L'Europe mathématique

Mathematical Europe

sous la direction de
Catherine Goldstein, Jeremy Gray
et Jim Ritter

[redacted]

[redacted]ons de la Maison des sciences de l'homme, Paris

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Comment apprécier les techniques indiennes pour dériver des formules mathématiques ?

Ce chapitre utilise les mathématiques indiennes pour montrer les erreurs qui résultent d'une vision anachronique et européocentrique de la nature des mathématiques. Un examen de la jyotiḥśāstra, *la branche des sciences védiques qui concerne la connaissance astronomique, révèle que les différences entre mathématiques indiennes et européennes se situent au moins à deux niveaux : d'une part, une différence évidente dans la présentation des résultats mathématiques (par exemple, la forme poétique adoptée), d'autre part une différence encore plus déterminante dans la méthodologie. Si les arguments déductifs étaient certainement présents dans de nombreux raisonnements mathématiques indiens, ils n'étaient pourtant pas nécessaires. Beaucoup de résultats furent trouvés et acceptés sur la base d'une technique* ad hoc.

Trois exemples de formules d'approximation incluant les fonctions trigonométriques développées par les mathématiciens indiens à partir des cordes grecques (deux réussies et une défectueuse) sont utilisés ici pour montrer la manière dont les résultats étaient trouvés, mettant en évidence les forces et les faiblesses propres à cette approche.

Considérer les mathématiciens indiens comme des Grecs incompétents, ainsi que le fit l'érudit musulman du XI^{e} *siècle al-Bīrunī, n'aide pas à comprendre les processus réellement en jeu. Il en va de même de la réhabilitation des mathématiciens indiens par des défenseurs cherchant avant tout à leur attribuer les mêmes connaissances déductives que celles connues en Occident. Ces évaluations, pour différentes qu'elles paraissent, négligent l'épistémologie propre à la mise en œuvre des textes indiens.*

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THE TASK OF SEPARATING THE MYTHS OF EUROPEAN MATHEMATICS from its history is made more difficult by the fact that some features of European mathematics, inherited from the Greek tradition and now universal in mathematical research, are often considered essential to the nature of mathematics itself. 'Mathematical thought' comes to mean the techniques that mathematicians now use, and the history of mathematics becomes the process of searching out the development of those techniques in the past. As a result, a mathematical tradition not sharing these features is likely to be perceived as mathematics queerly distorted, incompetently or improperly structured; or else as mathematics in disguise, with 'normal' mathematical processes concealed by a superficial difference in style. An example of the possible misunderstandings arising from such assumptions is furnished by some problems from Indian mathematics.

Indian mathematics and historians

The traditional Indian approach to the mathematics associated with *jyotiḥśāstra* (the branch of the Vedic sciences concerned with astronomical knowledge), which was prevalent among Hindu astronomers from earliest times through the nineteenth century of our era, is apt to appear peculiar and unreliable to the modern Western observer. Sometimes this tradition is criticized as inferior in quality to the Greek and Islamic works that shaped much of European mathematics; sometimes it is defended as essentially equivalent, or in some cases superior, to its Western counterpart, although dissimilar in some points of style. It is suggested here that in fact the difference between Indian and Western mathematics is twofold: in the first place, there is an obvious difference in the presentation of mathematical material, which serves to conceal an even more significant difference in methodology.

The most apparent feature of the difference in presentation is the fact that mathematical *jyotiḥśāstra* is preserved in the form of poetry. This corpus of

The research for this paper was carried out with the support of a Brown University Proctorship, and with the invaluable guidance and advice of Professor David Pingree of Brown University.

mathematical knowledge consists primarily of verse collections of briefly stated, and sometimes ambiguous, rules and formulas related to astronomical or astrological calculation. Since these computational rules are supposed to be memorized, the aim is to provide an aide-mémoire rather than an expository treatise. There are drawbacks inherent in this compressed format: owing to the demands of Sanskrit prosody, in these texts vagueness of terminology is a virtue, and precision a luxury. The form of a mathematical statement has to be flexible enough to fit the metre. In addition, there is no room provided in the written works for proof, so the statements in the verses are not backed up by a structure of deductive reasoning. The familiar hierarchy of the Euclidean tradition — definitions, axioms, and theorems progressing in complexity — is entirely absent from the treatises, and rare in supplementary works. Commentaries in prose and verse do sometimes derive or demonstrate the results they discuss, but their function is more usually limited to paraphrasing the text and glossing technical terms.

Thus a crucial task in the study of Indian mathematics is the problem of reconstructing the processes of mathematical discovery. How were these concise rules derived, or at least, why were they believed? What evidence was required to convince a mathematician of the truth of some result? The silence of the texts on this subject makes many reconstructions possible. But the prevailing tendency of the historian is to assume the existence of a demonstration, whether geometrical or algebraic, like those familiar in the Western tradition: that is, one in which any mathematical statement is logically related to some other statement known to be true. It is presumed that these derivations must have seemed as necessary to an Indian mathematician as they would to a Greek one. One historian claims that the presence of proofs in some commentaries "shows Indian mathematicians too were not satisfied unless they could prove the results they used". Where demonstrations are not given, the conclusion is that "the explanation and the rationale were left to oral instruction" (Sarasvati Amma 1979: 3). But it is hard to see how a principally oral tradition could have been depended upon to preserve proofs satisfactorily. Complicated demonstrations would almost certainly have had a short lifespan in such an environment, leaving the verse formulas without any justification of their validity, and hence useless to mathematicians who demanded such justification.

A somewhat different approach to considering the character of the Indian methodology is illustrated in this paper. It assumes that while deductive arguments were almost certainly present in much of Indian mathematical reasoning, they were by no means required; and many quite sound results were achieved and accepted on the basis of a more ad hoc, intuitive technique. Some arguments in favor of this approach can be drawn from the following examples of approximation formulas (which provided useful computational shortcuts in complicated calculations) involving the trigonometric functions developed by Indian mathematicians from the Greek chords.

Approximation formulas

Brahmagupta (b. 598) in his *Brāhmasphuṭasiddhānta* 14, 23–24 gives the following formula, of which a somewhat different version is found in his contemporary Bhāskara I's *Mahābhāskarīya*:

> *bhujakoṭyaṃśonaguṇā bhārddhāṃśās taccaturthabhāgonaiḥ* |
> *pañcadvīndukhacandrair vibhājitā vyāsadalaguṇitāḥ* ||23||
> *tajjye paramaphalajyāsaṅguṇitā tatphale vinā jyābhiḥ* |
>
> The degrees of half the circle diminished and multiplied by the degrees of *bhuja* or *koṭi* [half-chords], divided by 10125 diminished by a fourth part of that, multiplied by the radius, are the sines of those [*bhuja*/*koṭi* arcs], without sines. (Brahmagupta, *Brāhmasphuṭasiddhānta*: 243.)

From this we rewrite the rule as the following approximation to the sine function, well known in various forms from medieval Indian treatises, but never accompanied by a proof:

(1) $$\mathrm{Sin}\,\theta \approx \frac{4R\theta(180-\theta)}{40500-\theta(180-\theta)},$$

where R is the radius of the standard circle and $\mathrm{Sin}\,\theta = R\sin\theta$.

As noted by Hayashi (1991: 46), the above formula is consistently accurate to within 0.2% error. The excellence and ingenuity of this algebraic rule have inspired many conjectures as to its origin. A geometrical derivation has been offered by Inamdar (1950) and discussed by Gupta (1967), as well as by Hayashi. It relies on the fact that the length of an arc of a circle is greater than that of the chord subtending it to produce the following inequality:

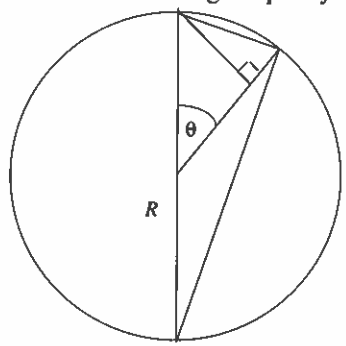

FIGURE 1

$$\frac{1}{\mathrm{Sin}\,\theta} = \frac{2R}{\mathrm{Crd}\,\theta\cdot\mathrm{Crd}(180-\theta)}$$

(2) $$> \frac{2R}{\frac{C\theta}{360}\cdot\frac{C(180-\theta)}{360}} = \frac{2R}{\theta(180-\theta)}\cdot\left(\frac{360}{C}\right)^2,$$

where C is the circumference of the circle.

The inventor of the rule is then assumed to have replaced the above inequality by the approximation

$$(3) \qquad \frac{1}{\operatorname{Sin}\theta} \approx p\frac{2R}{\theta(180-\theta)}\left(\frac{360}{C}\right)^2 + q$$

and then to have used the known values of Sin θ

$$\operatorname{Sin}\theta = R \quad \text{for} \quad \theta = 90,$$
$$\operatorname{Sin}\theta = \frac{R}{2} \quad \text{for} \quad \theta = 30,$$

to solve for the constants p and q as follows:

$$p = \frac{10125}{2R^2}\left(\frac{C}{360}\right)^2, \qquad q = -\frac{1}{4R}.$$

Substituting p and q in equation (3) and solving for Sin θ yields equation (1).

In short, we imagine the inventor considering the half-chord or sine function in terms of the geometrical relations among the sine and other chords and arcs of a circle, and then adjusting the resulting expression so that it is exact for $\theta = 30$ and $\theta = 90$. This is a very reasonable hypothesis, except that, as Hayashi points out, the initial idea of solving for the inverse of Sin θ is perhaps not an obvious one. Hayashi suggests that a clue may lie in Bhāskara I's description of his approximation as "the computation without [225], etc." This is a reference to the common practice in Indian treatises of providing values of Sin θ for θ at intervals of 225 minutes, or 3;45 degrees. The texts also frequently supply lists of Sine-differences, or values of $\operatorname{Sin}\theta - \operatorname{Sin}(\theta - 225')$, for use in linear interpolation. Relying on the small-angle approximation, they give $\operatorname{Sin}(225') = 225$, where $R = 3438$; so the first (non-zero) Sine and the first Sine-difference are both 225. Hence a "computation without [225], etc." is an expression for Sine that avoids use of linear interpolation with the Sine-differences. Hayashi notes that 40500 may be rewritten as $180 \cdot 225$, and proposes that "the origin of [this approximation] is ultimately related to Āryabhaṭa's theory of trigonometry", but no details are given. However, in view of the fact that Brahmagupta also characterizes this rule as a way to derive the sine "without sines," the actual presence of 225 in the formula may not indicate a direct theoretical relationship between this approximation and the function it replaces.

A different approach to the derivation of the same formula is proposed by Gupta (1967: 13). He notes that the behavior of the sine function is qualitatively similar to that of the function

$$(4) \qquad p = \theta(180 - \theta),$$

which in fact is explicitly defined by Bhāskara II in his version of this approximation given in the *Līlāvatī* (212–213). His sixteenth-century commentator Gaṇeśa points out that this function increases with the sine and reaches a maximum at the same point. Gupta suggests that the function $\frac{p}{8100}$ was accepted as a first approximation to $\frac{\operatorname{Sin}\theta}{R}$ and then modified to agree exactly with the known value at $\theta = 30$. Following Gaṇeśa's hint that "some *trairāśika* [rule of three] was applied", he rewrites the equation

$$(5) \qquad \frac{\dfrac{p}{8100} - \dfrac{\operatorname{Sin}\theta}{R}}{\dfrac{p\cdot\operatorname{Sin}\theta}{8100\cdot R} - \dfrac{\operatorname{Sin}\theta}{R}} = \frac{\dfrac{5}{9} - \dfrac{1}{2}}{\dfrac{5}{18} - \dfrac{1}{2}}$$

to yield equation (1).

This is a very plausible conjecture, although it is by no means the only possible one (Gupta in fact offers three other hypotheses besides the geometrical one mentioned before). Again, the source of the first intuitive step — in this case, realization of the resemblance between the sine function and p — remains a puzzling question.[1] Nonetheless, this derivation seems to be the simplest so far proposed for this approximation, and may be simplified even further by treating the procedure somewhat less formally, as follows: Suppose that the inventor of the formula, having arrived at the first approximation $\frac{p}{8100}$, computes the value for $\theta = 30$, producing $\frac{4500}{8100}$. Knowing that the desired value is $\frac{1}{2}$, he commences to modify the proportion by multiplying the numerator by some integer n and the denominator by $n + 1$. Successive experiments with the factors $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$ yield $\frac{9000}{24300}$, $\frac{13500}{32400}$, and $\frac{18000}{40500}$ respectively. But the denominator of the last result differs from the required value by 4500, which is just $p(30)$; so the inventor, perhaps after confirming his guess by checking other values of θ, accepts as a general rule the approximation

$$(6) \qquad \frac{\operatorname{Sin}\theta}{R} = \frac{4p}{5(8100) - p}.$$

This conjecture, like the others, though it may be reasonable, is impossible to prove. But this approach of hypothesizing a quite intuitive, experimental methodology is not only consistent with the apparent lack of emphasis on formal deduction, but can be usefully applied to the reconstruction of other discoveries.

One of these, another trigonometric approximation, is mentioned by Bhāskara II (b. 1114) in the *Karaṇakutūhala* (and, to my knowledge, nowhere else):

> *daśābdhyanvitākṣaprabhāṣaṣṭibhāgo*
> *'kṣakarṇānvitas tena bhaktākṣabhā sā |*
> *khanandāhatā dakṣiṇāḥ syuḥ palāṃśāḥ*
>
> A sixtieth part of the equinoctial shadow increased by 410 is increased by the equinoctial hypotenuse. The equinoctial shadow is divided by this [and] multiplied by 90. (Bhāskara II, *Karaṇakutūhala*: 2, 16 a–c and p. 17.)

1. It is intriguing to note that the fact that p reaches a maximum at $\theta = 180 - \theta$ might have been known to the authors of the ancient geometrical works, the *Śulbasūtras*, in relation to the maximization of the area of a rectangle; but that is only speculation.

The commentary which is provided by Ekanātha (*fl.* 1370) consists of a worked example (EKANĀTHA, *Karaṇakutūhalaṭīkā*: f. 16) that guides the interpretation of the verse to the following formula:

(7) $$\phi \approx \frac{90 s_0}{k_0 + \dfrac{s_0 + 410}{60}},$$

where ϕ is the local terrestrial latitude, s_0 the shadow at noon on the equinox of a standard gnomon of length 12 digits, and k_0 the hypotenuse of the right triangle formed by the shadow and the gnomon. This astronomical rule is easily seen to be equivalent to a more general approximate trigonometric identity:

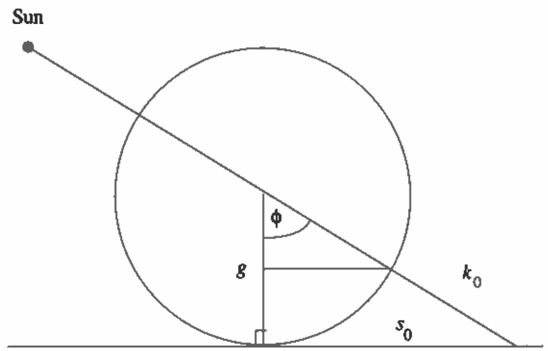

FIGURE 2

(8) $$\theta \approx \frac{90 \operatorname{Sin}\theta}{R + \dfrac{\operatorname{Sin}\theta + \frac{205}{6}\operatorname{Cos}\theta}{60}}.$$

The formula in equation (7), like that in equation (1), is an ingenious way of obtaining the desired quantity directly, without the use of a sine table, and in addition is extremely accurate. For very high terrestrial latitudes (about which no Indian astronomer would be concerned in any case), the error is somewhat worse than 1%, but up to $\phi = 60$ it is approximately 0.01%. An attempt has been made by a nineteenth-century editor of the *Karaṇakutūhala* to derive this rule from the previously given sine approximation (BHĀSKARA II, *Karaṇakutūhala*: 17–18), but the relationship between the two formulas is complicated and contrived. More simply, let us suppose that the inventor of this rule (who, as far as we know, is Bhāskara II himself), in searching for a useful way to relate ϕ and the length of the equinoctial shadow, chose as a first approximation the following:

(9) $$\frac{\phi}{90} \approx \frac{s_0}{k_0}.$$

Since, as is apparent from Figure 2, $\frac{s_0}{k_0} = \frac{\operatorname{Sin}\phi}{R}$, it is evident that this formula is quite crude. Bhāskara therefore sought to improve it by adapting it to fit specific cases. Choosing the simple cases of $\phi = 45$, $s_0 = g = 12$, and $\phi = 30$, $s_0 = k_0/2$, and using the customary sexagesimal notation for fractional quantities, he arrived at

$$\frac{s_0}{k_0} = \frac{6;56}{13;52} > \frac{\phi}{90} = \frac{1}{3}.$$

He recognized that this sine approximation to a linear function worsened for larger values of ϕ, increasingly overestimating the exact result. It was therefore necessary to add to the denominator some term that increased with ϕ. Such a term can be found in each case by solving the following expression for x:

(10) $$\frac{\phi}{90} = \frac{s_0}{k_0 + x}.$$

For the case of $\phi = 45$, this yields $x = 7;2$, and for $\phi = 30$, $x = 6;56$. But the discrepancy between these values is very close to one-sixtieth of the difference between the corresponding shadows; so Bhāskara simply added the required difference, 6; 50 plus one-sixtieth of s_0, to the denominator. Again, he presented this solution as general, producing the following formula for any ϕ:

(11) $$\frac{\phi}{90} \approx \frac{s_0}{k_0 + \dfrac{s_0}{60} + 6;50},$$

which when rewritten in decimal notation is just equation (7).

Analogy, rigour, result: another example

We have examined two samples of admirably successful results of a non-rigorous method of mathematical inference. We may wonder what happens when a resemblance between specific cases is misleading and the analogy inferred is false. In fact, the inability of this kind of method consistently to prevent such errors can perhaps be illustrated by a result preserved in the *Āryabhaṭīya* of Āryabhaṭa. This work, written in the late fifth century AD, is one of the earliest and best known of Indian mathematical treatises. In verse 6 of the second chapter, the author says:

> *tribhujasya phalaśarīraṃ samadalakoṭībhujārdhasaṃvargaḥ |*
> *ūrdhvabhujātatsaṃvargārdhaṃ sa dhanaḥ ṣaḍaśririti ||6||*
>
> The area of a triangle is the product of the perpendicular and half the base. Half the product of the area of that and the height, that amount is a six-edged [solid]. (ĀRYABHAṬA, *Āryabhaṭīya*: 39–40.)

This gives the following formulas for the area A of a triangle of base b and altitude h, and for the volume V of a triangular pyramid with a base of area A and height h:

$$A = \frac{1}{2}hb, \tag{12}$$

$$V = \frac{1}{2}hA. \tag{13}$$

While equation (12) is true, equation (13) is clearly incorrect: it should be $V = \frac{1}{3}hA$. This error, combined with the juxtaposition of the two formulas in one verse, suggests that in this case reasoning by analogy led Āryabhaṭa astray. Realizing that the area of a rectangle of base b and height h is just hb, and that the volume of a corresponding rectangular solid is hA, he reasoned that a triangular pyramid is to a rectangular solid as a triangle is to a rectangle, and erroneously took the volumes of the former to have the same proportion as the areas of the latter.

Some effects on perception and transmission

This manner of doing mathematics, therefore, if it is as we have reconstructed it, is both strengthened and weakened by its 'intuitive' character. Its flexibility is an advantage in that correct results are intuitively reasonable and simply expressed, and require no laborious proofs before they can be put to use. On the other hand, incorrect results that look equally reasonable and simple may not be detected as errors. A mathematics thus lacking the restrictions and the reliability of deductive proof is naturally difficult for a mathematician trained in the Euclidean tradition to accept. The eleventh-century Muslim scientist and scholar, al-Bīrūnī, who resided in India for several decades, had this to say about the indigenous science of the land of his exile:

> The Hindus had no men of this stamp [i.e., equal to the Greek philosophers] both capable and willing to bring sciences to a classical perfection. Therefore you mostly find that even the so-called scientific theorems of the Hindus are in a state of utter confusion, devoid of any logical order, and in the last instance always mixed up with the silly notions of the crowd [...] and I can only compare their mathematical and astronomical literature, as far as I know it, to a mixture of pearl shells and sour dates, or of pearls and dung, or of costly crystals and common pebbles. Both kinds of things are equal in their eyes, since they cannot raise themselves to the methods of a strictly scientific deduction. (AL-BĪRŪNĪ, *India*: I, 25.)

And he claims to have made Indians themselves realize the inferiority of their system to that of Greek science:

> On having made some progress [in understanding Indian learning], I began to show them the elements on which this science rests, to point out to them some rules of logical deduction and the scientific methods of all mathematics, and then they flocked together round me from all parts, wondering, and most eager to learn from me. (AL-BĪRŪNĪ, *India*: I, 23.)

But despite this demonstration of enthusiasm, al-Bīrūnī does not seem to have inspired a mathematical revolution among the Indians, who persisted in the use of their own techniques. As the preceding remarks make clear, al-Bīrūnī was far from seeking a sympathetic understanding of these techniques. It is doubtful whether he ever viewed Indian scientists as anything but rather incompetent Greeks; for in his presentation of the results of the treatises he studied, he commends those that are corroborated by the deductive system with which he is familiar and deplores those that are erroneous or 'unscientific'. For al-Bīrūnī, it appears, the successes of Indian science had to be the result of some attempt at what he knows as scientific deduction; and its failures were produced by the intervention of idolatry or stupidity that interfered with correct deduction. He did not believe — and by his own account, he did not think that even his Indian colleagues could believe — that a non-rigorous, non-deductive scientific method might possess any advantages or produce worthwhile results.

This attitude is interesting in light of the development of the Islamic mathematics that was later assimilated by Latin Europe. The influence of classical Greek mathematics on the structure, if not always the content, of these works is apparent in the presence of, e.g., geometrical proofs. This is hardly surprising, since much Islamic mathematics was directly inspired by the Greek works that the Muslims collected and translated. But it should be borne in mind that Indian mathematics too made its appearance early in Islam, with translations of Indian works into Arabic beginning in the eighth century. And Islamic works freely incorporated developments such as the Indian decimal numerals and the sine function, but without adopting the style of the Indian *siddhāntas*. There has been speculation that this may have affected the development of Islamic mathematics:

> From the time of Brahmagupta the Indians had a much better system of algebraical notation than the Greeks and had gone further that the Greeks in general methods for the solution of indeterminate equations. Arabic adoption of Indian methods in algebra would have led to a much more rapid development of algebra in Europe. Were they ignorant of these Indian methods, or were they attracted rather to the more practical and geometrical Greek form of algebra than to the more speculative and generalizing Indian algebra? (CLARK 1937: 368.)

The preference for a Greek over an Indian technique may be due at least in part to ease of comprehension for strangers to the tradition. As al-Bīrūnī complains about the Sanskrit verse treatises:

> Now it is well known that in all metrical compositions, there is much misty and constrained phraseology merely intended to fill up the metre and serving as a kind of patchwork, and this necessitates a certain amount of verbosity [...] From all this it will appear that the metrical form of

literary composition is one of the causes which make the study of Sanskrit literature so particularly difficult. (AL-BĪRŪNĪ, *India*: I, 19.)

The prose exposition of Greek treatises was doubtless easier to interpret. On the other hand, as the previous remark suggests, it may have been the mathematical style itself that influenced the choice. We do not know for certain what most Islamic scholars' opinions on the relative merits of Greek and Indian mathematics were; al-Bīrūnī, as he himself points out, is unusual in attempting to analyze rather than absorb Indian knowledge, and his views are no doubt influenced by his own training in the Greek methodology. But it seems reasonable to suppose that the axiomatic deductive method, with its capacity for systematic development, simply proved more appealing than an intuitive approach.

The influence of the axiomatic deductive method, now accepted as indispensable to mathematicians, has in its turn affected the perception and understanding of Indian techniques. As we have seen, many scholars expect logical foundations for mathematical results, and if such verification is not found, the results may be disparaged as incoherent jumbles of miscellanea. One historian, concurring with al-Bīrūnī's assessment, remarks that "the work of Āryabhaṭa is indeed a potpourri of the simple and the complex, the correct and the incorrect" (BOYER 1968: 233). This is undeniably true from the point of view of modern mathematics, but does not help us to understand the thought processes of Indian mathematicians, unless we rest satisfied with calling them confused and incomprehensible. The situation is not improved by 'defenders' of Indian science who try to ascribe its successes to a hidden foundation of very advanced deductive knowledge: e.g., the suggestion (KULKARNI 1988) that a fourth-century BC rule for the dimensions of a water-clock may indicate contemporary knowledge of hydraulics and integral calculus! In effect, these different evaluations reflect the same premise: namely, that the Indian and Western traditions share in essence the same criteria for mathematical thought, though they may not be equally successful in applying them. The texts themselves, being generally silent on the question of origins and methods, cannot explicitly contradict this view. But one should be very cautious about imposing an essentially Greek mathematical philosophy upon them, whether to their credit or discredit. It may be more just to say of the traditional Indian mathematician what the mathematician J. E. Littlewood said of his colleague Ramanujan: "if a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further."

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