Elliptic functions and elliptic curves

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Recap last week

elliptic function

- is inverse of $u = \int^x \frac{dt}{\sqrt{p(t)}}$ (where p of degree 3,4)
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Example:

$$u = \int^x \frac{dt}{\sqrt{1-t^4}} = \int^x \frac{dt}{\sqrt{(1-t^2)(1+t^2)}}$$

periods ω, $i\omega$ ($\omega =$ length of half a lemniscate)
i.e., $x(u) = x(u + n\omega + mi\omega)$ for any $n, m \in \mathbb{Z}$.

Issues

to be discussed in next slides

1. $\sqrt{p(t)}$ is many-valued in $\mathbb{C}$
2. integration in $\mathbb{C}$ was hardly developed at all

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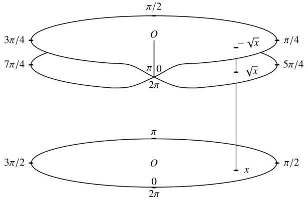

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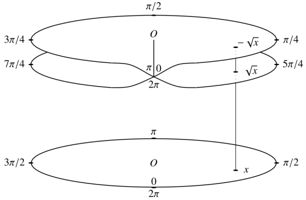
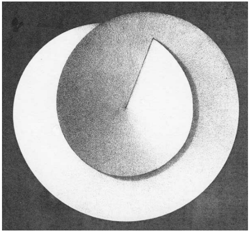
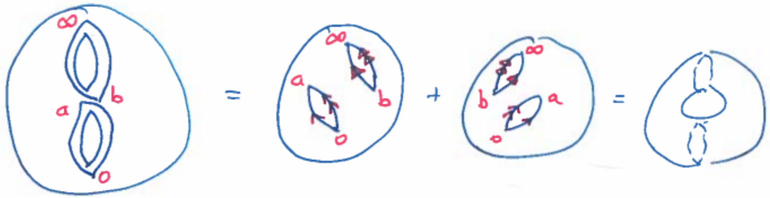




On a Riemann surface, many-valued functions are turned into single-valued ones. best considered on $\mathbb{C} \cup \infty$
Major innovation due to elliptic functions

Riemann surface for $\sqrt{z(z-a)(z-b)}$

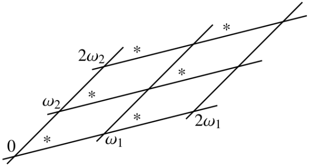
Branch points at $z = 0$, $z = a$, $z = b$, $z = infty$.
Two sheets because of $\sqrt{}$-operation:



i.e., $u = \int^x \frac{dz}{\sqrt{z(z-a)(z-b)}}$ naturally lives on a torus.

double periodicity = torus

Consider the period lattice $n\omega_1 + m\omega_2$, for $n, m \in \mathbb{Z}$:



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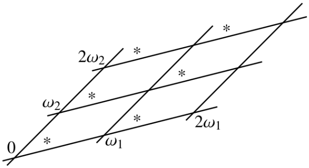

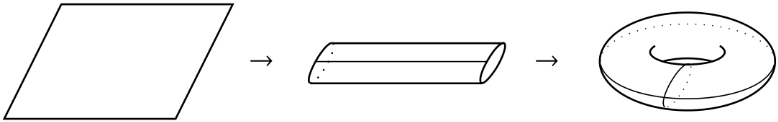

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- Weierstrass:
  $$\wp(z) = \frac{1}{z^2} + \sum_{nm\neq 0} \left(\frac{1}{(z+n\omega_1+m\omega_2)^2} - \frac{1}{(n\omega_1+m\omega_2)^2} \right)$$

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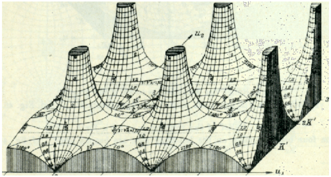
- again: major innovation due to elliptic functions!

Weierstrass $\wp$-function



- now most common and easily understood elliptic function
- satisfies differential equation:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

  for specific constants g_2 and g_3

- for diehards: $g_2 = 60\sum \frac{1}{(n\omega_1+m\omega_2)^4}$, $g_3 = 140\sum \frac{1}{(n\omega_1+m\omega_2)^6}$

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- Morale: circle can be parametrised by circle functions

$$(x, y) = (u^{-1}, (u^{-1})')$$

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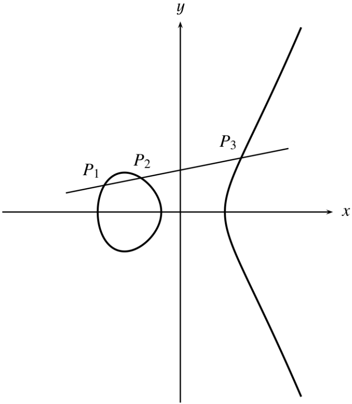
- Hence the modern definition: an elliptic curve is any curve parametrised by elliptic functions.
- Note that Weierstrass function $\wp$ satisfies

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

and therefore it parametrises the elliptic curve $y^2 = 4x^3 - g_2x - g_3$.

Clebsch and group structure



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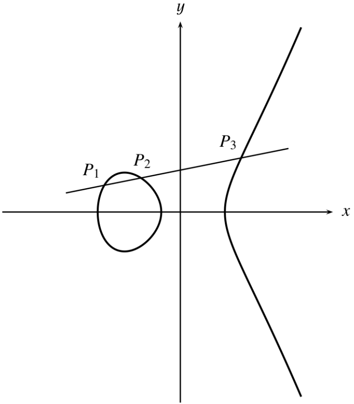

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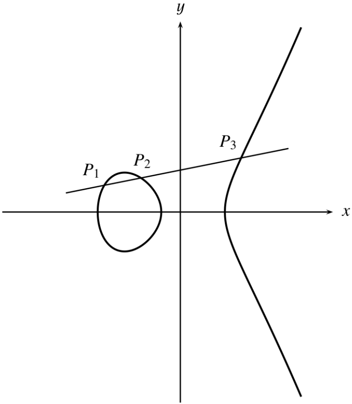

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end of 18th Century: "The mine of mathematics is exhausted"

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Niels Henrik Abel 1802-1829

- poor personal circumstances
- had read everything available to him on math
- received a hint to turn to elliptic integrals
- warmly received by Crelle in Berlin
- poorly received in Paris
- mostly jobless and dies early (tbc)

Carl Gustav Jacob Jacobi 1804–1851

- happier circumstances, though career still insecure
- inspired by Euler to work on elliptic functions
- sees applications of elliptic functions to number theory; but he also works in math physics, determinants, history of math
- novelty: research seminars
- dies of smallpox

Legendre to Jacobi

"It gives me a great satisfaction to see two young mathematicians such as you and [Abel] cultivate with success a branch of analysis which for so long a time has been the object of my favourite studies and which has not been received in my own country as well as it would deserve. By these works you place yourselves in the ranks of the best asnalysts of our era."

Abel prize

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The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2019 to
Karen Keskulla Uhlenbeck
University of Texas at Austin, USA
“for her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and for the fundamental impact of her work on analysis, geometry and mathematical physics.”
The Abel Prize Laureate 2019
The Abel Prize Laureate 2019 International Page
Welcome to the Abel Prize
The Abel Prize was established on 1 January 2002. The purpose is to award the Abel Prize for outstanding scientific work in the field of mathematics. The prize
The Abel Prize 2019 Events Registration
The registration for the Abel Prize 2019 events is now open.

Assignment 5

Read (part of) any one of these publications:

- Abel; Recherches sur les fonctions elliptiques (English available) – introduction has a programmatic general view
- Riemann: Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse (English or German) including his introduction of Riemann surfaces
- Legendre, Traité des fonctions elliptiques (French)
- Gauss, Tagebuch (Latin with German commentary)
- anything from Euler (various languages and translations)
- any other mathematical research published between 1800 and 1900 to which you feel attracted (explain why)

After reading, write your own personalised review. E.g., describe how the content of this historical material relates to your own advances in mathematics; consider the style and type of mathematics; express your wonder, amazement or disdain, etc. Your review should contain both non-trivial (not necessarily hard) mathematics and personal reflection.

Assignment 6

Another topic deserving some attention is to look at the community of mathematicians and how they cooperate (or not). The 19th Century sees the founding of a number of important journals that still exist, like Annales de M. (Gergonne), Comtes Rendus, Crelle, Liouville, Acta Mathematica (Mittag-Leffler), and Mathematische Annalen (Clebsch, Neumann). A well-known example of non-cooperation is described in Van Dalen. See if you can find other examples (pre 1900) of extreme (non)cooperation.