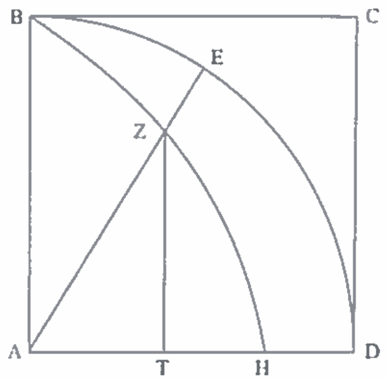
Pappus Collectio IV

Props. 26–29: Quadratrix[1]

Genesis and Symptoma of the Quadratrix

#30 For the squaring of the circle a certain line has been taken up by Dinostratus and Nicomedes[2] and some other more recent (mathematicians). It takes its name from the *symptoma* concerning it. For it is called "quadratrix" by them, and it has a *genesis* of the following sort.



Set out a square ABCD and describe the arc BED of a circle with center A, and assume that AB moves in such a way that while the point A remains in place, <the point> B travels along the arc BED, whereas BC follows along with the traveling point B[3] down the <straight line> BA, remaining parallel to AD throughout, and that in the same time both AB, moving uniformly, completes the angle BAD, i.e.: the point B <completes> the arc BED, and BC passes through the straight line BA, i.e.: the point B travels down BA.[4] Clearly it will come to pass that both AB and

[1] The Latin word "quadratrix" (i.e., squaring line) translates the Greek name (τετραγωνίζουσα) for the transcendent curve that will be the subject of Props. 26–29. The Latin version is commonly used as the standard name for this particular curve, though the term can have other meanings, too.

[2] The common author Nicomedes connects the passages on the conchoid and quadratrix curves. Dinostratus was a late fourth century BC mathematician, the brother of Menaechmus, who invented the conics as locus curves. On the authorship concerning the curve quadratrix and its *symptoma*-mathematics see the commentary.

[3] συνακολουθείτω; the basic verb is, once again "ἀκολουθέω" = follow along in order. As in the other instances in *Coll.* IV, it does not have the connotation of strict logical derivation – on the contrary (see below). On the use of "ἀκολουθεῖν" compare the remarks on analysis-synthesis in the introduction to Props. 4–12.

[4] This generation via synchronized motions is reminiscent of the *genesis* of the spiral in Prop. 19; the connection between these two curves has been emphasized by Knorr (e.g., Knorr 1978a, 1986).

BC reach the straight line AD at the same time. Now, while a motion of this kind is taking place, the straight lines BC and BA will intersect each other during their traveling in some point that is always changing its position together with them. By this point a certain line such as BZH is described in the space between the straight lines BA and AD and the arc BED, concave in the same direction <as BED>, which appears to be useful, among other things, for finding a square equal to a given circle.[1]

And its principal *symptoma* is of the following sort. Whichever arbitrary <straight line> is drawn through in the interior toward the arc, such as AZE, the straight line BA will be to the <straight line> ZT as the whole arc <BED is> to the arc ED. For this is obvious from the *genesis* of the line.

Criticism of the Quadratrix Under the Description via Motions (Sporus)

<u>#31</u> Sporus, however, is with good reason displeased with it, on account of the following <observations.>[2]

For, first of all, he[3] takes into the assumption the very thing for which it <i.e., the quadratrix> seems to be useful. For how is it possible when two points start from B, that they move, the one along the straight line to A, the other along the arc to D, and come to a halt <at their respective end points> at the same time, unless the ratio of the straight line AB to the arc BED is known beforehand? For the velocities of the motions must be in this ratio, also.[4] Also, how do they think that they[5] come to a halt simultaneously, when they use indeterminate velocities, except that it might happen sometime by chance; and how is that not absurd?

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[1] The quadratrix can be used also for the division of an angle in any given ratio (probably its original use), and for problems related to this construction. Cf. Props. 35–38.

[2] The passage taken from Sporus differs significantly from the mathematical expositions in *Coll.* IV. Note, e.g., the rhetorical questions and the polemical style. Co p. 88 replaces the name "Sporos" with the Latin word "spero." His paraphrase means: "I expect, however, that this line justifiedly and deservedly does not satisfy, for the following reasons." The replacement changes the meaning of the introductory sentence, and indeed of the whole passage criticizing the quadratrix considerably.

[3] The Greek text uses the third person singular. It is unclear whom Sporus' argument targeted.

[4] The use of the notion "velocity" is not quite precise here. However, it is clear what Sporos means, and his argument is valid. In order to synchronize the two motions as required, one must know π – or else use an approximation to stand in for it. However, π is exactly what the curve is supposed to exhibit in construction. Co p. 88 paraphrases "motuum velocitates." Hu 254, 7 emends A's elliptical "ἀναγκαῖον." For a parallel construction, without emendation, see, however 270, 11/12 Hu.

[5] The reading πῶς οἴονται (how do they think) as given in A, was kept. Both Hultsch and Treweek reject it in favor of the reading πῶς οἷόν τε (254, 8 Hu + app/ Tr. 109, 11), attested in the minor manuscripts. Co p. 88 paraphrases "quo pacto arbitrantur."

Furthermore, however, its endpoint, which they use for the squaring of the circle, i.e.: the point in which it intersects the straight line AD, is not found <by the above generation of the line>. Consider what is being said, however, with reference to the diagram set forth. For when the <straight lines> CB and BA, traveling, come to a halt simultaneously, they will <both> reach AD, and they will no longer produce an intersection in each other. For the intersecting stops when AD is reached,[1] and this <last> intersection would have taken place as the endpoint of the line,[2] the <point> where it meets the straight line AD. Except if someone were to say that he considers the line to be produced, as we assume straight lines <to be produced>, up to AD. This, however, does not follow from the underlying principles, but <one proceeds> just as if the point H were taken after the ratio of the arc to the straight line had been taken beforehand.[3] Without this ratio being given,[4] however, one must not,[5] trusting in the opinion of the men who invented the line, accept it,[6] since it is rather mechanical.[7] Much rather, however, one should accept the problem that is shown by means of it.[8]

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[1] Restoring A's reading πρὸς (when) instead of Hultsch's πρὸ (= before; cf. 254, 16 Hu app).

[2] Restoring, with Tr 109, 20, the reading of A.

[3] For an extension of the quadratrix to the base line one needs to know the direction. As the quadratrix does not have a constant direction, or even curvature, one needs, in the end, to know the position of H, and it would have to be determined beforehand, using the ratio of radius and circumference (π). My translation differs from Hultsch's Latin interpretation. Co has the following Latin paraphrase, rejected by Hultsch (p. 89 Co): Sed ut cumque sumatur punctum …, praecedere debet proportio circumferentiae ad rectam lineam.

[4] The Greek word (δοθῆναι) is the technical term from geometrical analysis. It is not certain (in fact perhaps unlikely) that Sporus, whom Pappus paraphrases here, intended it that way. What is certain, however, is that Pappus is going to interpret it in this strict technical sense for Props. 28 and 29. See below, and see the commentary on Props. 26–29.

[5] Accepting Hultsch's emendation οὐ for the difficult manuscript reading ἦ, kept in Tr 109, 26. Co p. 89 keeps the manuscript reading, and paraphrases as a question: Or should we… ? The disadvantage is that in that case one would have expected the question particle at the beginning of the sentence.

[6] I.e.: accept it as fully geometrical. The quadratrix itself (in the motion description) is not fully accepted; but note the upcoming remark on the mathematics *about* it. It is quite possible that Sporus and Pappus have different opinions on this matter. The issue cannot be pursued here.

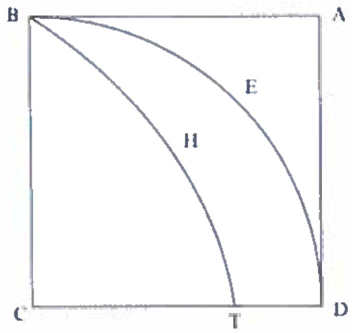
[7] Greek: μηχανικωτέραν. This word, used for the curve itself here, and not just for the way in which it is generated, is different from the label "ὀργανικῶς", i.e., "describable with an instrument". The latter was used in connection with Nicomedes' conchoid (cf. footnotes above). Hultsch deletes the phrase "and it is put to use by the students of mechanics for many problems" as an interpolation (254, 24–256, 1+app. Hu). There is indeed no evidence that the quadratrix played a major role in mathematical treatises on mechanics. A similar phrase occurs at 244, 20 Hu. See the introduction to Props. 19–30 in the commentary on "mechanical."

[8] Hultsch has changed the transmitted text considerably. His Latin paraphrase means: "but before I must report (assuming παραδοτέον) the problem that is solved on account of it." With Tr 110, 1–2, I keep the transmitted text. Co's paraphrase on p. 89 is compatible with this reading. See the commentary.

Prop. 26: Rectification of the Arc of a Quadrant

When a square ABCD is <given>, and the arc BED with center C,[1] and when the quadratrix BHT has come to be[2] in the above said way, it is shown that as the arc DEB <is> to the straight line BC, so <is> BC to the straight line CT.[3]

a

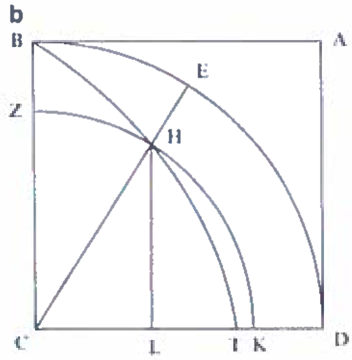


For:

If it is not <in that ratio to CT>, it will be <in that ratio> either to a <straight line> larger than CT or to one smaller.[4]

Assume first that, if this is possible, it is so to a larger <straight line> CK, and describe the arc ZHK with center C, intersecting the line in H, and <draw> HL as a perpendicular <onto CD>, and produce CH, after it has been joined, to E.

b



[1] Note the change of lettering in the diagram. Perhaps Prop. 26 was taken from a different source (Nicomedes, as opposed to Dinostratus, or else Sporus, for the curve's *genesis*?).

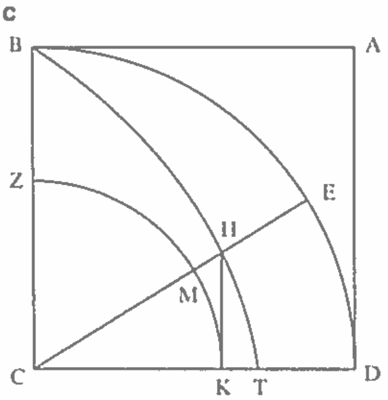
[2] Note that the quadratrix is posited at the outset. The upcoming argument will keep the problematical *genesis* of the curve out of sight, and use its *symptoma* only.

[3] This proportion will yield the construction of a straight line equal to arc DEB (Prop. 27).

[4] We get a classical proof via double reductio (so-called method of exhaustion). Apart from the (short and straightforward) alternative argument for the inverse of Prop. 13, this is the first, and the only, example for this argumentative technique in *Coll.* IV. On Prop. 26 see also Heath (1921, I, pp. 226–229).

Now, since as the arc DEB <is> to the straight line BC, so is BC, i.e.: CD, to CK,[1] whereas as CD <is> to CK, <so is> the arc BED to the arc ZHK (for as the diameter of a circle <is> to the diameter <of a second circle>, <so is> the circumference of the circle to the circumference <of the second circle[2]>), it is obvious that the arc ZHK is equal to the straight line BC.[3] And since, on account of the *symptoma* of the line, BC is to HL as the arc BED <is> to the arc ED, therefore, as the arc ZHK <is> to the arc HK, so <is> the straight line BC to HL,[4] also. And it has been shown that the arc ZHK is equal to the straight line BC. Therefore, the arc HK is equal to the straight line HL as well, which is absurd.[5] Therefore, it is not the case that as the arc BED <is> to the straight line BC, so is BC to a <straight line> larger than CT.

#32 I say, however, that it <i.e., BC> is not <in that ratio> to a <straight line> that is smaller, either.



[1] By assumption.

[2] This theorem is also used in Props. 36, 39, and 40, and a similar one in Prop. 30 (cf. notes ad locum). An explicit proof is given in *Coll.* V, 11 and *Coll.* VIII, 22. A possible justification might proceed as follows. XII, 2: circles have the ratio of the squares over their diameters; *Circ. mens.* I: circles have the ratio of the rectangles with radius and circumference as sides; V, 16 and VI, 1: circumferences have the ratio of diameters. V, 15: similar arcs have the ratio of diameters. The frequent occurrence of this motif may indicate that it is part of the special "jargon," a kind of basic tool within the "analytic track" of *symptoma*-mathematics of the third kind. Specifically, it might be a typical tool of Nicomedes. Nicomedes apparently systematically exploited properties of spiral lines, taking Archimedean arguments as a starting-point. Compare Pappus' remarks on the study of spiral lines and quadratrices as a central branch of geometry of the linear kind in the upcoming meta-theoretical passage.

[3] BC:CK = CD:CK = arc BED:BC (assumption); CD:CK = arc BED:arc ZHK ⇒ BC = arc ZHK (V, 9).

[4] arc BED:arc ED = BC:HL (*symptoma*). arc BED:arc ED = arc ZHK:arc HK (equal parts).

[5] arc ZHK:arc HK = BC:HL; arc ZHK = BC ⇒ arc HK = HL (V, 9). This is not possible, because 2HL is a chord under two times arc HK.

For if this is possible, assume that it is <in that ratio> to KC, and describe the arc ZMK with center C, and <draw> KH at right angles to CD intersecting the quadratrix in H, and produce CH, after it has been joined, to E. Similarly to what has been written above, then, we will show both that the arc ZMK is equal to the straight line BC, and that as the arc BED <is> to the <arc> ED, i.e.: <as> the <arc> ZMK <is> to the <arc> MK, so <is> the straight line BC to the <straight line> HK.[1] From these <observations> it is obvious that the arc MK will be equal to the straight line KH, which is absurd.[2] Therefore, it will not be the case that as the arc BED <is> to the straight line BC, so is BC to a <straight line> smaller than CT.

It has been shown, however, that it is not <in that ratio> to a larger one, either. Therefore, it <is in that ratio> to CT itself.

Prop. 27: Squaring the Circle

It is obvious, also, however, that when a straight line is taken as the third proportional to the straight lines TC and CB, it will be equal to the arc BED, and its fourfold to the circumference of the whole circle.[3] When, however, a straight line equal to the circumference of the circle has been found, it is very clear that it is rather easy indeed to put together a square equal to the circle itself. For the rectangle between the circumference of the circle and the radius is two times the circle, as Archimedes has shown.[4]

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[1] Just as in the first part of the "exhaustion," one gets: CD:CK = arc BED:BC (assumption); arc BED:arc ZMK = CD:CK ⇒ arc ZMK = BC (V, 9), arc BED:arc ED (= arc ZMK:arc MK) = BC:HK (*symptoma*).

[2] HK must be larger than arc MK. I am not aware of an elementary geometrical argument in ancient geometry for this (correct) statement. Hultsch and Ver Eecke (1933b) ad locum refer to an argument that can be reconstructed from (Ps.-) Euclid, *Catoptrics* 8.

[3] Construct the third proportional s for TC and CB (VI, 11): TC:BC = BC:s; TC:BC = BC:arc BD (Prop. 26 with V, 16) ⇒ s = arc BD. Then 4 s is equal in length to the circumference of the circle.

[4] *Circ mens*. I. This rectangle can be transformed into a square via II, 14.