The Greeks often thought of the factors of a number as its "parts." Thus for example the number 4 represented justice since it is the smallest number made up of two equals, $4 = 2 \times 2$. The number 7 is special from this point of view also, as Aristotle explains (fragment 203):

> Since the number 7 neither generates [in the sense of multiplication] nor is generated by any of the numbers in the decad [i.e., the first ten numbers], they identified it with Athene. For the number 2 generates 4, 3 generates 9, and 6, 4 generates 8, and 5 generates 10, and 4, 6, 8, 9, and 10 are also themselves generated, but 7 neither generates any number nor is generated from any; and so too Athene was motherless and ever-virgin.

When the factors of a number are considered its parts it becomes natural to ask whether all numbers are the sum of its parts. In fact this is not so; very few numbers are "perfect" enough to have this pleasant property, as Nicomachus (c. 100) explains:

> When a number, comparing with itself the sum and combination of all the factors whose presence it will admit, it neither exceeds them in multitude nor is exceeded by them, then such a number is properly said to be perfect, as one which is equal to its own parts. Such numbers are 6 and 28; for 6 has the factors 3, 2, and 1, and these added together make 6 and are equal to the original number, and neither more nor less. 28 has the factors 14, 7, 4, 2, and 1; these added together make 28, and so neither are the parts greater than the whole nor the whole greater than the parts, but their comparison is in equality, which is the peculiar quality of the perfect number.
>
> It comes about that even as fair and excellent things are few and easily enumerated, while ugly and evil ones are widespread, so also are the superabundant and deficient numbers found in great multitude and irregularly placed, but the perfect numbers are easily enumerated and arranged with suitable order; for only one is found among the units, 6, only one among the tens, 28, and a third in the ranks of the hundreds, ____, and a fourth within the limits of the thousands, 8128.

Euclid proved that if p is a prime and $2^p - 1$ is also prime then $2^{p-1}(2^p - 1)$ is perfect. This is the grand finale of Euclid's number theory (*Elements* IX.36). The theorem amounts to a recipe for finding perfect numbers: in a column list the prime numbers; in a second column the values $2^p - 1$; cross out all rows in which the second column is not a prime number; for the remaining rows, place $2^{p-1}(2^p - 1)$ in the third column. Then the numbers in the third column are perfect numbers.

2.2. Find the perfect number omitted in the Nicomachus quote above using Euclid's recipe. What prime p did you need to use?

The following is essentially Euclid's proof of the theorem. If $2^p - 1$ is prime, it is clear that the proper divisors of $2^{p-1}(2^p - 1)$ are $1, 2, 2^2, \dots, 2^{p-1}$ and $(2^p - 1), 2(2^p - 1), 2^2(2^p - 1), \dots, 2^{p-2}(2^p - 1)$. So these are the numbers we need to add up to see if their sum equals the number itself.

2.3. (a) Show that $1 + 2 + 2^2 + \dots + 2^{p-1} = 2^p - 1$ by adding 1 at the very left and gradually simplify the series from that end.

(b) Use a similar trick for the remaining sum, and thus conclude the proof.

§ 3. Origins of geometry

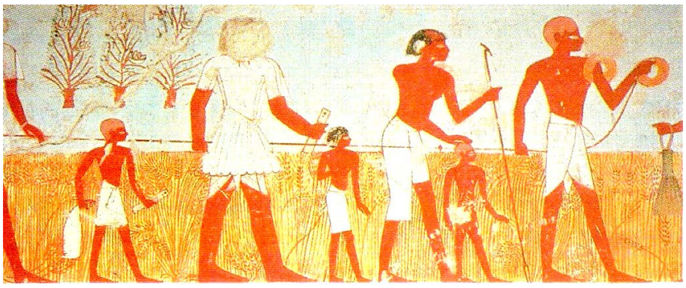


Figure 5: Egyptian geometers, or "rope-stretchers" as they were called, delineating a field by means of a stretched rope.

"Geometry" means "earth-measurement," and indeed the subject began as such, according to ancient sources such as Proclus and Herodotus, as we see in the readings. This was necessitated by the yearly overflowing of the Nile in Egypt: the flooding made the banks of the river fertile in an otherwise desert land, but it also wiped away boundaries between plots, so a geometer, or "earth-measurer," had to be called upon to redraw a fair division of the precious farmable land. In fact the division was perhaps not always so fair, as Proclus also suggests, for one can fool those not knowledgable in mathematics into accepting a smaller plot by letting them believe that the value of a plot is determined by the number of paces around it.

3.1. Prove that a square has greater area than any rectangle of the same perimeter.

3.2. Discuss what general point about history we can learn from the following paraphrase of Proclus's remark in Heath's *History of Greek Mathematics* (1921): "[Proclus] mentions also certain members of communistic societies in his own time who cheated their fellow-members by giving them land of greater perimeter but less area than the plots which they took themselves, so that, while they got a reputation for greater honesty, they in fact took more than their share of the produce." (206–207)

Among the first things one would discover in such a practical context would be how to draw straight lines and circles. In fact you need nothing but a piece of string to do this.

3.3. Explain how.

3.4. Problem 3.1 shows that it is important to be able to construct squares. How would do this with your piece of string?

3.5. In the Rhind Papyrys (c. −1650) the area of a circular field is calculated as follows: "Example of a round field of a diameter 9 khet. What is its area? Take away $\frac{1}{9}$ of the diameter, namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. Therefore it contains 64 setat of land." What is the value of π according to the Rhind Papyrys?

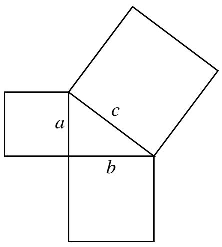
People soon recognised the austere beauty of geometrical constructions and began using it for decorative and especially religious purposes. Indeed, Egyptian temples are very geometrical in their design; the famous pyramids are but the most notable cases. One of the first decorative shapes one discovers how to draw when playing around with a piece of string is the regular hexagon.

3.6. Show how this is done.

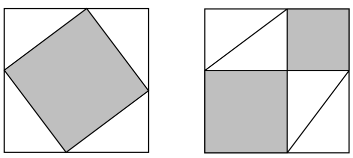
The hexagon has great decorative potential since it can be used to tile the plane. Hexagonal tiling patterns occur in Mesopotamian mosaics from as early as about -700.

3.7. Show that the hexagon contains even more area than a square of the same perimeter. As Pappus explains in the readings, bees seem to know this.

The step from this kind of decorative and ritualistic pattern-making to deductive geometry need not be very great. In fact, two of the most ancient theorems of geometry could quite plausibly have been discovered is such a context. Take for instance the Pythagorean Theorem. Its algebraic form "$a^2 + b^2 = c^2$" seems to be the only thing some people remember from school mathematics, but classically speaking the theorem is not about some letters in a formula but actual squares:

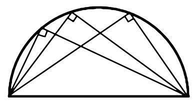


The simplest case of the theorem, when the two legs are equal ($a = b$), is very easy to see when looking at a tiled floor, as we see in the reading from Plato's *Meno*. Inspired by this striking result, ancient man might have gone on to consider the case of a slanted square, and then discovered that with some easy puzzling the theorem is easily generalised to this case as well:



3.8. Explain how this proves the theorem.

The Greek tradition has it that Thales (c. −600) was the first to introduce deductive reasoning in geometry. One of the theorems he supposedly dealt with was "Thales' Theorem" that the triangles raised on the diameter of a circle all have a right angle:



3.9. Explain how Thales' Theorem can very easily be discovered when playing around with making rectangles and circles. Hint: Construct a rectangle; draw its diagonals; draw the circumscribed circle.

Thus we see a fairly plausible train of thought leading from the birth of geometry in practical necessity, to an appreciation for its artistic potential, to the discovery of the notions of theorem and proof.

Another indication of the use of constructions is the engineering problem of digging a tunnel through a mountain. Digging through a mountain with manual labour is of course very time-consuming. It is therefore desirable to dig from both ends simultaneously. But how can we make sure that the diggers starting at either end meet in the middle instead of digging past each other and making two tunnels?

3.10. Solve this problem using a rope. (You may assume that the land is flat except for the mountain.)

Such methods were used in ancient times. On the Greek island of Samos, for example, a tunnel over one kilometer in length was dug around year −530, for the purpose of transporting fresh water to the capital. It was indeed dug from both ends.

§ 4. Babylonia