si in *primaria* pro omnibus curvis *æquatione* naturam earum communem explicante plures extent variabiles, necesse est dari alias *æquationes accessorias*, coefficientium variabilium dependentiam inter se exprimentes, qua-

rum ope omnes variabiles ex æquatione primaria tolli possint præter unam. Cæterum pro concursu duarum linearum proximarum, sua intersectione punctum curvæ quæsitæ (quam & tangere intelliguntur,) determinantium, manifestum est, concurrentes quidem, adeoque lineam ex concursu *formatam* tangentes, esse geminas; intersectionis autem seu concursus punctum esse unicum, adeoque & ordinatam ei respondentem unicam esse; cum alioqui in investigatione solita linearum propositam tangentium, rectarum vel curvarum (velut circulorum, parabolarum &c.) ex datæ curvæ ordinatis quærendarum, ordinatæ *geminæ*, tangentes *unicæ* concipiantur. Itaque quoad præsentem calculum, quo ipsæ ex tangentibus rectis vel curvis positione datis investigantur ordinatæ (contra quam in communi) manent coordinatæ x & y in hoc transitu (a proximo ad proximum) invariatæ, adeoque sunt indifferentiabiles; at coefficientes, (quæ in communi calculo indifferentiabiles censentur, quia constantes,) quatenus hic variabiles sunt, differentiantur. Notabile est autem, *si omnes infitæ coefficientes sint permanentes*, curvæque adeo ordinatim concurrentes sint congruæ inter se; perinde fore, ac si intelligantur *esse vestigia ejusdem lineæ motæ*; curvaque earum concursu formata lineam motam perpetuo durante motu tanget. Unde in hoc casu oritur connexio quædam cum generatione *trochoëidum*; nam & basis, super qua volvitur generatrix trochoëidis, generatricem durante motu tangit.

Calculus autem ita instituetur: assumatur aliquis angulus rectus fixus, cujus crura utcunque producta constituere intelligantur duos axes relationis curvarum, seu axem cum axe conjugato; in quos demissæ normales ex puncto curvæ quocunque erunt ordinata, x, & ordinata conjugata seu abscissa, y; uno verbo: *coordinatæ*, x & y; quarum relationem ex datis quærendo habebitur *æquatio*, (1) quam paulo ante appellavimus *primariam*, cum sit cuilibet cujuslibet curvarum ordinatim sumtarum puncto communis. Quod si æquationi 1, infunt plures coefficientes variabiles, ut b, c, dabitur earum dependentia per *secundariam* æquationem, (2) unam vel plures; atque ita ex æq. 1 tollendo coefficientes variabiles, præter unam b prodibit æq. (3) Hanc æquationem differentiando, ut prodeat æq. (4) cum in ea sola affutura sit differentialis ipsius b, evanescet differentialitas,

adeoque habemus æq. 4 ordinariam, cujus ope ex æq. 3 tollendo variabilem residuam b, habebitur æquatio, (5) in qua præter x & y tantum supererunt coefficientes invariabiles (ut a) quæ erit *æquatio* ad curvam quæsitam concursu seriei linearum formatam, adeoque *ad seriei linearum tangentem communem*. Sed & aliter institui potest calculus, prout facilitas invitabit, non tollendo statim variabiles, sed servando. Nempe datis æq. (1) primaria, & æq. (2) secundaria (una vel pluribus pro explicanda dependentia coefficientium variabilium inservituris) differentientur æq. 1, ut prodeat (3), & æq. 2, ut prodeat (4) (una vel plures, si pro æq. 2 affuerint plures.) Ita habebimus plures quantitates differentiales, sed tamen habebimus & æquationes sufficientes ad eas tollendas; & quidem modo tolli possint differentiales quantitates usque ad unam; etiam residua ista evanescet per se, & sic prodibit æq. (5) ordinaria, seu carens quantitate differentiali; quam conjungendo cum æq. 1 & 2 tolli poterunt variabiles omnes, & prodibit æq. (6) naturam exprimens curvæ quæsitæ, linearum concursu formatæ, quæ erit eadem cum æq. 5 calculi prioris.

Hac jam methodo solvi possunt innumera problemata sublimioris Geometriæ, hactenus non habita in potestate, pertinentiaque ad tangentium conversam; ex quibus nonnulla in specimen indicabo, magnæ utique generalitatis. Veluti: data relatione inter AT & AΘ resegmenta axium per curvæ tangentem CT facta, invenire curvam CC; nam rectæ curvam tangentes ordinatim positione dantur, adeoque & curva quæsita, quippe quæ earum concursu formatur. Vel si dato puncto axis T, detur lineæ datæ EE punctum E, sic ut juncta TE, si opus producta, quæsitam curvam CC tangat, patet ex dictis curvam CC præscripta hic methodo haberi. Similiter data relatione inter AP & AΠ, resegmenta axium facta per curvæ perpendicularem PC, licet invenire curvam CC: nam ob rectas PΠ ordinatim positione datas, etiam datur linea FF formata per earum concursum; hujus vero evolutione describetur curva CC quæsita. Unde hic quidem infinitæ curvæ satisfacientes dari possunt, omnes scilicet *parallelæ* inter se, quæ ejusdem lineæ evolutione *condescribuntur*. & data relatione inter AP & AΠ dari potest *curva quæsita* non tantum satisfaciens, sed & *transiens per punctum datum*. Interim hoc casu curva CC non semper est ordinaria, quoniam scilicet non ipsamet,

sed ipsius per evolutionem generatrix rectarum positione datarum concursu formatur. Certe cum ipsa curva formatur concursu, habetur *determinata*, nec in arbitrio est punctum, per quod transeat; quæ distinctio utilis est in hac doctrina.

Sed exemplum calculi dabimus in problemate itidem generali, ad aliquam tamen specialem lineam applicato: *data relatione perpendicularis* PC *ad proprium ab axe resegmentum* AP, *invenire lineam* CC. Patet enim datis positione punctis C, nempe centris circulorum, & radiis PC datis magnitudine (ob datam relationem ad AP) *dari ordinatim circulos lineam* CC *tangentes*, adeoque lineam ipsam circulorum concursu formatam haberi posse, id quod jam verbulo indicaveramus olim in *Actis* 1686 *mense Junio* p. 300, sub schediasmatis finem. Itaq; centro P, radio PC magnitudine dato, describatur circulus CF. Ut ergo methodum paulo ante positam huc applicemus: ex puncto circuli quocunque F agantur normales ad crura anguli recti PAH, seu coordinatæ FG, y, & FH, x (quæ in casu concursus duorum circulorum incidunt in CB, CL) fit AP, b, & PC, c;

fiet ex natura circuli, $xx + yy + bb \overset{(1)}{=} 2bx + cc$ æquatio primaria omnibus nostris circulis & cuique cujusque puncto communis. Quoniam autem datur relatio inter AP & PC, dabitur curva EE, cujus ordinata PE æquetur ipsi PC; hæc curva ponatur (exempli gratia) esse parabola, cujus parameter, a, & fiat $ab \overset{(2)}{=} cc$, quæ æquatio secundaria exhibet relationem seu dependentiam inter c & b. Hujus ope tollendo c, ex æq. 1, fiet $xx + yy + bb \overset{(3)}{=} 2bx + ab$; patet autem in æq. 1. præter coordinatas x & y, adesse coefficientes c, b, a; ex quibus c & b sunt in uno circulo constantes, & c quidem est circulo infita, cum ejus radium designet; b est extranea, quippe situm centri designans; ambæ variatis circulis sunt variabiles, sed a est constantissima sive permanens, cum non unius tantum circuli omnibus punctis, sed & pro omnibus circulis nostris in æquatione maneat eadem. Reducta jam æquatio 3 ad unam coefficientem variabilem b, differentietur, secundum b (solam in ea differentiabilem) & fiet $2bdb = 2xdb + adb$, seu (evanescente db) fit $b \overset{(4)}{=} x + a : 2$ (qui calcu-

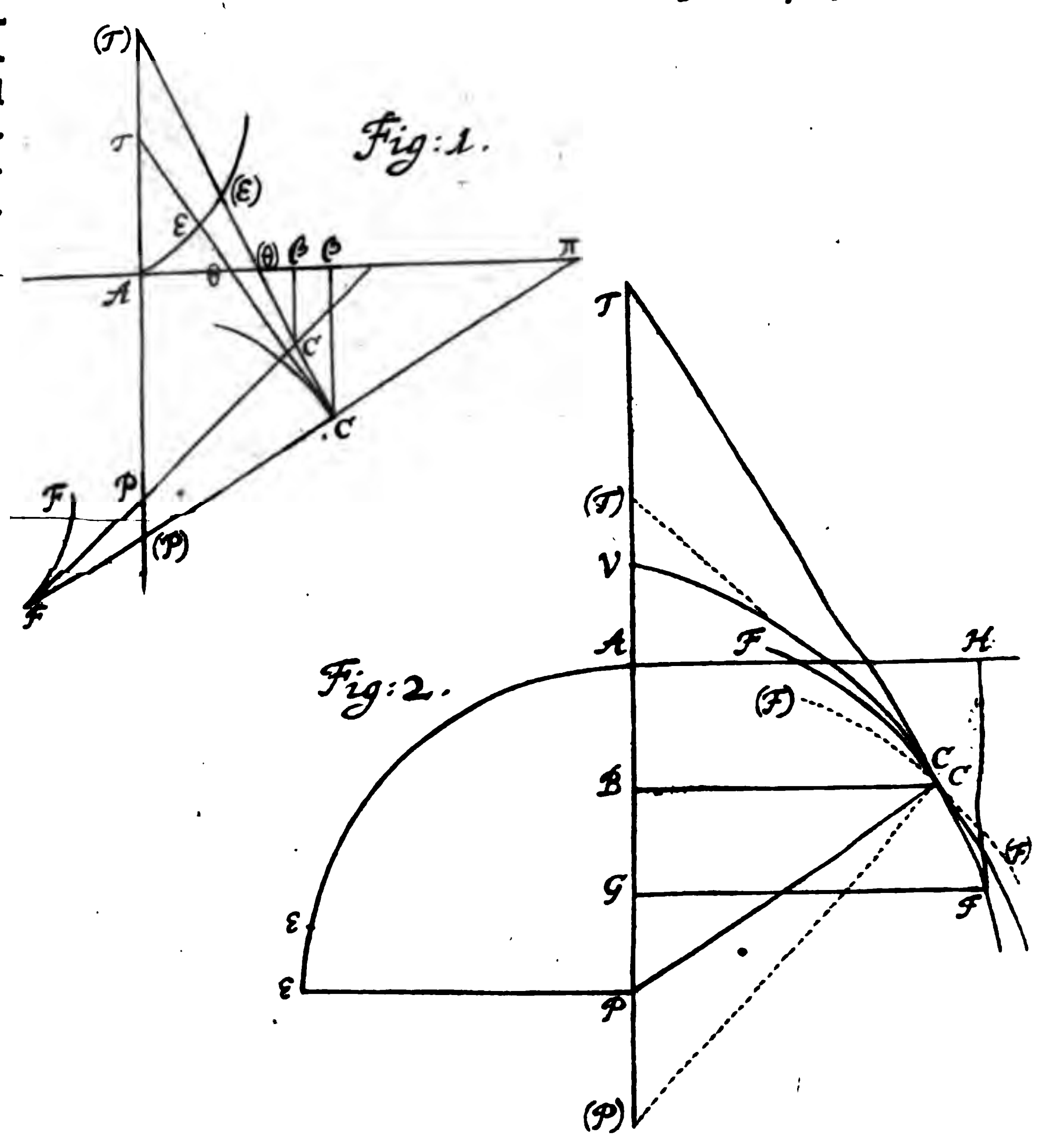
lus in casu unius differentiabilis in effectu coincidit cū methodo vetere de maximis & minimis a *Fermatio* propofita, ab *Huddenio* promota, sed quæ tantum est corollarium nostræ.) Jam ope æq. 4. tollendo residuam coefficientē variabilem b, ex æq. 3 fiet $ax + aa : 4 \overset{(5)}{=} yy$, quæ est æquatio ad curvam cc quæsitam. Idque indicio est eam esse parabolam, ipsi datæ AE congruentem, sed paulo tantum aliter sitam; continuata enim CC vertice suo V incidet in axem AP, sed supra datæ AE verticem A, ita ut distantia verticum AV sit communis lateris recti pars quarta. Si alteram calculandi rationem malis, per plures differentiales; resumtæ æquationes 1 & 2 differentientur, & ex 1 fiet $bdb \overset{(3)}{=} + db + cdc$, sed ex 2 fiet $adb \overset{(4)}{=} 2cdc$, quarum (3 & 4.) ope tollendo dc, evanescet simul & db, & fiet $b \overset{(5)}{=} x + a :$ 2 ut paulo ante. Unde jam per 1, 2, 5 tollendo c & b coefficientes variabiles; prodibit $ax + aa : 4 \overset{(6)}{=} yy$ pro æquatione lineæ quæsitæ, ut ante.

Atque ita docuimus data relatione perpendicularis PC ad proprium ex axe resegmentum AP exhibere lineam CC, quia ordinatim dantur circuli lineam tangentes. Sed data relatione rectæ tangentis TC ad proprium ex axe resegmentum AT (seu circulis normalibus ad lineam ordinatim datis) invenire lineam, CC; alterius est methodi, & *constructione tractoria* talis linea haberi potest, a nobis in his *Actis Sept. anni superioris mense* explicata. Hujus autem præsentis methodi nostræ maximus præterea est usus ad complura alia problemata Geometriæ superioris, aut etiam ad mechanica vel physica applicata. Cum enim id agitur, ut figura formetur, in quovis puncto dato suæ lineæ terminantis præstans aliquid desideratum, persæpe consequimur quæsitam formando ipsam concursu linearum, quarum quævis in aliquo puncto satisfacit, ipsomet sc. puncto concursus. Hac ratione jam olim in schediasmate *de Lineis Opticis* inveni modum lineas exhibendi, quæ radios ordinatim positione datos, seu a datæ figuræ speculo venientes, reddant convergentes, aut divergentes, aut parallelos. Formatur enim talis linea ellipsium concursu, si radii debeant fieri convergentes; eademque methodus valet, si reddendæ sint parallelæ aut divergentes.

P. S.

Solutionem suam Problematis Bernoulliani mense nupero Majo una cum objectione Anonymi Actis Eruditorum insertam, Dn. Marchio Hospitalius Autor defendere non distulit, ostenditque, ut intellexi, Anonymum, si calculum suum ad finem perduxisset, ipsummet solutionis datæ successum fuisse deprehensurum. Cæterum Anonymus ille aliam solutionem non dedit, neque id secundum Analysin vulgarem facile præstari potest. Nostra autem nova, adeoque & Dn. Marchionis ac Dominorum Bernoulliorum Methodus, non hoc tantum, sed &, quemadmodum jam mense Julio p. 313. in Actis anni superioris est admonitum, innumera similia solvit, sive absolute pro re nata, sive per quadraturas. Et generale Problema sic concipi potest: *Data ratione inter duas Functiones invenire Lineam.* Data ratio intelligitur, quæ est inter duas datas, velut m & n. *Functionem* voco portionem rectæ, quæ ductis ope sola puncti fixi & puncti curvæ cum curvedine sua dati rectis, abscinditur. Tales sunt: Abscissa AB vel Aβ; ordinata BC vel βC; tangens CT vel Cθ; perpendicularis CP vel Cπ; subtangentialis BT vel βθ; subperpendicularis BP vel βπ; per tangentem resecta AT vel Aθ; per perpendicularem resecta AP vel Aπ; corresecta PT vel πθ; radius osculi seu curvedinis CF; & aliæ innumeræ.

TAB. VII. ad A. 1694. pag. 311.



Leibniz's 1694 paper on envelope rule:

- Rule thoroughly explained with formulas, figures and examples.
- "New", "of no little importance":

NOVA Applicatio

non parvi ad Geometriam augendam momenti

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sed ipsius per evolutionem generatrix rectarum positione datarum concursu formatur. Certe cum ipsa curva formatur concursu, habetur *determinata*, nec in arbitrio est punctum, per quod transeat; quæ distinctio utilis est in hac doctrina.

Sed exemplum calculi dabimus in problemate itidem generali, ad aliquam tamen specialem lineam applicato: *data relatione perpendicularis* PC *ad proprium ab axe resegmentum* AP, *invenire lineam* CC. Patet enim datis positione punctis C, nempe centris circulorum, & radiis PC datis magnitudine (ob datam relationem ad AP) *dari ordinatim circulos lineam* CC *tangentes*, adeoque lineam ipsam circulorum concursu formatam haberi posse, id quod jam verbulo indicaveramus olim in *Actis* 1686 *mense Junio* p. 300, sub schediasmatis finem. Itaq; centro P, radio PC magnitudine dato, describatur circulus CF. Ut ergo methodum paulo ante positam huc applicemus: ex puncto circuli quocunque F agantur normales ad crura anguli recti PAH, seu coordinatæ FG, *y*, & FH, *x* (quæ in casu concursus duorum circulorum incidunt in CB, CL) sit AP, *b*, & PC, *c*;

fiet ex natura circuli, $xx + yy + bb \overset{(1)}{=} 2bx + cc$ æquatio primaria omnibus nostris circulis & cuique cujusque puncto communis. Quoniam autem datur relatio inter AP & PC, dabitur curva EE, cujus ordinata PE æquetur ipsi PC; hæc curva ponatur (exempli gratia) esse parabola, cujus parameter, *a*, & fiat $ab \overset{(2)}{=} cc$, quæ æquatio secundaria exhibet relationem seu dependentiam inter *c* & *b*. Hujus ope tollendo *c*, ex æq. 1, fiet $xx + yy + bb \overset{(3)}{=} 2bx + ab$; patet autem in æq. 1. præter coordinatas *x* & *y*, adesse coefficientes *c, b, a*; ex quibus *c* & *b* sunt in uno circulo constantes, & *c* quidem est circulo insita, cum ejus radium designet; *b* est extranea, quippe situm centri designans; ambæ variatis circulis sunt variabiles, sed *a* est constantissima sive permanens, cum non unius tantum circuli omnibus punctis, sed & pro omnibus circulis nostris in æquatione maneat eadem. Reducta jam æquatio 3 ad unam coefficientem variabilem *b*, differentietur, secundum *b* (solam in ea differentiabilem) & fiet $2bdb \overset{(4)}{=} 2xdb + adb$, seu (evanescente *db*) fit $b \overset{(4)}{=} x + a : 2$ (qui calcu-

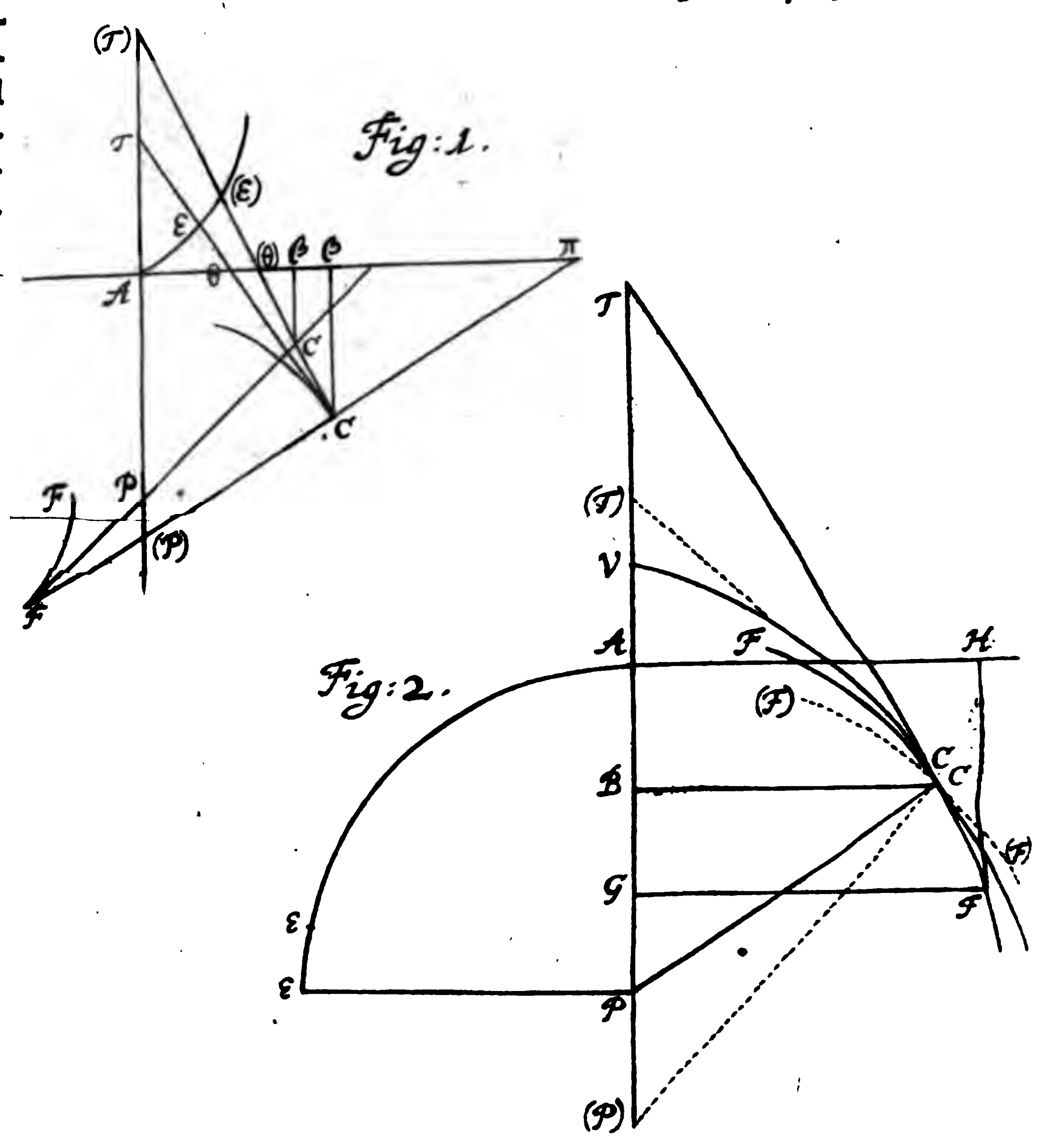
lus in casu unius differentiabilis in effectu coincidit cũ methodo veterum de maximis & minimis a *Fermatio* proposita, ab *Huddenio* promota, sed quæ tantum est corollarium nostræ.) Jam ope æq. 4. tollendo residuam coefficientẽ variabilem *b*, ex æq. 3 fiet $ax + aa : 4 \overset{(5)}{=} yy$, quæ est æquatio ad curvam *cc* quæsitam. Idque indicio est eam esse parabolam, ipsi datæ AE congruentem, sed paulo tantum aliter sitam; continuata enim CC vertice suo V incidet in axem AP, sed supra datæ AE verticem A, ita ut distantia verticum AV sit communis lateris recti pars quarta. Si alteram calculandi rationem malis, per plures differentiales; resumtæ æquationes 1 & 2 differentientur, & ex 1 fiet $bdb \overset{(3)}{=} xdb + cdc$, sed ex 2 fiet $adb \overset{(4)}{=} 2cdc$, quarum (3 & 4) ope tollendo *dc*, evanescet simul & *db*, & fiet $b \overset{(5)}{=} x + a : 2$ ut paulo ante. Unde jam per 1, 2, 5 tollendo *c* & *b* coefficientes variabiles; prodibit $ax + aa : 4 \overset{(6)}{=} yy$ pro æquatione lineæ quæsitæ, ut ante.

Atque ita docuimus data relatione perpendicularis PC ad proprium ex axe resegmentum AP exhibere lineam CC, quia ordinatim dantur circuli lineam tangentes. Sed data relatione rectæ tangentis TC ad proprium ex axe resegmentum AT (seu circulis normalibus ad lineam ordinatim datis) invenire lineam, CC; alterius est methodi, & *constructione tractoria* talis linea haberi potest, a nobis in his *Actis Sept. anni superioris mense* explicata. Hujus autem præsentis methodi nostræ maximus præterea est usus ad complura alia problemata Geometriæ superioris, aut etiam ad mechanica vel physica applicatæ. Cum enim id agitur, ut figura formetur, in quovis puncto dato suæ lineæ terminantis præstans aliquid desideratum, persæpe consequimur quæsitam formando ipsam concursu linearum, quarum quævis in aliquo puncto satisfacit, ipsomet sc. puncto concursus. Hac ratione jam olim in schediasmate *de Lineis Opticis* inveni modum lineas exhibendi, quæ radios ordinatim positione datos, seu a datæ figuræ speculo venientes, reddant convergentes, aut divergentes, aut parallelos. Formatur enim talis linea ellipsium concursu, si radii debeant fieri convergentes; eademque methodus valet, si reddendæ sint parallelæ aut divergentes.

P. S.

Solutionem suam Problematis Bernoulliani mense nupero Majo una cum objectione Anonymi Actis Eruditorum insertam, Dn. Marchio Hospitalius Autor defendere non distulit, ostenditque, ut intellexi, Anonymum, si calculum suum ad finem perduxisset, ipsummet solutionis datæ successum fuisse deprehensurum. Cæterum Anonymus ille aliam solutionem non dedit, neque id secundum Analysin vulgarem facile præstari potest. Nostra autem nova, adeoque & Dn. Marchionis ac Dominorum Bernoulliorum Methodus, non hoc tantum, sed &, quemadmodum jam mense Julio p. 313. in Actis anni superioris est admonitum, innumera similia solvit, sive absolute pro re nata, sive per quadraturas. Et generale Problema sic concipi potest: *Data ratione inter duas Functiones invenire Lineam.* Data ratio intelligitur, quæ est inter duas datas, velut m & n. *Functionem* voco portionem rectæ, quæ ductis ope sola puncti fixi & puncti curvæ cum curvedine sua dati rectis, abscinditur. Tales sunt: Abscissa AB vel Aβ; ordinata BC vel βC; tangens CT vel Cθ; perpendicularis Cp vel Cπ; subtangentialis BT vel βθ; subperpendicularis BP vel βπ; per tangentem resecta AT vel Aθ; per perpendicularem resecta AP vel Aπ; corresecta PT vel πθ; radius osculi seu curvedinis CF; & aliæ innumeræ.

TAB. VII. ad A. 1694. pag. 311.



Leibniz's 1694 paper on envelope rule:

- Rule thoroughly explained with formulas, figures and examples.
- "New", "of no little importance":

NOVA
Applicatio

non parvi àd Geometriam augendam momenti

- No pseudonym:

G. G. L.

Leibniz’s two envelope papers

1692

MENSIS APRILIS A. M DC XCII.

DE LINEA EX LINEIS NVMERO INFINITIS ordinatim ductis inter ſe concurrentibus formata, easq; omnes tangente, ac de novo in ea re Analyſis infinitorum uſu.

Autore O. V. E.

Some marginal matter in optics treated disparagingly.

1694

MENSIS JULII A. MDCXCIV.

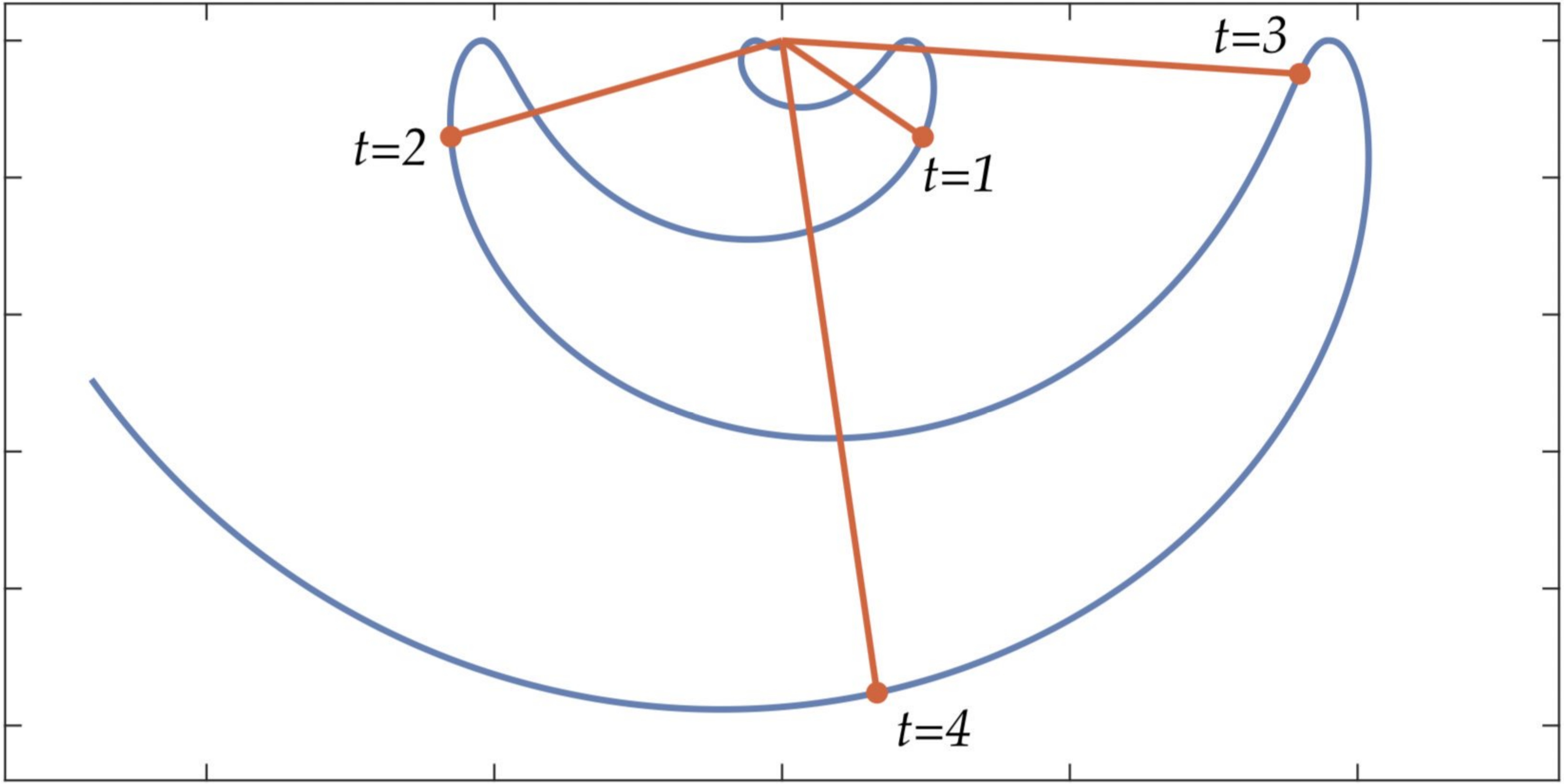
G. G. L. NOVA CALCULI DIFFERENTIALIS Applicatio & uſus, ad multiplicem linearum conſtructionem, ex data tangentium conditione.

“New application” of “no little importance for geometry” praised and trumpeted.

1693:
Realisation that envelopes relevant for the representation of transcendental curves.

The Paracentric Isochrone

$dr/dt = constant$

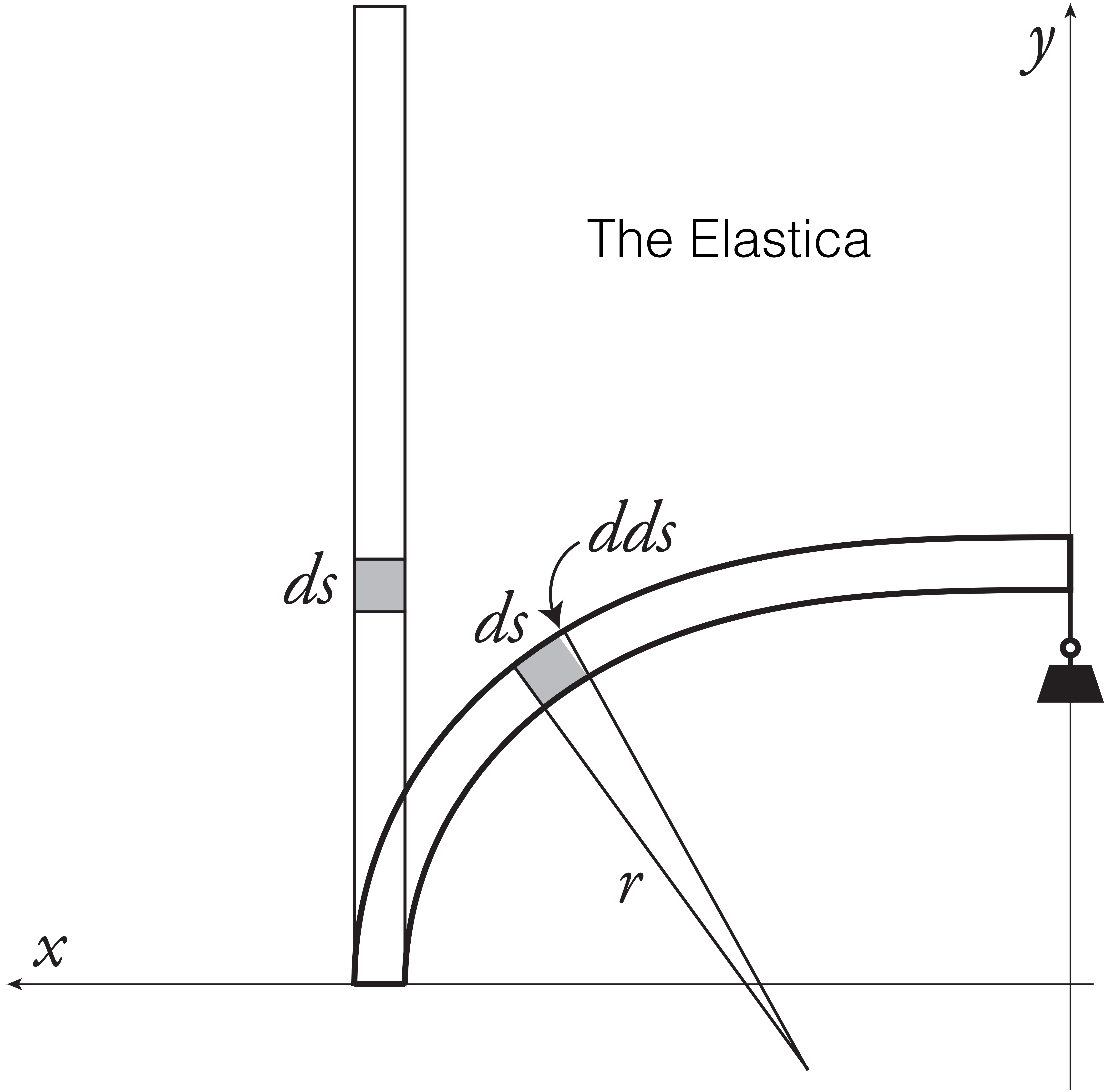


reduces to integrating $1/\sqrt{1-x^4}$

$$u(x_1) = \int_0^{x_1} \frac{dx}{\sqrt{1-x^4}}$$

Wanted:
Simple geometrical characterisation of *u*.

rectification of the elastica reduces to integrating $1/\sqrt{1-x^4}$



The extension *dds*

is proportional to the force (by Hooke's Law)

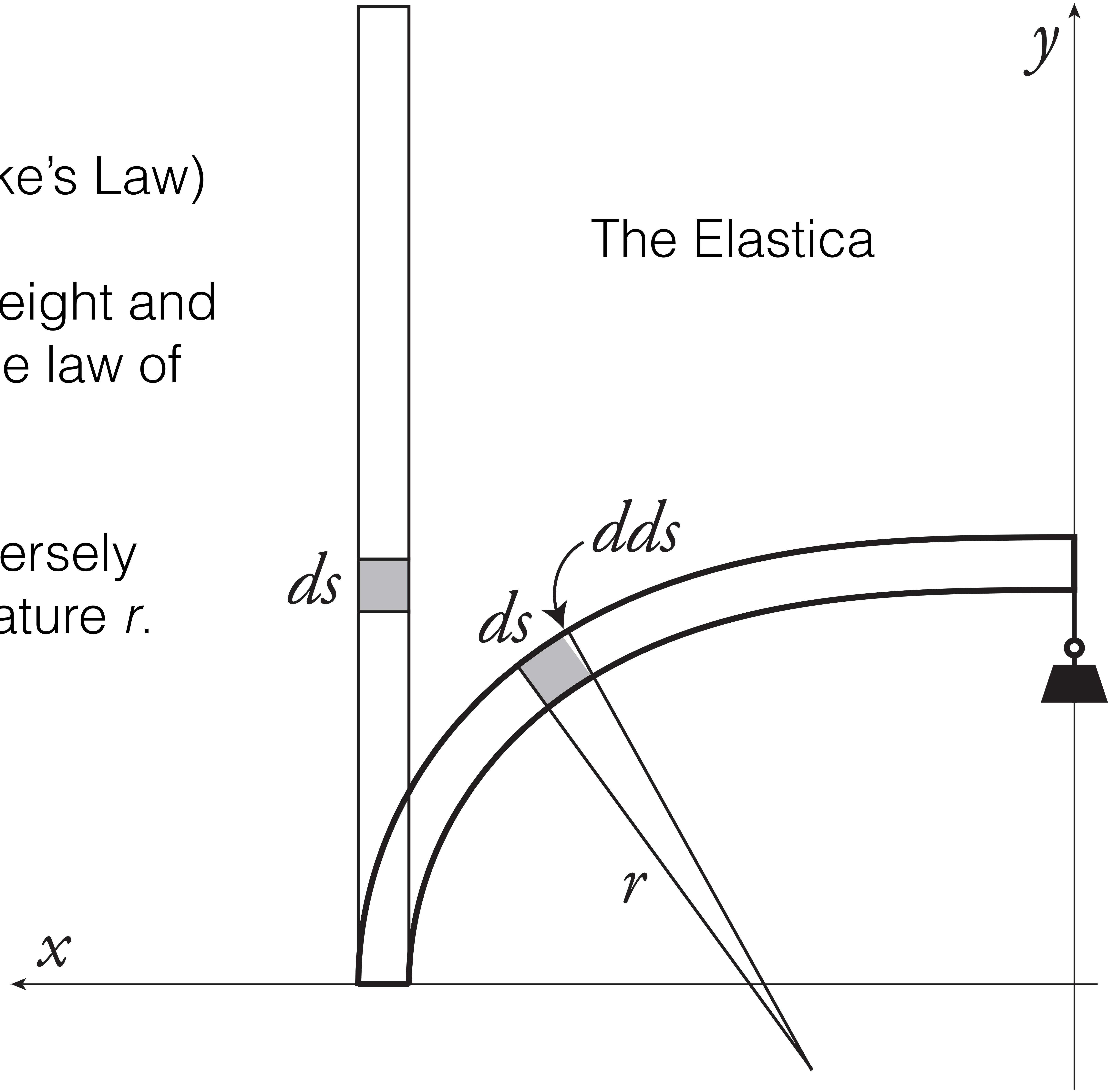
which is the product of the (fixed) weight and the length of the lever arm *x* (by the law of the lever).

But the extension *dds* is also inversely proportional to the radius of curvature *r*.

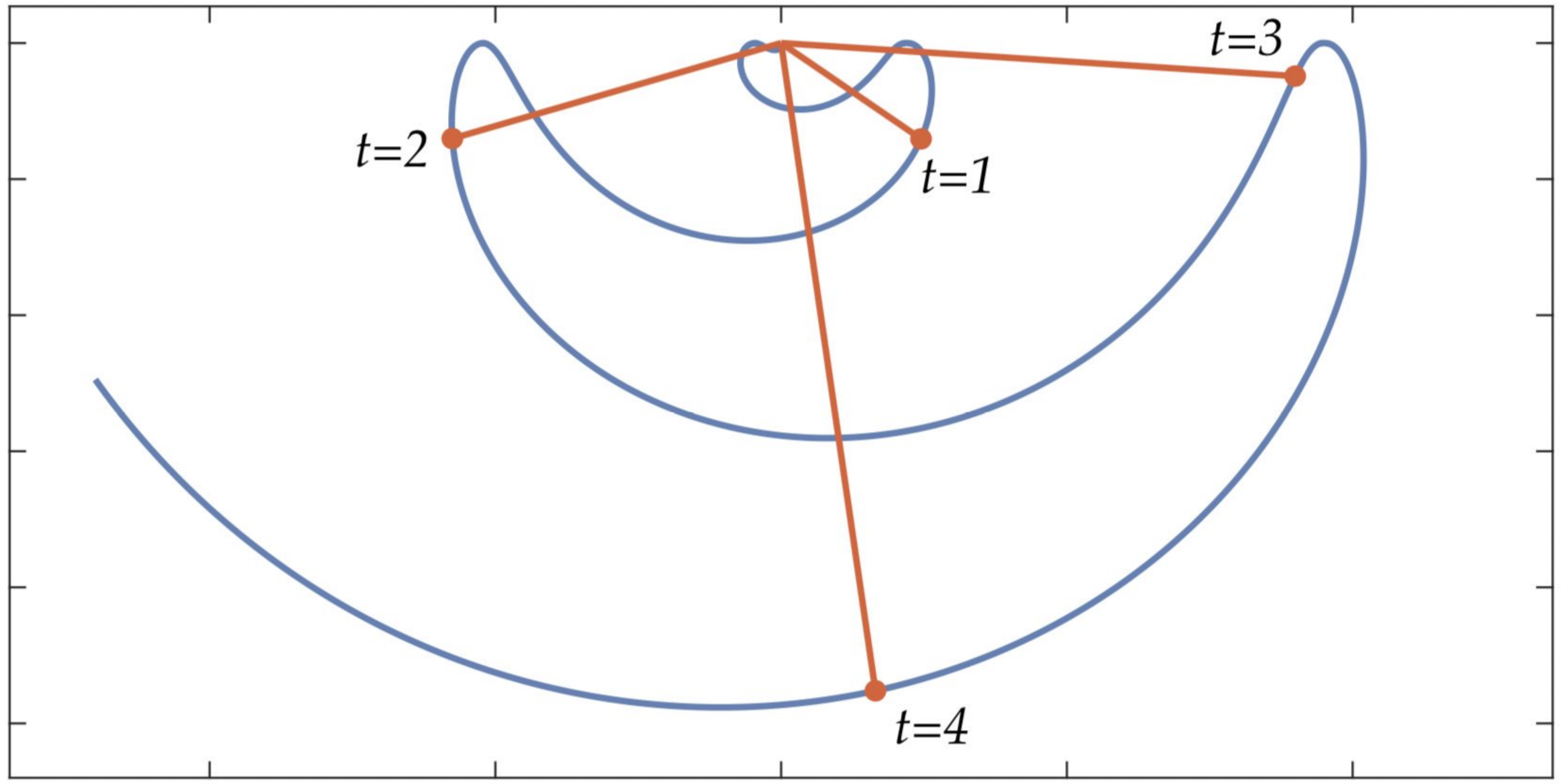
So *x* is proportional to *r*,

$$ax = -\frac{d^2y}{dx\,ds}$$

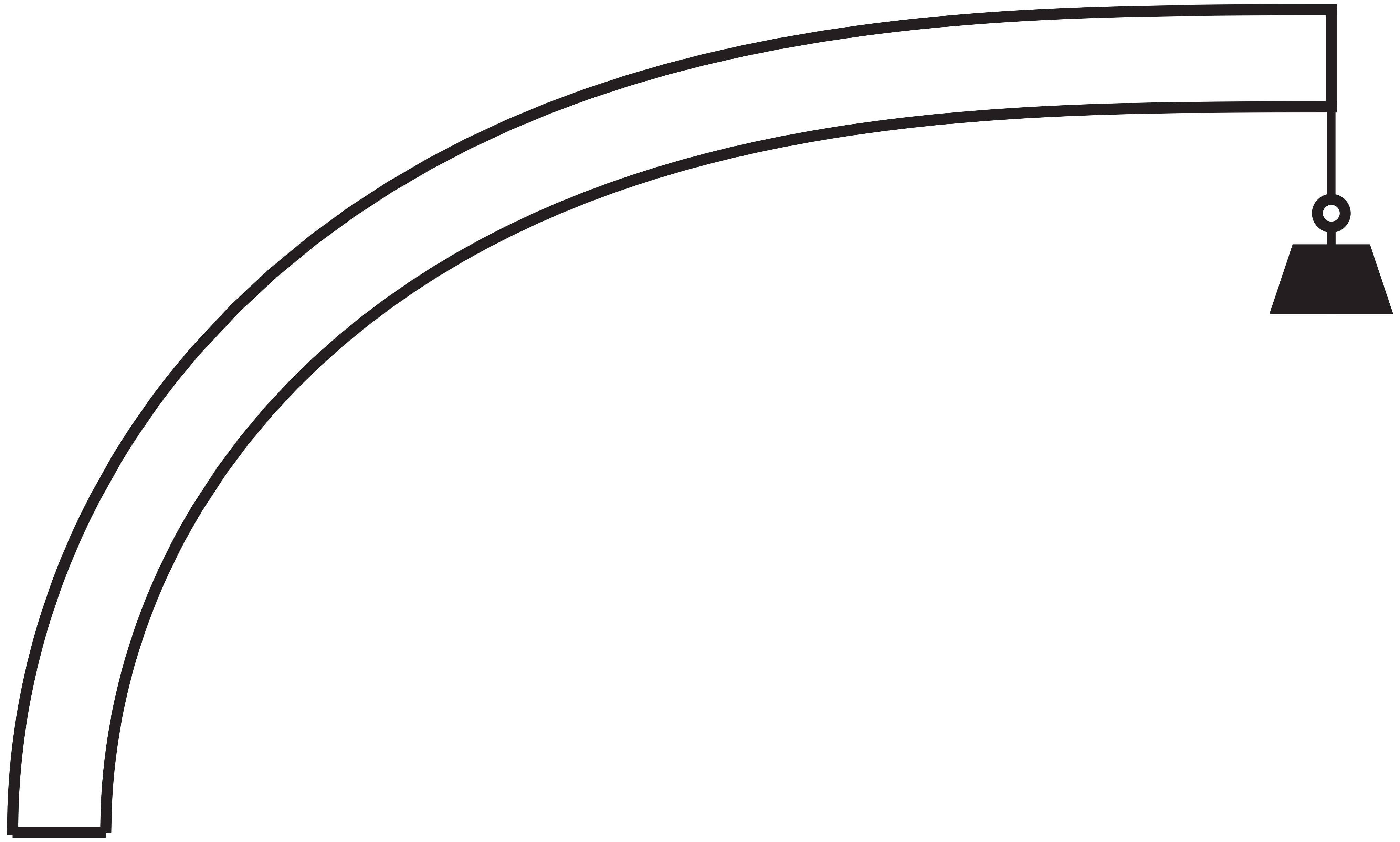
The Elastica



To construct the Paracentric Isochrone,



rectify the Elastica:



Jacob Bernoulli:

1691:
Announces that he has solution to elastica.

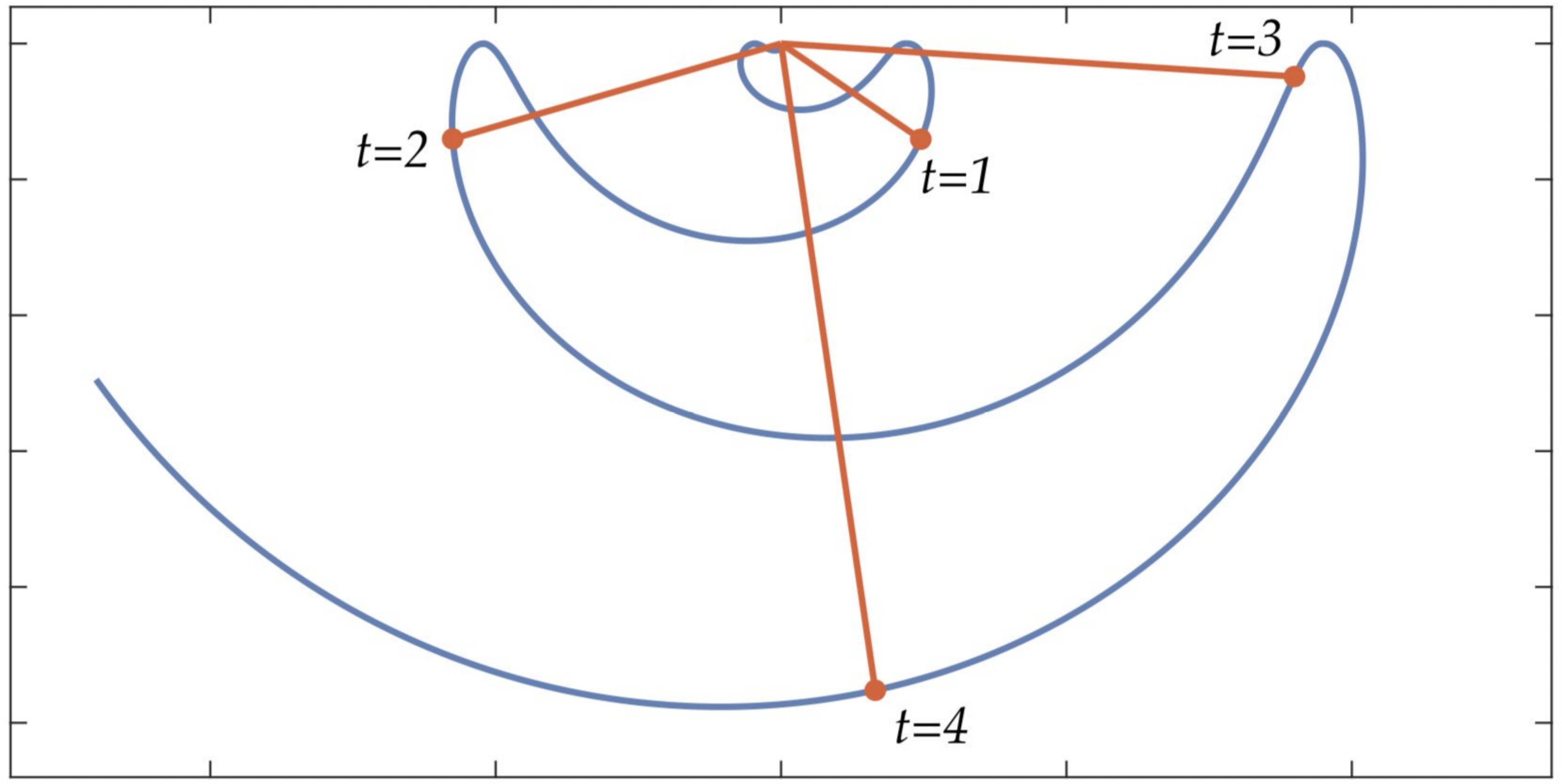
SPECIMEN ALTERUM CALCULI DIFferentialis in dimetienda Spirali Logarithmica, Loxodromiis Nautarum, & Areis Triangulorum Sphaericorum: una cum Additamento quodam ad Problema Funicularium, aliisque, per I. B.

1694:
Gives solution, followed immediately by application to paracentric isochrone.

JAC. B. CURVATURA LAMINÆ ELAsticæ. Ejus Identitas cum Curvatura Lintei a pondere inclusi fluidi expansi. Radii Circulorum Osculantium in terminis simplicissimis exhibiti, una cum novis quibusdam Theorematis huc pertinentibus, &c.

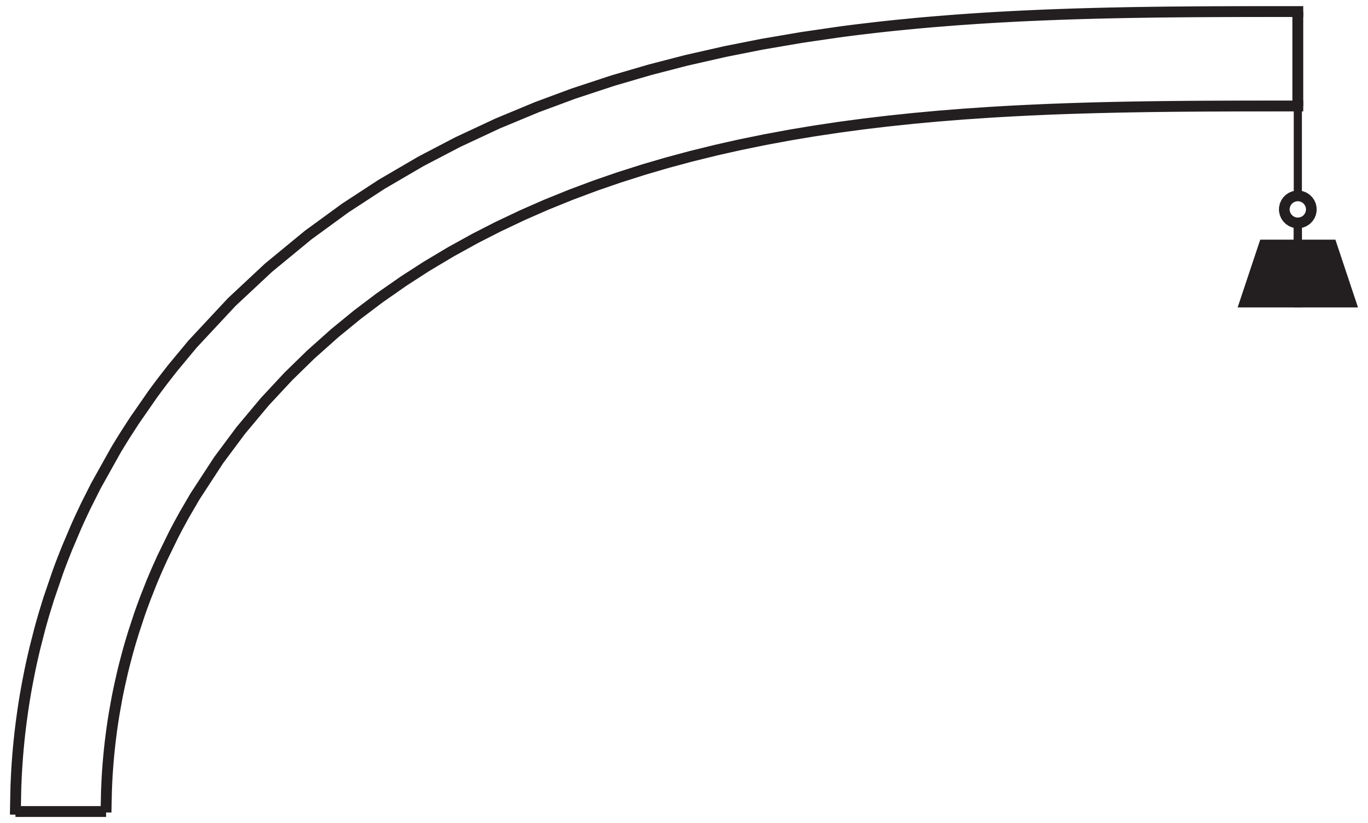
JAC. B. SOLUTIO PROBLEMATIS LEIBnitiani de Curva Accessus & Recessus æquabilis a puncto dato, mediante rectificatione Curvæ Elasticæ.

To construct



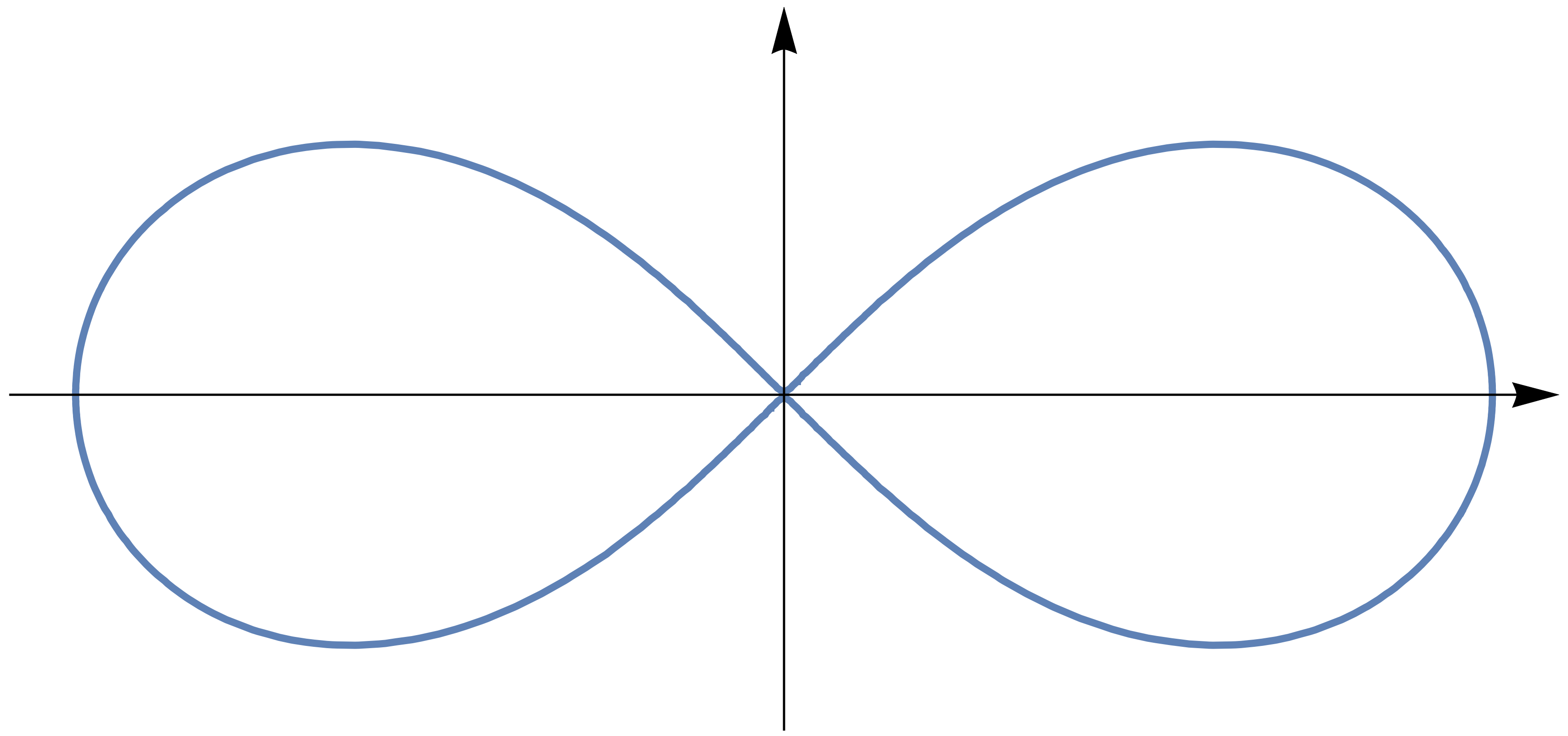
paracentric isoschrone
dr/dt = constant

rectify

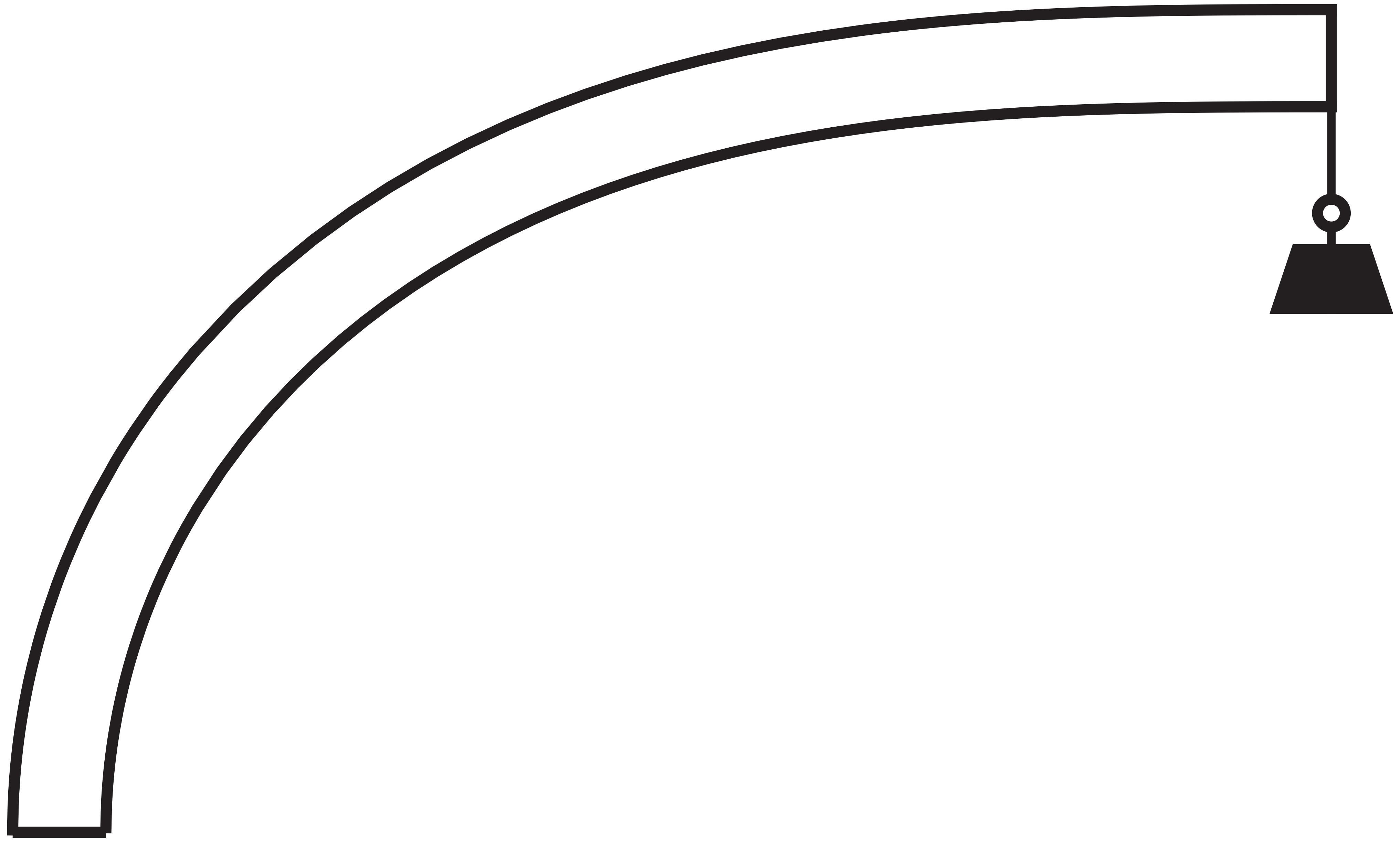


shape of elastic beam

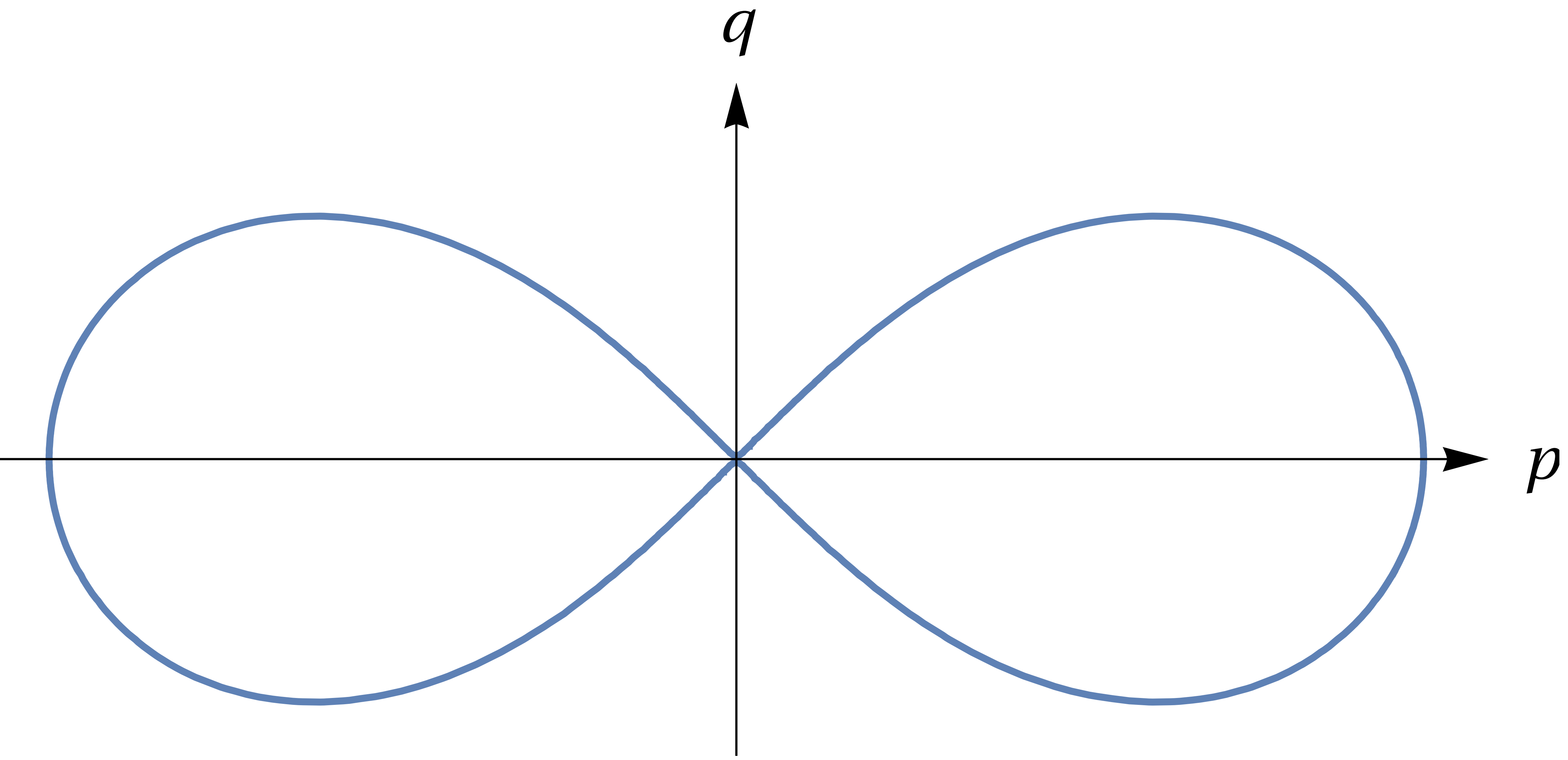
or rectify



lemniscate



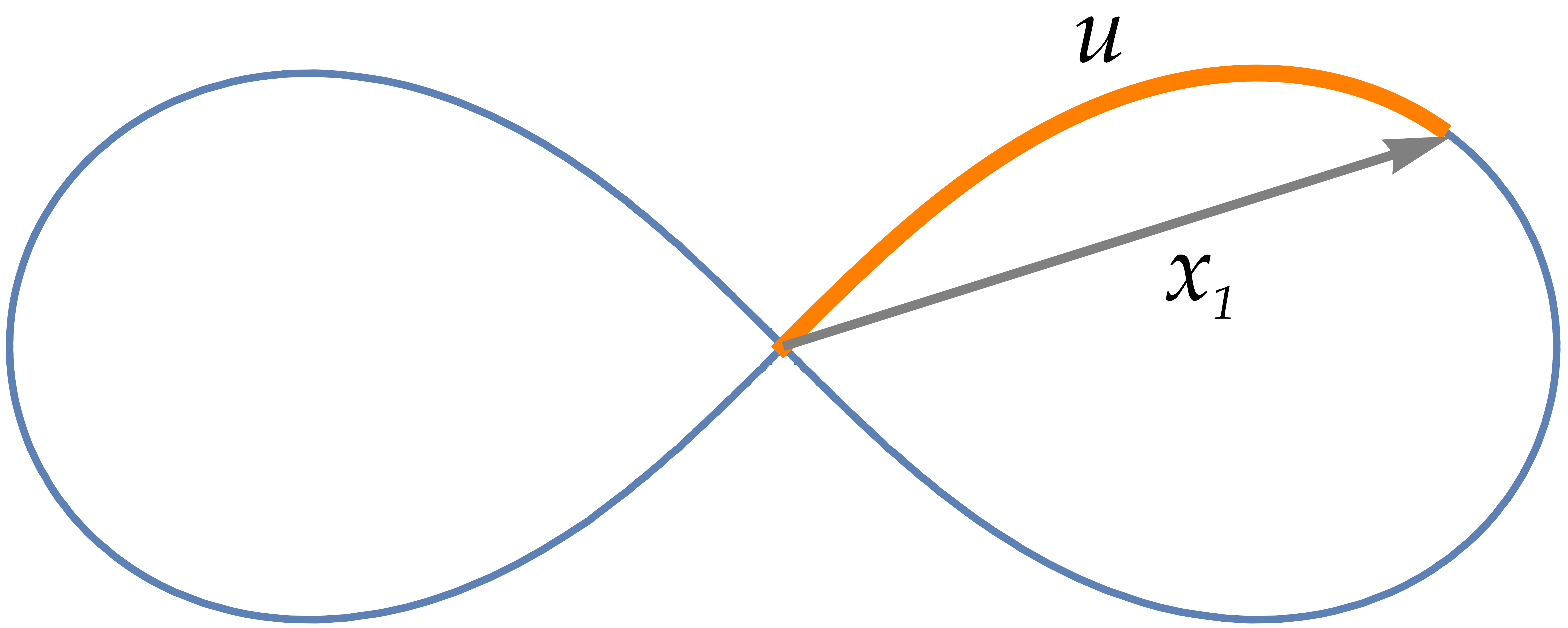
Jacob Bernoulli:
"The best method is that which uses a curve that Nature herself produces."



$$4(p^2 - q^2) = (p^2 + q^2)^2$$

Johann Bernoulli: "If the curve can be algebraic, he sins against the laws of geometry who has recourse to a mechanical one."

$$u(x_1) = \int_0^{x_1} \frac{dx}{\sqrt{1 - x^4}}$$



Lemniscate finds *u*
in simple terms:

u can also be expressed
in terms of arc length of ellipse, but messier.

$$\int_0^{x_1} \sqrt{\frac{a^4 + (b^2 - a^2)x^2}{a^4 - a^2 x^2}}\, dx \qquad x^2/a^2 + y^2/b^2 = 1$$

RECTIFICATION OF QUADRATURES

Better than leaving unknown quadratures such as

$$\int \sqrt{a^4 + x^4}\, dx$$

is to find an α such that

$$\int y\, dx = \int \sqrt{1 + (\alpha')^2}\, dx$$

$$\int \frac{1}{1+x^2}\, dx = \arctan x$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x$$