Master's thesis

Mathematics of life and death

The history of life tables and life annuities with an emphasis on the work of Nicolaas Struyck (1686-1769)

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Chapter 1

Introduction

The history of life tables and life annuities is fascinating in many ways. It is filled with mathematically interesting discoveries, many of which provide an insight into the interests and motivations of the people behind them, and the historical situation in which they were made. It contains contributions by historically interesting people like statesman Johan de Witt and astronomer Edmond Halley, whose work inspired many later mathematicians. On top of this, the data from historical life tables provide us with an insight into life expectancy and mortality rates in varying periods in history.

In this thesis, we will examine all of these facets to provide a complete overview of the history of life tables and life annuities. In doing this, we will pay extra attention to the work of Nicolaas Struyck, an eighteenth-century Dutch mathematician. He was one of the first people in the Netherlands to create a life table, and he was the first worldwide to publish separate life tables for men and women. He also used his tables as a basis for detailed calculations of values of life annuities. Still, little research has been done into his work. By studying it closely and also looking at the methods of his predecessors, we will investigate exactly how Struyck performed his calculations. We will then be able to assess Struyck's significance in the history of demography.

Inspiration for the subject of this thesis came from the article '*Lijfrentes in de zeventiende en achttiende eeuw*' by prof. Jan Hogendijk. He wondered how Nicolaas Struyck had calculated his life tables. While attempting to answer this question, I quickly discovered that the work of earlier mathematicians could provide important clues to his methods, and that the work of later mathematicians could help in analyzing Struyck's work and its legacy. The more I read about the history of life tables and life annuities, the more I was struck by how interesting this subject is, both mathematically and historically, and above all, how very human. I have greatly enjoyed researching both the mathematics and the lives of the fascinating mathematicians involved in this history, and finding out the close links that often existed between the two. I found it particularly exciting to be able to find and read original seventeenth- and eighteenth-century sources, whether in the university library or online, and to analyze seventeenth- and eighteenth-century mortality data using twenty-first-century methods. If only the pioneers of the mathematics of life tables could have known.

The general structure of this thesis will be as follows. First, I will introduce and define life tables and life annuities in chapter 2. Then, in chapters 3 to 9, we will look at major developments in the history of life tables and life annuities, starting with the publication of the very first life table, and ending with recent mathematical work on the subject of human mortality. The conclusion will tie all the previous chapters together. In it, we will take a closer look at the people behind the science, and examine the motivation and historical context behind each of the developments we have seen.

Appendix A, included at the end of this thesis, contains the (life) tables referred to in the text.

This thesis is typeset in LaTeX; graphs were rendered in Wolfram Mathematica.

Chapter 2

Introduction to life tables and life annuities

In this chapter, the two central concepts of this thesis will be introduced: life tables and life annuities. I will also give definitions of some related concepts, which we may encounter later on.

2.1 Life tables

A life table, or mortality table, is a table of statistical data describing the survival rate of a certain group of people[1] as a function of age. In what form the data are given, can vary: for example, the table could give the expected number of people who will survive to a certain age out of some initial group, or it could give the probability that a person of a certain age will die within the next year (or month, or decade). Both tables would be called 'life tables' because each of these types of data can be calculated from the other.

2.2 Life annuities

Simply said, a life annuity is a contract by which sums of money are paid out in regular intervals (usually yearly, hence the word 'annuity') for as long as a person lives, in return for some initial investment.

To make this more precise, let us first take a look at the parties involved in such a contract. Theoretically, a life annuity may involve four parties:

- The *issuer*, for example an insurance company
- The *owner*; this is the person who makes the initial investment

[1] Actually, life tables exist not just of people, but of many different animals and other organisms. However, in this thesis, we will encounter only life tables of humans.

- The *annuitant*; this is the person whose lifespan will determine the duration of the annuity
- The *beneficiary*; this is the person receiving the yearly payments.

To buy a life annuity, the prospective owner first pays a sum of money to the issuer, and chooses an annuitant and a beneficiary for the annuity. In return, the issuer agrees to pay a fixed amount of money to the beneficiary every year, for as long as the annuitant lives.

The owner, annuitant and beneficiary need not be three distinct people. In fact, it is so common for all three to be the same person, that the word 'annuitant' is sometimes used for all three roles. Modern life annuities often take the form of pensions, providing the owner (who is also the annuitant and beneficiary in this case) with a steady income until their death, thus basically ensuring that they will not run out of money if they live longer than expected.

However, historically, all four parties might well be distinct. In the seventeenth and eighteenth centuries in the Netherlands, the annuitant was quite commonly chosen to be a young child, often even completely unknown to the owner/beneficiary[2]. This way, the beneficiary could hope the payments would continue for a long time. If the beneficiary died before the annuitant, his or her heirs would continue to receive the payments until the death of the annuitant.

As an aside, I have found no mention of this same practice in the United Kingdom. Perhaps this explains why the Dutch language has distinct words for 'person whose lifespan determines the duration of a life annuity' and 'beneficiary of a life annuity' (namely 'lijf' and '(lijf)rentenier', respectively) that cannot be used interchangeably, while English only has the word 'annuitant' to describe both.

A very interesting question concerning life annuities is how high the initial investment should be, given the desired yearly payout. I will call the amount of the initial investment the *value* of the annuity. This value should, of course, depend on the remaining life expectancy of the annuitant: this remaining life expectancy determines the expected number of times the issuer will have to pay out, which should somehow be 'fair' in comparison with the initial investment the issuer has received. To determine the value of a life annuity, therefore, two things are needed: some definition of what is 'fair' in this case, and the remaining life expectancy of the annuitant.

Perhaps surprisingly, life annuities were sold long before any accurate calculation was made of life expectancy. Records survive of corrodies[3] from the fourteenth century of which the prices did depend on the age of the annuitant, but this dependance was based on experience, or possibly guesswork, rather than on actual data[13]. It was only

[2] [44], page 9.

[3] A corrody is much like a life annuity, except that it was rarely paid out in money, but rather in a combination of commodities like food, drink, clothing or accommodation, usually by a monastery or other religious institution.

in the seventeenth century that life tables were first used as a basis for calculations of values of life annuities.

In the sixteenth and seventeenth centuries, the sale of life annuities and redemption bonds[4] was an important way to raise money for cities and governments. Annuities were not sold continuously, but only at times when money was needed quickly. The city or government in question would start a so-called *negotiation*[5] of life annuities by publishing prices for the annuities. In the registration period that followed, civilians could buy annuities for these prices. When enough money had been raised, the negotiation would end.

Commelin [19] writes that when the United Provinces were being invaded by the French army in 1672, the government passed a resolution containing two important measures. The first was to institute days and hours of prayer, in the hope that the wrath of God would be averted. The second measure was the sale of life annuities and redemption bonds to raise money for the defense of the country. This illustrates the importance of life annuities and bonds for governments; without them, wars might even have ended differently.

[4]A bond is a kind of loan, in return for which, the issuer pays interest to the holder of the bond. In a redemption bond, the issuer pays back the loan, over some agreed period of time.

[5]The sale of a life annuity involved no negotiation in the sense of discussing or bargaining: the buyer of the annuity had no say in deciding the price. Rather, I think the word 'negotiation' is used here in its original Latin meaning of 'doing business'.

Chapter 3

The first life table

3.1 Bills of Mortality

The circumstances that led to the creation of the very first life table were not related to annuities or insurances, but were rather more grim in nature.

The Black Death swept across Europe in a devastating pandemic between 1346 and 1353, killing between 30% and 60% of the population. From that time on until around 1667, the disease stayed present in Europe, flaring up into widespread epidemics every few years [32].

In the period 1592-1594, one of these outbreaks occurred in the city of London. During this time, weekly 'Bills of Mortality' were published. These Bills were weekly accounts of the number of burials, arranged by cause of death and by parish, and the number of christenings by parish, in greater London in that week. This way, if the number of deaths from the plague increased significantly, the authorities would soon be aware of this and could consider taking measures to prevent further infections. Civilians could buy the Bills too, and perhaps retreat to the countryside before the risk of infection became too great in London.

After 1594, the Bills were discontinued until 1603, when another outbreak hit London. The Bills were published continuously from then on [26].

The men who discovered that the Bills were useful for many other purposes than just keeping track of the plague, were John Graunt and his friend Sir William Petty. In this chapter, we will first look at their backgrounds and then examine their work.

3.2 John Graunt (1620-1674)

Figure 3.1: A portrait claiming to be of John Graunt. However, I have not been able to verify this, and the style suggests it may be from the 19th century. There are no other known portraits of Graunt.

John Graunt was born in London on April 24, 1620 as the son of a draper. He himself became a haberdasher[1] and apparently gained quite a good reputation by his integrity, because at the age of thirty, he had enough influence to have his friend William Petty appointed professor of music at Gresham College, London. He was a member of the Common Council (a governing body of London) for two years, captain and later major of the trained band (local militia regiment) [21].

In his free time, Graunt practiced mathematics. Remarkably, even though no such thing as statistics had been invented yet, he noticed that interesting calculations could be made with the Bills of Mortality. In January or February of 1662[2], he published his *'Natural and Political Observations made upon the Bills of Mortality'*. In this book, he analyzed the bills in many ways, estimating for example the population of London, the in- or decrease of several diseases and the ratios between men and women at birth and at death. The book also contained the first known life table, although this was probably the work of his friend William Petty (see also the section about Petty). The work was so groundbreaking that some consider Graunt to be the founder of statistics [58]. On the 26th of February, Graunt was admitted to the Royal Society, having been recommended by King Charles II personally.

In 1666, the year of the Great Fire of London, Graunt was admitted into the management of the New River Company[3]. The Great Fire was subject of many conspiracy theories, one of the strongest being that it might have been the result of a Catholic plot. Apparently, it was this that led John Graunt, who had converted from Puritanism to Catholicism, to be accused of having had a hand in the Great Fire. He had allegedly closed off the water supply to the city. This accusation would later be proven baseless, as Graunt only started working for the Company 23 days after the outbreak of the Fire. However, with his reputation damaged and both his house and shop destroyed by the Great Fire, Graunt went bankrupt. He died in poverty, of jaundice, on April 18, 1674 [21].

[1] A haberdasher sold small articles for sewing, like needles, thread and buttons.

[2] The dedication in Graunt's book is dated '25 January $166\frac{1}{2}$' (*sic*). Until 1752, Great Britain (except for Scotland) used a calendar in which the new year started on March 25 ('Old Style'). What was January 1661 in England, would therefore have been January 1662 in continental Europe, where the new year started on January 1 ('New Style'). I will use the New Style.

[3] The New River is a canal that was dug to supply London with fresh drinking water.

3.3 William Petty (1623-1687)

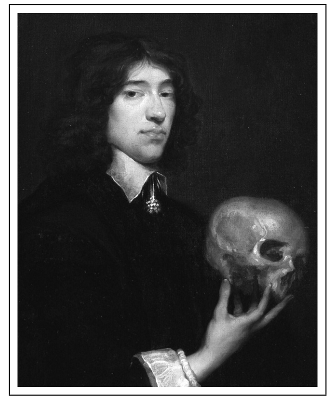

Figure 3.2: Portrait of William Petty, by Isaac Fuller, ca. 1649

William Petty was born in Romsey, in the south of England, on May 26, 1623, as the son of a clothier. He went to sea at an early age, but his fellow sailors left him behind on the coast of France after he had broken his leg on board. He stayed in France for a few years, teaching English and navigation and studying at the Jesuit College in Caen. He later returned to England and joined the navy, but he returned to continental Europe in 1642, when the English Civil War broke out. This time, he went to the Netherlands, where he studied medicine at the universities of Utrecht, Amsterdam and Leiden. In 1645, Petty went to Paris to spend time in the company of Marin Mersenne, who corresponded with many of the great scientists of that time. Petty returned to England in 1646. He finished his studies of medicine in Oxford, where he gained quite a reputation by 'reviving' convicted murderess Anne Greene, who had been hanged and was presumed dead until Petty discovered she was still breathing[4]. In 1651, Petty became professor of anatomy; by this time, he was also professor of music in London, a position he secured with the help of his friend John Graunt.

The following year, Petty travelled to Ireland as a physician for the army. There, he managed to get a contract for making an accurate land survey of Ireland, an enormous task which took him several years, but paid well in both money and large areas of land.

In the late 1650's, Petty returned to England, where he renewed his contacts with fellow scientists and intellectuals, including naval official and politician Samuel Pepys, now famous for the diary he kept. Petty was one of the founding members of the Royal Society, and was knighted on April 22, 1662, at the first incorporation of the Royal Society.

Petty remained a friend of John Graunt and helped him to write and improve his *'Observations'*. After Graunt's death, Petty published the fifth edition of the book, which he had improved and enlarged so much that he sometimes referred to it as his own work. This, coupled with the simple facts that Petty was an accomplished scientist with a knighthood while Graunt was a haberdasher and suspected terrorist,

[4]Anne Greene (or Green) (1628-ca.1665) was a domestic maid. She became pregnant after she had been seduced by her master's grandson. The child was stillborn, as would later be confirmed by medical evidence. However, she was tried and convicted for murder of the child; she was hanged in Oxford on December 14th, 1650. After her body was taken down from the gallows, it was transported to William Petty's house, to be dissected by Petty and fellow doctor Thomas Willis. There, they discovered she still showed signs of life. Greene was nursed back to health, and within a month, she made a full recovery. Because her 'revival' was considered to be a direct act of God, she was then pardoned and released. Anne Greene lived on for another 15 years, during which she married and had three children.[55, 35]

might have caused the wide-spread belief that Petty was the original author of the *'Observations'*. Edmond Halley (1656-1742) wrote:

> The contemplation of the mortality of mankind has, besides the moral, its physical and political uses, both which have been some years since most judiciously considered by the curious Sir William Petty, in his natural and political observations on the Bills of Mortality of London, owned by Capt. John Graunt.[5]

However, it was later convincingly argued[6] that Graunt was in fact the original author of the work, although Petty probably contributed some parts, including the conclusion and the life table. For this reason, I will sometimes refer to the life table as Petty's table in the rest of this thesis, while keeping in mind that Graunt must have been closely involved as well.

Sir William Petty died on December 16, 1687, in London and was buried in his place of birth, Romsey [22].

3.4 William Petty's life table

The question of exactly how Graunt and Petty's life table was calculated, has inspired many scientific publications, but no definite answer has been found. Remember that the age at which a person died, was not reported in the Bills of Mortality; only the cause of death was noted. Therefore, Petty had only very limited data to base his life table on.

In the *'Observations'*, only the calculation of the expected number of newborns to survive to six years old is explained. Petty first counted the number of deaths from causes associated with children: "Abortive, and Stil-born, or died of Teeth, Convulsion, Rickets, or as Infants, and Chrysoms[7]". From this, he concluded that about 36 out of 100 people died before their sixth birthday: the first entry in table A.1. He then estimated, presumably from experience, that about 1 of 100 people survived until age 76. No method was mentioned for calculating the other entries in the table. It is now generally assumed that Petty simply chose some more or less regularly decreasing sequence of numbers.

In the book, the life table was used for only one purpose, namely to estimate the number of 'fighting men' in London; that is, the number of men between ages 16 and 56, who were assumed to be able to fight. I imagine that this may have been an idea of Graunt's, since he was an officer in London's trained band. No more mention was made of the table in the rest of the *'Observations'*.

[5]The first sentence of Halley's *'Estimate'* [30]; see also the section about Halley.

[6]Prominently by Wilcox [58], who wrote 'Petty and Graunt were of different mental stature. (...) To the trained reader Graunt writes statistical music; Petty is like a child playing with a new musical toy which occasionally yields a bit of harmony.'

[7]A chrisom (or chrysom) child is a child that dies before it is baptized.

Chapter 4

Calculation of life expectancy

4.1 The Huygens brothers

In March of 1662, only two months after Graunt's *'Observations'* was published, the famous Dutch scientist Christiaan Huygens received a copy of the book from his English acquaintance Sir Robert Moray[1], who thought it might interest him. Huygens read it, but apparently was not tempted to do his own research on the subject: his only response to Moray was that he liked the book and thought it was useful and well written[2].

But then, seven years later, Christiaan Huygens' brother Lodewijk brought up the *'Observations'* in a letter to Christiaan[3]. Lodewijk also added 'a few calculations' he had done: he had made a table of the remaining life expectancy of people of different ages. This is very remarkable, as it is the first known calculation of life expectancy. Perhaps it was this application of the concept of expectation, which he himself had introduced, that sparked Christiaan's interest. In any case, this time he responded with some calculations of his own. Among other things, he invented the concept of median remaining lifetime and drew a graph based on Petty's life table, which we now regard as the first graph of any probability distribution function. We will look at these inventions in more detail in a later paragraph.

Perhaps surprisingly, neither of the Huygens brothers ever published their findings. Consequently, their work remained unknown until 1895, when their correspondence was published in '*Oeuvres complètes de Christiaan Huygens*'[36].

Let us first look at the Huygens brothers' biographies before further discussing his work.

[1] In a letter dated March 6, 1662 [36].
[2] Letter from Huygens to Moray, dated June 9, 1662 [36].
[3] Dated August 22, 1669 [36].

4.2 Christiaan Huygens (1629-1695)

Figure 4.1: Portrait of Christiaan Huygens, by Caspar Netscher, 1671

Christiaan Huygens was born on April 14, 1629, the second of the five children of well-known poet and composer Constantijn Huygens. He was educated by a private tutor until he was sixteen. Even at this early age, Christiaan apparently had talent for mathematics: the mathematician Descartes, who was a friend of Constantijn, occasionally visited the Huygens family and was much impressed by Christiaan.

Christiaan went to Leiden University in 1645, to study mathematics and law. Here, he was the best pupil of renowned professor Frans van Schooten jr. After two years, Christiaan transferred to the newly established College at Breda. He still studied law there, although he spent much of his time solving mathematical problems.

During this time, Christiaan started corresponding with Marin Mersenne, who also corresponded with men like Descartes, Pascal and Fermat. Mersenne described problems these great mathematicians had encountered, and Christiaan enjoyed attempting to solve them.

Christiaan Huygens' first work *'Cyclometriae'* appeared in 1651, when he was only twenty-two. It concerned the problem of squaring the circle and other curves. After completing his studies, Christiaan, instead of pursuing a career in law or politics, spent all his time practicing mathematics, physics and astronomy. He learned how to grind optical lenses, and discovered a moon of Saturn using a telescope he had made himself.

In 1655, Christiaan and his older brother Constantijn travelled through France for several months to complete their education. Such a voyage, called a Grand Tour, was popular among young men from well-to-do families. On their travels, both brothers obtained a doctorate of law at the university of Angers. Although five years of study would usually be required for such a degree, this university sold them for the price of fifty guilders. This would have been quite a hefty sum for ordinary people, but as Andriesse [7] points out: "One paid more for a wig."[4]

During his time in France, Christiaan was presumably introduced to the problems that French mathematicians Pascal and Fermat were working on. The question was how staked money could be fairly distributed between players of a game of chance, if the game was broken off prematurely. In correspondence between Pascal and Fermat, they had already solved some problems relating to this, but they had not published them. After his return to the Netherlands, Christiaan in 1657 wrote the treatise '*De ratiociniis in ludo aleae*' ('*Of reckoning in games of chance*'), first published as an appendix to a book by Van Schooten, later as a stand-alone book. This work was

[4] From [7], page 132.

the first publication in the field of probability theory, of which Christiaan is now recognized as one of the founders.

Also in 1657, Christiaan Huygens invented the pendulum clock. This earned him instant fame and recognition, because the new type of clock was an attempt to find a solution to the age-old and important problem of how to keep track of one's longitude at sea. It turned out later that the movements of a ship caused too much disturbance in the pendulum for accurate timekeeping, but by that time, Christiaan's reputation was already established. Later, among many other things, he discovered the nature of the rings of Saturn, derive a formula for the centrifugal force and conduct thorough studies in optics. He was made a Fellow of the Royal Society in London and a prominent member of the *Académie Royale des Sciences* in Paris.

Christiaan never married, and died on the 8th of July 1695 [14, 31, 46].

4.3 Lodewijk Huygens (1631-1699)

Lodewijk Huygens, younger brother to Christiaan and the third of Constantijn Huygens' children, was born on March 13, 1631. Like his older brothers, he was educated at home for some time. In 1647, like Christiaan, he went to the College at Breda. Lodewijk appears to have been more defiant than his brothers; an incident in 1649, when he took part in a duel, gained him the disapproval of his father and the school authorities.

In September of 1672, Lodewijk Huygens was appointed drossaard[5] of Gorinchem. Four years later, he was convicted for taking bribes and sentenced to a fine of 6000 guilders. However, he kept his post for a few more years, during which time there were more disagreements with the mayor of Gorinchem. Later, he moved to Rotterdam, where he was a naval official. He was married to Jacoba Teding van Berkhout, with whom he had four sons [61].

Throughout his life, Lodewijk maintained a correspondence with his brother Christiaan. Remarkably, it was him, not Christiaan, who was the first ever to calculate life expectancy from life tables. However, the brothers never published their calculations, so this remained unknown until the correspondence was published in 1895 [36].

There is no known portrait of Lodewijk Huygens, except one of him as a nine-year-old boy.

[5]A local official who could perform various functions, depending on local law.

4.4 The Huygens brothers on Graunt and Petty's work

On August 22, 1669, Lodewijk Huygens wrote to his brother Christiaan:

> (...) J'aij fait une table ces jours passez du temps qu'il reste à vivre, à des personnes de toute sorte d'aage. C'est une consequence que j'aij tiré de cette table du livre Anglois of the Bils of mortalitij, de la quelle je vous envoije icij une copie[6], afin que vous preniez la peine de faire un peu les mesmes supputations, et que nous puissions voir comme nos calculs s'accorderont.[7]

Lodewijk did not initially show his calculations, but after his brother asked to see them, he added them as an appendix to his next letter[8]. They are as follows.

From Graunt and Petty's first table (table A.1), it can be seen that of 100 newborns, an expected 36 will die between the ages of 0 and 6. Take 3 to be the average age at which these people die, then the total number of years these 36 people have lived is 36×3. Likewise, take 11, 21, ... for the average ages of the age categories 6-16, 16-26, ..., and calculate the total number of years the 100 people have lived as $36 \times 3 + 24 \times 11 + 15 \times 21 + \ldots + 1 \times 81 = 1822$. Now divide this by 100 (the number of people considered); the life expectancy of a newborn, rounded to the nearest two months, turns out to be 18 years and 2 months[9].

The remaining life expectancy for people of ages 6, 16 etcetera can be calculated in a similar way. Lodewijk's results are displayed in table A.3.

When Christiaan Huygens responded to this letter[10], he agreed these values were correct. He also introduced another concept, which we now call the median remaining lifetime. He first considered an extreme situation in which 90 out of 100 people died before their sixth birthday, but the remaining 10 each lived on to be 155 years and 2 months old. In this case, Lodewijk's calculation of life expectancy yields $\frac{90\times3+10\times(155+\frac{2}{12})}{100}$, which, rounded to the nearest two months, gives 18 years and 2 months, like before. Christiaan then argued that since the average person in this situation would never reach the age of 18 years and 2 months, another value was of

[6] Copies of tables A.1 and A.2 were enclosed in an appendix to the letter.

[7] From [36], page 482. Freely translated: '(...) I've made a table these past few days of the remaining time to live, for persons of all sorts of ages. It's a consequence I've derived from this table from the English book 'Of the Bills of Mortality' [by John Graunt], of which I hereby send you a copy, in order that you would take the trouble of making some of the same calculations, and we could see how our calculations agreed.'

[8] Dated October 30, 1669; [36], pages 515-519.

[9] Rounded to the nearest month, the result would actually be 18 years and 3 months. There are more values in the table where the result differs by a month (both above and below) from the values I'd expect. My best guess would be that Lodewijk rounded to the nearest two months, with the exception of one value which returned exactly 20 years and 3 months.

[10] November 21, 1669; [36], pages 524-532.

importance, namely the age that a person had a 50% probability of reaching. To explain his method, he first drew a graph interpolating the values in Graunt's life table (table A.2). This graph, displayed in figure 4.2, is the first known graph of what we know now as a distribution function[11].

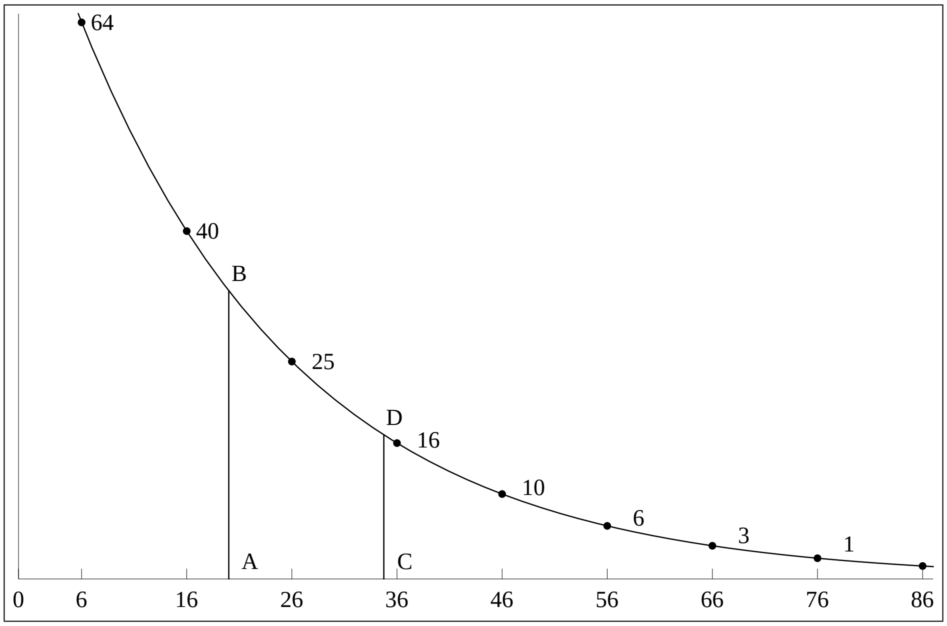

Figure 4.2: Graph interpolating the values from William Petty's life table (table A.2), as drawn by Christiaan Huygens in [36], unnumbered page between 530 and 531. The horizontal axis displays ages (in years); the vertical axis displays the number of people remaining out of 100 newborns.

Christiaan drew the point A at the age he wanted to consider (in his example, 20 years), and the point B above it on the graph. The length $|AB|$ now represented the number of people still alive at 20 years of age, out of 100 newborns. Christiaan then found the age C and the corresponding point on the graph, D, such that $|CD|$ is half of $|AB|$. Age C (in the example, about 36 years) is thus the age at which half of the twenty-year-olds will still be alive, so a twenty-year-old has equal probability of dying within the next 16 years or surviving them.

Christiaan used the extreme example we just mentioned to illustrate the difference between the remaining life expectancy and the median remaining lifetime: if 90 out of 100 people die before their sixth birthday, and we assume the deaths are evenly distributed, then 50 out of these 100 will have died before $\frac{50}{90} \times 6$ years, which returns 3 years and 4 months as the median remaining lifetime for a newborn. This is very different from the life expectancy, which is 18 years and 2 months.

[11] To be precise, the graph in figure 4.2 displays a tail distribution function: if X stands for the age at which a person dies, then the graph displays $P(X > x)$ for ages $6 \leq x \leq 86$. A tail distribution function is the complement of a cumulative distribution function, which would be of the form $P(X \leq x)$.

The distinction between the remaining life expectancy and the median remaining lifetime is a subtle but important one, and the two are still often confused. For example, it is easy to assume that if the average life expectancy (at birth) in a society is 50 years, then it will be very rare for a person to reach the age of 90. But this is not necessarily true: the low life expectancy might be caused by a high infant mortality, in which case a child that has survived its first year, might have a much higher life expectancy than 50, and might have a higher than expected probability of living to 90. The median remaining lifetime at birth would be influenced much less by infant mortality, and would give a better estimate of how rare it was to live to 90.

Christiaan himself gives a similar motivation for introducing the concept of median remaining lifetime. Lodewijk had ended his first letter with the words

> Selon mon calcul vous vivrez environ jusqu' l'aage 56. ans et demij. Et moij jusqu'a 55.[12]

Christiaan pointed out that this is not entirely true. While the remaining life expectancy of a 38-year-old[13] may indeed be around 17 years, this is not a good indicator of the chances he has of actually reaching the age of 55. To estimate the most probable age at which a person will die, Christiaan argued, it is better to use the median remaining lifetime. He then ended his letter to his brother on a teasing note:

> (...) je vois par exemple qu'a vostre aage de 38 ans, vous pouuez encore faire estat de 19 ans et 4 mois environ. Mais si vous vous amusez a faire appeller souuent des gens pour vous battre, il faut encore en retrancher quelque chose.[14]

[12] In Lodewijk's letter dated August 22, 1669. Roughly translated: "According to my calculations, you will live until age 56 and a half. And I until 55."

[13] Lodewijk, having been born in March of 1631, was 38 years old at this time.

[14] In Christiaan's letter dated November 21, 1669. Roughly translated: "(...) I see, for instance, that at your age of 38 years, you may still count on 19 years and 4 months or thereabouts. But if you amuse yourself by regularly asking people to beat you, something has to be subtracted."

Chapter 5

Calculations on life annuities

5.1 Introduction of Johan de Witt

In 1669, Christiaan Huygens remarked to his brother that data from life tables could be useful for calculations on life annuities[1]. However, neither of the brothers actually pursued this subject. It was Johan de Witt, a Dutch statesman and acquaintance of Christiaan Huygens, who first developed a method to calculate the value of a life annuity using mortality data. His treatise '*Waerdye van lyf-rente naer proportie van los-renten*' ('*The worth of life annuities compared to redemption bonds*'), published in July of 1671, contained a clear and thorough explanation of how to calculate how much interest should be given on life annuities so that they would be worth as much as bonds. We will take a closer look at these calculations in section 5.3.

First, however, I would like to include a biography of this remarkable man, who found time to produce very useful mathematical work while holding the highest office in Dutch politics, and who was killed just over a year later.

[1] In a letter dated November 28, 1669 [36].

5.2 Johan de Witt (1625-1672)

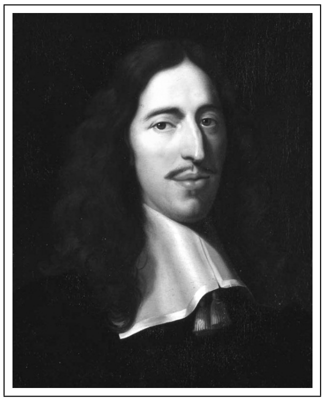

Figure 5.1: Portrait of Johan de Witt, by Jan de Baen, 1670

Johan de Witt was born on September 25, 1625[2] as the youngest surviving child of the mayor of Dordrecht. He was a descendent of a distinguished family: not only had both his father and his grandfather been elected as mayor many times, but his uncle Andries de Witt had been Grand Pensionary of Holland[3] from 1618 to 1621.

Johan had two sisters, and one brother, Cornelis, who was two years older. The brothers were close enough in age that they were in the same class at the Latin school of Dordrecht, since renamed the Johan de Witt Gymnasium. They also went to university together, studying law in Leiden. Beside law, Johan also informally studied mathematics with professor Frans van Schooten jr., who also taught Christiaan Huygens.

After their studies, the brothers embarked on a Grand Tour of France and England. The journey of the De Witt brothers lasted 21 months, of which 3 were spent studying in Angers, France, where they were both awarded a doctorate. Presumably, like the Huygens brothers less than a decade later, they bought these degrees[4]. After the brothers' return to the Netherlands in 1647, Johan started training as a solicitor at a prominent law firm in The Hague, while Cornelis moved back to Dordrecht to enter local politics there.

At that time, most of what is now the Netherlands was contained in the Republic of the Seven United Provinces. Although a republic, the country did have a *de facto* head of state: the Stadtholder (Dutch: Stadhouder), Prince Willem II of Orange. The most important governing body of the republic was the States General (Staten-Generaal), consisting of delegates from the States Provincial, the local governments of each of the seven provinces. Of these seven provinces, Holland was the largest and the most influential. For this reason, the position of Grand Pensionary (Raadspensionaris), head of the delegation from Holland to the States General, was a very important one. The Grand Pensionary was the second most influential person in the country, after the Stadtholder.

[2]Although there has been some discussion about De Witt's date of birth and no baptismal record has survived, Rowen [43] writes that nearly all evidence supports this date.

[3]The Grand Pensionary (Raadspensionaris) of Holland was the political leader of the province of Holland, the most powerful of the Seven United Provinces.

[4]I have found no hard evidence for this. However, we know from the Huygens brothers that law degrees could be bought in Angers. From this and the fact that the De Witt brothers obtained doctorates within three months of arriving in Angers, I conclude that they probably paid for their degrees as well.

In 1649-1650, a disagreement arose between the Stadtholder and the province of Holland about the size of the army. Holland wanted to dismiss a number of companies to save money, but the prince would not allow this. Neither party was willing to give in to the other, so in 1650, Prince Willem took action by having the six representatives to the States of Holland that opposed his plans most actively, arrested. One of these six representatives was Jacob de Witt, Johan's father.

The imprisonment did not last long as within a month, the States of Holland gave in and agreed to the prince's terms. Jacob was released, but he resigned his political functions. For some time, it looked like his sons had to give up on any political aspirations as well.

In November of the same year, Prince Willem II died, while his wife was still pregnant with their only child. This started the 'First Stadtholderless Period' in Dutch history: during this time, the powers of the Stadtholder effectively (though not officially) transferred to the Grand Pensionary of Holland.

A month later, when there was a vacancy for a representative of Dordrecht to the States of Holland, the city of Dordrecht appointed Johan de Witt. He performed this task well, and when the Grand Pensionary of Holland, Adriaan Pauw, was sent on a diplomatic mission to London in 1652, De Witt took over his work. The diplomatic mission, which was meant to prevent war, failed. The First Anglo-Dutch War started in May of 1652. A year later Pauw died, and De Witt succeeded him as Grand Pensionary.

Thanks in part to De Witt's negotiations, a peace treaty between England and the Netherlands was signed in 1654. This treaty contained a secret clause, the Act of Seclusion, which stated that the young Prince Willem III (the son of Willem II) could never become Stadtholder, like his father was. The clause was secret in the sense that on the Dutch side, only the States of Holland signed it, while the States General signed the rest of the treaty, without even knowing about the extra clause.

From 1665 to 1667, there was a Second Anglo-Dutch War. Because this war ended with a very successful Dutch attack on the English fleet, known as the Raid on the Medway, De Witt was able to negotiate an advantageous peace treaty. Still, the discussion about the position of Willem III was ongoing. De Witt supported a proposition to give the boy a high military position, as long as all representatives of all Dutch cities agreed that this position could never be combined with the position of Stadtholder. The proposition was accepted and became known as the Perpetual Edict (Eeuwig Edict) of 1667.

In his private life, De Witt married Wendela Bicker on January 15, 1655, and they had three daughters and one son. Two more daughters were born after that, but both died before the age of two. Only days after this latest tragedy, on the first of July, 1668, Wendela herself died. Johan de Witt was grief-stricken and did not attend meetings for two weeks. From then on, he raised his children with the help of female family members.

In 1670-1672, France invaded the Netherlands and formed an alliance with England. Panic ensued in the Netherlands. De Witt became highly unpopular because he had strengthened the navy while weakening the army, which could hardly do anything against the overwhelming force of the French army. De Witt tried to rectify the situation by funding the building of fortifications and recruitment of large numbers of soldiers.

To finance the impending war, money had to be raised. A common way to do this would be the sale of 'lijf- en losrenten', that is, life annuities and bonds. In July of 1671, De Witt published his '*Waerdye van lyf-rente naer proportie van los-renten*'. In this treatise, addressed at the States of Holland, he explained how the value of a life annuity should be calculated. De Witt's conclusion was that the state was paying annuitants too much. He proposed lower interest rates, which would still make life annuities more profitable to the buyer than bonds, but would save the state a lot of money compared to the old rates.

On June 21, 1672, De Witt was wounded in an attack by four Orangists. He survived, but was bedridden for weeks. Apparently encouraged by this news, riots erupted all over Holland and Zeeland to force each city's leaders to revoke the Perpetual Edict. On July 3, the States of Holland appointed Willem III as Stadtholder.

About halfway through July, when Johan de Witt had recuperated enough to resume work, his position had changed dramatically. Effectively, the Stadtholder was now running the country and the Grand Pensionary no longer had any political power. On the fourth of August, De Witt resigned his post and instead was given a seat on the High Council[5].

However, the damage had been done. Large numbers of pamphlets were printed and distributed accusing De Witt of treason and collaboration with the French, undaunted by the fact that there was no evidence to support this. On the 24th of July, Johan's brother Cornelis de Witt was arrested on suspicion of plotting to murder Willem III. The case against Cornelis was not strong and even under torture he did not confess to anything, yet on the morning of the 20th of August, he was sentenced to exile from Holland. A short time later, Johan received a letter requesting him to pick up Cornelis from the Gevangenpoort[6], where he had been held.

A furious mob was waiting outside the Gevangenpoort, riled up further by Cornelis' accuser Willem Tichelaer, who was far from satisfied with Cornelis' sentence. Johan was let in, but the threat of violence was so strong that the brothers could not go back out. That afternoon, the mob broke into the Gevangenpoort, dragged the De Witt brothers outside, lynched them and cut up their bodies.

There is still a strong suspicion that the killings were the result of an Orangist conspiracy. In any case, the murderers were never prosecuted, and Willem Tichelaer was granted a pension and employment by Willem III himself [43].

[5] The highest court in the Netherlands.

[6] A prison in the center of The Hague.

Even after the violent slayings of the brothers, many pamphlets were issued denouncing them. The following quote is from one of these pamphlets.

> Leimen gelt af, men seide niet dat het was om daar door vrinden en sig selfs te zegenen, en d'arme Renteniers te benauwen, maar om te bewijsen dat het Land rijk wierdt: (...)
>
> 't Was goet sien dat Meester Jan een goet Mathematicus was, en wel cijfferen kon, dat het beter voor hem was, sijn penningen te beleggen en andre daar door te obligeeren, als den Burger dat te laten genieten.[7]

The writer of this pamphlet might well have been referring to De Witt's '*Waerdye*'. The fact that De Witt argued that the prices of annuities should be higher, can not have helped his popularity. Remarkably, the prices were not changed for some time after De Witt's publication; even in 1674, life annuities were still sold at the old prices[8] [54]. We will come back to this fact in the section on Johannes Hudde.

[7] From [8]. 'If money was made, they did not say that this was to bless friends and themselves, and to distress the poor annuitants, but to prove that the State was becoming rich: (...) It was easy to see that Meester Jan ['Meester' is a title for holders of a law degree; 'Jan' is short for 'Johan'] was a good mathematician, and could calculate well that it was better for him to invest his money, obligating others, than to let the citizen benefit from it.' Note: the meaning of the verb 'afleggen' (of which 'lei (...) af' is a past tense) in this context is not entirely clear to me. According to a historical dictionary [37], 'een schuld afleggen' in the 17th century meant 'to pay off a debt'; applied to state finance, I presume it means the state had money left over (at the end of a year) to pay off debts.

[8] We will see these prices later, in table 5.1. According to this table, the price of an annuity on a three-year-old was ten times the yearly payout, while De Witt had proved very clearly that it should at be at least sixteen times the yearly payout.

5.3 Johan de Witt's calculation of the values of life annuities

5.3.1 A conservative estimate of mortality

In his '*Waerdye*' [59], De Witt considered a life annuity for a three-year-old annuitant. To find the value of such an annuity, he first needed an estimate of the probability that a three-year-old annuitant would die at a certain age. The estimate he used is a bit complicated. De Witt's explanation was roughly as follows.

- The probability that a three-year-old will die within a certain half year, is constant for all half years between ages 3 and 53.
- The probability that a three-year-old will die within a certain half year, is constant for all half years between ages 53 and 63; this probability equals two-thirds of the probability for ages 3-53.
- The probability that a three-year-old will die within a certain half year, is constant for all half years between ages 63 and 73; this probability equals one half of the probability for ages 3-53.
- The probability that a three-year-old will die within a certain half year, is constant for all half years between ages 73 and 80; this probability equals a third of the probability for ages 3-53.
- The probability that a three-year-old will live past 80, is 0.

In a modern notation, we could define a valuation $v(n)$ for all ages $3 \leq n \leq 79.5$ (where n is a multiple of 0.5) as follows:

- for all $3 \leq n \leq 52.5$, let $v(n) = 1$;
- for all $53 \leq n \leq 62.5$, let $v(n) = \frac{2}{3}$;
- for all $63 \leq n \leq 72.5$, let $v(n) = \frac{1}{2}$;
- for all $73 \leq n \leq 79.5$, let $v(n) = \frac{1}{3}$.

Now the probability that a three-year-old will live to be n years old, but die before $n + \frac{1}{2}$, is $\frac{v(n)}{128}$, where 128 is the total of $v(n)$ for all values of n.

This estimate is visualized in the graph below, which shows the total probability a three-year-old will live to (at least) age n, plotted against n. Although the fact that the plot is a continuous line is not strictly accurate, it does help to show clearly that the graph is linear over each of the four age intervals De Witt distinguished. Clearly, this estimate is not a very realistic one. But then, De Witt did not intend for it to be realistic: he only needed a very conservative estimate of the life expectancy of an

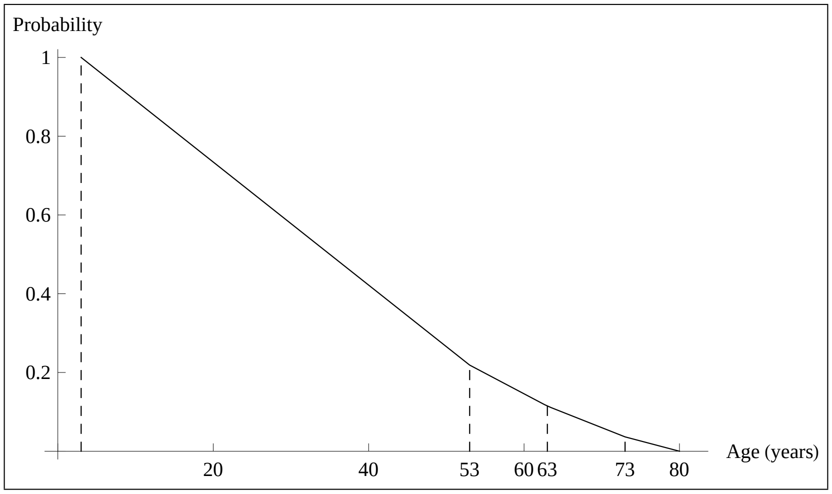

Figure 5.2: Graph of Johan de Witt's estimate

annuitant. In this way, when he calculated a fair price for an annuity based on his estimate, he was able to argue that this price would still be beneficial to the buyer, because the annuitant would probably live longer than calculated.[59]

5.3.2 Comparison of De Witt's estimate to real life tables

To see whether the probability of surviving to a certain age according to De Witt's estimate was indeed lower than actual observed probabilities, I will now compare this estimate to the life table created by Nicolaas Struyck, to whom I will devote a later chapter.

Struyck's life table for men begins with 710 five-year-old boys. To compare De Witt's estimate with Struyck's life table for men, therefore, we need to calculate how many men are expected to survive to each age, out of an initial group of 710 five-year-olds, according to the estimate.

For all $b \geq a$, let $P(b|a)$ denote the probability that a person of age a, will survive to (at least) age b. Then according to De Witt's estimate, the probability that a three-year-old will survive to the age of five, is $P(5|3) = \frac{124}{128}$[9]. Therefore the probability of a five-year-old surviving to a certain age $n \geq 5$, is equal to the probability of a three-year-old surviving to that age, divided by $\frac{124}{128}$; in mathematical notation, $P(n|5) = P(n|3)/P(5|3) = P(n|3)/\frac{124}{128}$. So out of an initial population of 710 five-year-old children, the expected number to survive to age n is equal to the probability of a three-year-old surviving to that age, divided by $\frac{124}{128}$ and then multiplied by 710. We can now easily compare these values with the values from Struyck's tables in a combined graph[10], as seen in figure 5.3. As in the previous graph, De Witt's estimate

[9]Namely $\frac{124}{128} = 1 - (v(3) + v(3.5) + v(4) + v(4.5))$, which is 1 minus the probability that the child will die before age 5.

[10]Struyck's life table for women begins with 711 five-year-old girls rather than 710, but because

is displayed as a continuous line for practical purposes.

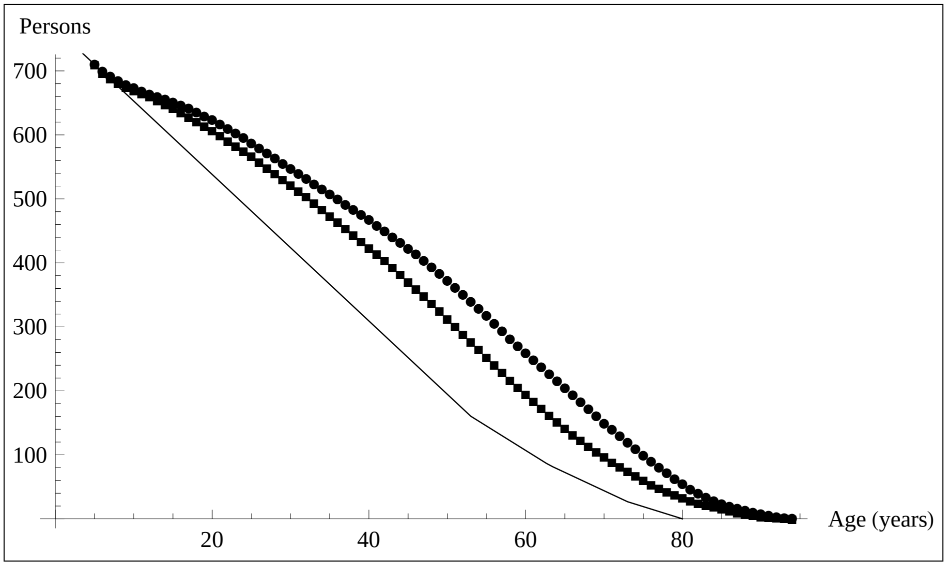

Figure 5.3: Graphs of Struyck's tables for men (squares) and women (circles) and De Witt's estimate (line)

Assuming that mortality rates in Struyck's life tables were representative for the mortality rates of annuitants in the Netherlands[11], it is clear that De Witt's estimate was indeed a conservative one, as he intended.

However, Johan de Witt of course did not have Struyck's life table available. A table that he probably would have known, is Graunt and Petty's table[12]. Let us therefore compare these two tables as well.

Figure 5.4 shows a plot of Graunt and Petty's life table, together with a life table calculated from De Witt's estimate, which has been chosen to give an estimate of the number of survivors out of 82 three-year-olds[13]. Perhaps surprisingly, De Witt's estimate of the number of survivors is actually higher than Graunt and Petty's table indicates.

---

the difference between 710 and 711 is hardly significant, all three data sets can be compared in one graph.

[11]Struyck's tables were based on the lives of Amsterdam annuitants registered in the period 1672-1674; De Witt needed an estimate of mortality rates of future annuitants (from 1671 onwards) in the Netherlands in general.

[12]De Witt's '*Waerdye*' was published within two years after Christiaan Huygens remarked to his brother that a life table could be used for calculations on life annuities. As far as we know, nobody else had made this connection before De Witt's publication. It therefore seems likely to me that Christiaan Huygens, with whom De Witt exchanged letters, inspired him to perform these calculations, or that the two men at least had some contact on this topic. Since the Huygens brothers' work was based on Petty's life table, De Witt would then almost certainly have come in contact with the table.

[13]Petty's table contains 100 newborns of which 64 survive to age 6, so for this visual comparison, I have simply taken the number of three-year-olds to be $(100 + 64)/2 = 82$.

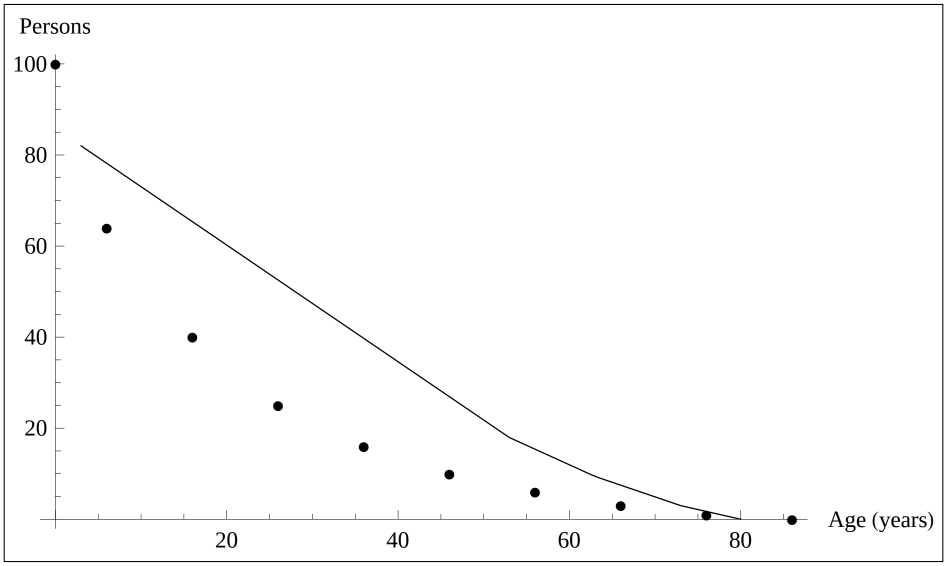

Figure 5.4: Graphs of Graunt and Petty's life table (circles) and De Witt's estimate (line)

It is unclear to me how De Witt knew, then, that his estimate was lower than reality. Possibly he had access to another life table, or he could have even created his own life table for this purpose without publishing it, or maybe his intuition was simply that good.

5.3.3 Calculation of life annuity values using this estimate

Johan de Witt considered a life annuity which pays out one million guilders every year[14] in half-yearly terms, that is, ten million stuivers[15] every six months. De Witt wanted to calculate a fair purchase price for such an annuity, given the age of the annuitant and the yearly interest rate on savings, which he took to be 4%.[16] His reasoning was as follows.

If the annuitant dies within half a year after the annuity was bought, no money is paid out, so the actual value of the annuity is 0.

If the annuitant dies after half a year but before a full year has passed, one payment of ten million stuivers is made. The value of the annuity, in this case, is $10\,000\,000 \cdot \sqrt{100/104} \approx 9\,805\,807$ stuivers. This is the amount of money that, if put in a savings account with a yearly interest rate of 4%, will be worth 10 000 000

[14]Such a life annuity would have a value of several million guilders. Considering that one million guilders in 1671 is equivalent to 11 million euros in 2012 (calculated using the CPI, see also the next section of this thesis), we see that the investment needed for such an annuity would be enormous. Clearly, therefore, De Witt did not choose this value because it was realistic, but rather to be able to calculate accurate results without fractions.

[15]1 guilder ('gulden')=20 stuivers

[16]Note that although payments are made every half year, the 4% interest rate is per year; the interest per half year, therefore, is $(\sqrt{104/100} - 1) \times 100 \approx 1.98$ percent.

stuivers after half a year.

Likewise, if the annuitant dies after a year but before one-and-a-half years after the annuity is bought, two payments are made, the first after half a year and the second after a year. The values of these two payments at the moment the annuity is bought, are approximately $9,805,807$ and $10\,000\,000 \cdot (\sqrt{100/104})^2 \approx 9\,615\,385$ respectively, so the (rounded) value of the annuity is 19,421,191 stuivers.

De Witt continued calculating these values up until the case where the annuitant died between 100 and 100.5 years after the start of the annuity.

Up until this point, the calculations were still quite general, so that anyone could easily replicate these calculations for a different yearly payout or a different interest rate.

Next, he considered a life annuity for a three-year-old annuitant and introduced the estimate of mortality given earlier in this section. Using the estimate, De Witt could now find the expected value of a life annuity on a three-year-old child by multiplying each of the probabilities that the annuitant would die at a certain age, with the corresponding total value of the annuity, and summing these. He found 320 032 139 stuivers, or 16 001 607 guilders, and concluded that a fair price to pay for a life annuity would be sixteen times the yearly payout.

De Witt argued that the value of the life annuity would actually be higher than he had calculated: firstly, his estimate of mortality was purposely higher than true mortality rates; secondly, the interest on savings was in reality lower than 4%, and thirdly, the buyer would generally choose a healthy child, who would have a lower probability of dying within a short period of time. Therefore, the calculated price would still be a good deal for the buyer.

5.4 Inflation

Interestingly, although De Witt must have been well-informed about matters relating to the economy, he did not take inflation into account. However, while certainly of influence, this factor was not as important as one might think. During the seventeenth century, the inflation rate in the Netherlands fluctuated wildly. For example, the yearly inflation rates for the years 1638-1639, ..., 1642-1643 were, respectively, around -3.4%, 5.2%, -3.6%, -3.5% and 5.6%; on balance between 1638 and 1643, therefore, there was actually a slight deflation of around 0.2%. It is only on a much larger scale that overall trends in the inflation rate become visible.

Figure 5.5 is a graph of the Consumer Price Index (CPI)[38] in the Netherlands for each year in the century leading up to De Witt's publication, that is, the period 1571-1671. This index is a measure for the value of money at a certain time, relative to a reference point. The CPI in this graph has been calculated relative to 1995=1000, which means that in any given year, the amount of guilders equal to the CPI for that year in the graph, would have been worth 1000 guilders in 1995. The inflation rate can be calculated from the CPI as the percentage with which the CPI has grown over a certain time period.

Figure 5.5: Graph of the Consumer Price Index (CPI) in the Netherlands in the period 1571-1671, from [38], relative to 1995=1000. The dashed lines indicate years of some major events in Dutch history.

In this figure, the letters stand for the following historical events:

a 1621: End of the truce in the Eighty Years' War

b 1628: Capture of the Spanish treasure fleet

c 1648: End of the Eighty Years' War

It is visible in the figure that the CPI rose fairly steadily from 1571 until around 1625. Apparently, therefore, there was a significant inflation during this period. For the period 1625-1671, it is much harder to see such a general trend. Although there were still periods of strong inflation, they were nearly canceled out by periods of strong deflation. The increase in the CPI between 1625 and 1671, and thus the total inflation over this period of 46 years, was only around 5.6%. Thus the average inflation during De Witt's entire lifetime would have been very small.

Furthermore, the fact that several of the periods of drastic inflation correspond to major events in Dutch history, suggests a strong correlation. Therefore, to predict the CPI for some future year with any accuracy, one would have to somehow predict the unfolding of wars, the changing prices of valuable metals and more. This seems impossible to me, or at least a very daunting task.

Therefore, assuming De Witt knew of the phenomenon of inflation at all, even if he would have had accurate records of the inflation over the previous century, he might easily have been forgiven for concluding that inflation was not a significant factor in determining the value of a life annuity, or that it was simply too unpredictable to take into account.

5.5 Johannes Hudde (1628-1704)

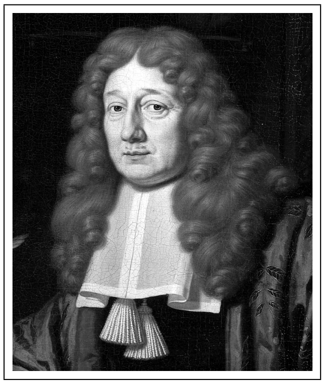

Figure 5.6: Portrait of Johannes Hudde, by Michiel van Musscher, 1686

Johannes Hudde was born in Amsterdam in April of 1628[17] as a son of well-to-do merchant Gerrit Hudde. Around the same time as Johan de Witt, Hudde studied law at the university of Leiden and informally also studied mathematics under Frans van Schooten. Van Schooten apparently valued Hudde's talent, because he published several of Hudde's short works on geometry and algebra in Van Schooten's '*Exercitationes mathematicae*' in 1657.

After his studies, Hudde probably went on a Grand Tour of France and Switzerland[18], spending some time at the university of Saumur, in France. I could not find out whether he obtained a degree there, but it does not seem unlikely that like Huygens and De Witt, he bought a doctorate of law[19].

After his return to Holland, Hudde maintained contact with his former fellow students: around 1665, he corresponded with Huygens about astronomical observations and with both Huygens and De Witt about the mathematics of games of chance. Around the same time, he also collaborated with well-known philosopher Baruch Spinoza, who had a keen scientific interest, in creating lenses for telescopes.

From 1663, Hudde involved himself in Amsterdam city politics. He held several different positions, one of which was leading a large-scale project to build locks in the Amstel river.

Before Johan de Witt published his '*Waerdye*' in July of 1671, Hudde checked his calculations for him. The work apparently sparked Hudde's interest, because on the 18th of August, he sent a letter to Christiaan Huygens on the subject of life tables. In the letter he included a large table containing the ages and life spans of 1495 annuitants from annuities sold by the city of Amsterdam between 1586 and 1590 (table A.4). Whether or not this table qualifies as a life table is perhaps a matter of

[17] Baptized on April 25th [1].

[18] Not much is known about this journey, except for some letters sent by Hudde and his travel companion, mathematician Hendrik van Heuraet.

[19] I have found only circumstantial evidence to support this. Firstly, the town of Saumur is less than 45 kilometers removed from the city of Angers, where the Huygens and De Witt brothers obtained their doctorates. The small Protestant university of Saumur may therefore have had close ties to the larger, Catholic university of Angers. Doctorates were definitely being sold at Angers, so perhaps the same practice was common at Saumur. Secondly, it seems to have been generally accepted for wealthy young men from the Netherlands to buy a doctorate in France; it would seem odd to me if Hudde and Van Heuraet returned from France without degrees, or if they had spent years studying for their degrees if they could be so easily bought. Thirdly, I imagine that a doctorate of law would be very useful for Hudde's career as a politician.

debate[20], but with a little work, a life table can certainly be calculated from it. This means that at the very least, Hudde was the first known person to collect accurate data to base a life table on.

Interestingly, Johannes Hudde's political affiliations were very different from De Witt's; this might explain why Hudde sent his table to Huygens instead of De Witt, who would probably have had more use for it. Whereas Johan de Witt was, as Grand Pensionary, the main political opponent of prince Willem III, Hudde supported the prince. After the prince became Stadtholder in July of 1672, he even appointed Hudde as one of the four burgomasters of Amsterdam on September 15th, less than a month after the murder of the De Witt brothers.

In his new position, Hudde still managed to include his scientific interests in his work. In 1672 and 1673, he was asked for his advice on the sale of new life annuities in Amsterdam, to raise money for the defense of the city in the ongoing wars with France and England. Later, he became very influential in the financial management of the city, and was even involved in the finances of the province of Holland. He also continued to be interested in canals and locks: among many other things, he invented a way to circulate fresh water through the city's dirty canals to improve hygiene, and he had marker stones placed in the walls of canals to keep track of the water level, leading to the invention of 'Normaal Amsterdams Peil' (Normal Height Datum)[21].

A position as burgomaster could only be held by the same person for two out of every three years, and until 1703 Hudde held the position the maximum possible number of times. He married Debora Blauw on February 21, 1673; they did not have any children.

Johannes Hudde died in Amsterdam on April 15th, 1704. Although his only mathematical publications were those published by Van Schooten when Hudde was still a student, the German mathematician Gottfried Leibniz recorded in his personal notes that Hudde had written a lot of unpublished work[22]. However, most of that work has since been lost [56, 15].

[20]The table contains the same information as a life table, and strictly speaking, it meets the definition of a life table given at the beginning of this thesis: 'a table of statistical data describing the survival rate of a certain group of people as a function of age'. Still, I do not feel it is a proper life table, because the data are still nearly in the form Hudde would have found them in; they are data on a set of individuals, instead of having been combined to provide statistical data on a group of people.

[21]Normaal Amsterdams Peil, NAP for short, refers to a reference height used in measuring water level. The system is widely used in the Netherlands and has been adopted by several other countries, including Belgium and Germany [57].

[22]Leibniz and Hudde knew each other's work well; Leibniz referred to Hudde many times in his own publications, and wrote to a friend that he was disappointed when Hudde decided to concentrate on politics instead of mathematics. When Leibniz was traveling from London to Hannover in 1676, he made a stop in Amsterdam to meet Hudde in person. It was on this visit that he noticed the amount of unpublished mathematical work Hudde had produced.

5.6 Hudde's table of data

Hudde's data are displayed in table A.4. The table gives the age of each annuitant when the annuity was bought (at the top of each column) and the number of years the annuitant survived after that (in the body of the table). The age 49 is not included: presumably no annuities were bought on 49-year-olds in Amsterdam between 1586 and 1590.

Commelin [19] writes that Hudde was influential in setting the prices for the life annuities sold in Amsterdam in 1672 and 1673. Therefore, it might be interesting to look at the prices more closely, and compare them to the table of data Hudde had sent to Huygens. The annuity prices are displayed in table 5.1.

Van 1 tot 20 Jaren		1000 gl.	
20-30		959 gl.	(*sic*)
30-40		900 gl.	
40-45		850 gl.	
45-50	Exclu-	800 gl.	100 guld.
50-55	sive	750 gl.	jaarlijkse
55-60	van	675 gl.	Renten.
60-65		600 gl.	
65-70		500 gl.	
70-75		400 gl.	
75-80 en daarbove		300 gl.	

Table 5.1: Prices for the life annuities sold in Amsterdam in the period 1672-1674, from [19]. The left column contains age categories; the right one gives the corresponding prices in guilders of a life annuity with a yearly payout of 100 guilders.

Although Johannes Hudde did not create a life table based on his data, it is not hard to do so now.

As an example, note that column 5 in Hudde's table of data (table A.4) starts with the numbers $0, 3, 5, 5, \ldots$. The 0 indicates that one of the five-year-old annuitants sadly died within a year. In total, the column contains 96 entries. Since 1 of the 96 5-year-old annuitants died within a year, based on the data in this column, the probability that a 5-year-old will live to age 6 is $\frac{95}{96}$.

However, for greater accuracy, we can involve more data from the table in this calculation. We do not know whether Hudde would have done this, but for the sake of accuracy, let us do so. Look at column 4, which starts with the numbers $1, 2, 2, 2, \ldots$ and has a total of 89 entries. It follows that all 89 4-year-olds survived to age 5, and 88 of them survived to age 6. Therefore, based only on the data in this column, a five-year-old has a probability of $\frac{88}{89}$ to live to age 6. We can also combine the data from columns 4 and 5 to see that based on these columns, a total of $95 + 88 = 183$ out of $96 + 89 = 185$ five-year-olds survived to age 6. In the same way, data from columns 3, 2 and 1 can be taken into account as well to find that a total of $95 + 88 + 90 + 63 + 60 = 396$ out of $96 + 89 + 93 + 63 + 60 = 401$ annuitants who were alive at age 5, survived to age 6. Therefore the probability for a 5-year-old to survive the next year, based on Hudde's data, is $\frac{396}{401}$. For shorter notation, let $P(b|a)$, with $b \geq a$, denote the probability that a person of age a will live to (at least) age b; then we get $P(6|5) = \frac{396}{401}$.

Similarly, for any age n, the probability $P(n+1|n)$ can be calculated from the data in all columns up to n. This would have been quite a bit of work in Hudde's time, but luckily we can nowadays use a computer to quickly do the calculations for us. Now given any two ages $b \geq a$, we can find $P(b|a)$ by taking the product $P(b|a) = P(b|b-1) \cdot P(b-1|b-2) \cdots P(a+1|a)$ (and by defining $P(b|b) = 1$ for all b).

To find a life table based on Hudde's data, we need to choose a 'starting point': an initial number of people of some given age, so that we can then find the expected number of these people to survive to any later age. As the youngest annuitants in Hudde's table are 1 year old, it makes sense to start our life table with age 1. Let's therefore take 1000 one-year-olds as the starting point of the life table. Now the expected number of people that will survive to some age b, out of 1000 one-year-old children, is simply equal to $1000 \cdot P(b|1)$. The life table that we obtain by rounding off these numbers to the nearest integer, is displayed in table A.5. We will come back to this table later, in section 7.3.1.

Define $Q(b|a)$, with $b \geq a$, as the probability that a person of age a will live to age b but not to age $b+1$. Then $Q(b|a) = P(b|a) \cdot (1 - P(b+1|b))$. Now that we have these probabilities, we can use De Witt's method to calculate the value of a life annuity with a yearly payout of 100 guilders. I will first assume, like De Witt, that the interest rate on savings is 4%. Because the data in Hudde's table are given in whole years, instead of half years, I will assume the annuity pays out only once per year (instead of in half-yearly terms, like in De Witt's calculation). In figure 5.7, the results of this calculation are displayed as dots, while the real prices (from table 5.1, displayed earlier) are displayed as a line.

Clearly, the real prices and calculated values are very different. The value of an annuity on a one-year-old is (rounded off) 1841 guilders according to the calculation above, while the price that was actually asked for such an annuity, was 1000 guilders. As a quick check of our calculation, we can take a look at the work of De Witt, who calculated that the price for an annuity on a three-year-old should be at least sixteen times the yearly payout. This is true for our calculated values, but not for the actual prices of the 1672-1674 annuities.

Of course, it may be that Hudde used a slightly different calculation than the one we used. As a result, his outcomes may have been closer to the prices from table 5.1. Let us investigate this possibility by looking at different choices we could have made in the calculation above.

For a start, we could have chosen to use only column n for calculations on an n-year-old, instead of using all columns up to n. We could also have assumed the annuity pays out in half-yearly terms instead of once per year. However, it turns out

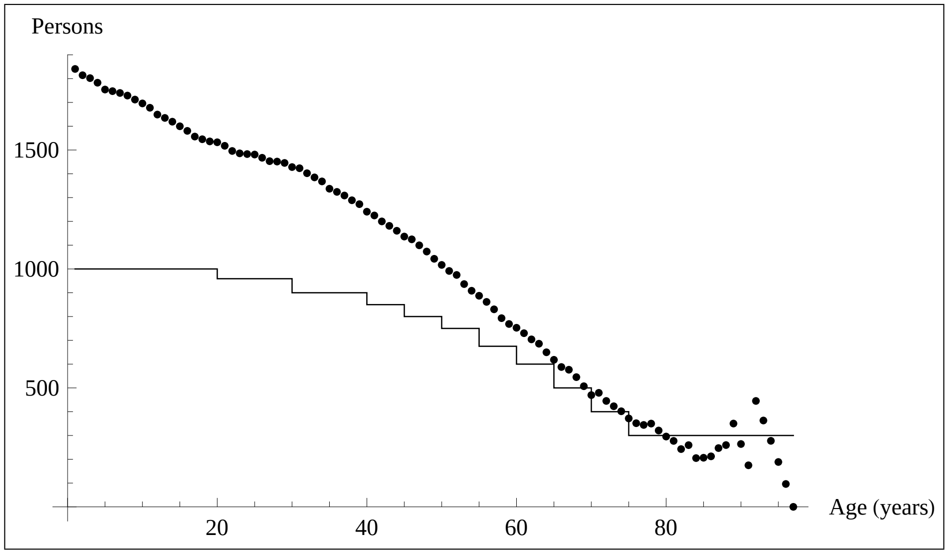

Figure 5.7: Graphs of the prices of 1672-1674 life annuities (line) and the prices calculated from Hudde's data (dots) assuming an interest rate of 4%, according to age of the annuitant

that neither of these choices has a significant impact on the outcomes.

We could also have assumed there was a tax on annuity payouts, so that the beneficiary of the annuity does not actually receive 100 guilders per year, but perhaps only 80. This was true in the time of Nicolaas Struyck, whom I will study in a later chapter of this thesis. But taking this into account would only have the effect of multiplying each outcome with a constant, and as is easy to see from the different shapes of the graphs in figure 5.7, this cannot make the graphs correspond well.

The only remaining option is that the interest rate was different from 4%. Changing this value can make the calculated values correspond reasonably well with the actual prices, but only if the interest rate is chosen to be close to 8%. The results are visible in figure 5.8.

An interest rate of 8%, however, does not sound very realistic, especially considering the fact that around the same time, De Witt used a value of 4% for his calculations and mentioned that the real interest rate was lower. Furthermore, it was not in the interest of the Amsterdam city authorities to sell life annuities of which the payouts were so high that they were comparable to a savings account with an 8% interest rate. The only reason I can think of for why they would choose these values, is to benefit the buyers on purpose. It might be that the city was so short of money that it sold very cheap life annuities as a quick fix. But even then, 8 percent seems like rather a lot; surely, 6% would still have persuaded people to buy an annuity.

The most likely answer, it seems to me, lies not in the mathematics, but in history and the political situation of 1672. Johan de Witt, who had pointed out that life annuities were often sold at prices that were much too low, had just been lynched. There also was a war going on, and Amsterdam had to be defended, which would cost a lot of money very quickly. For those reasons, it was probably not the best time to

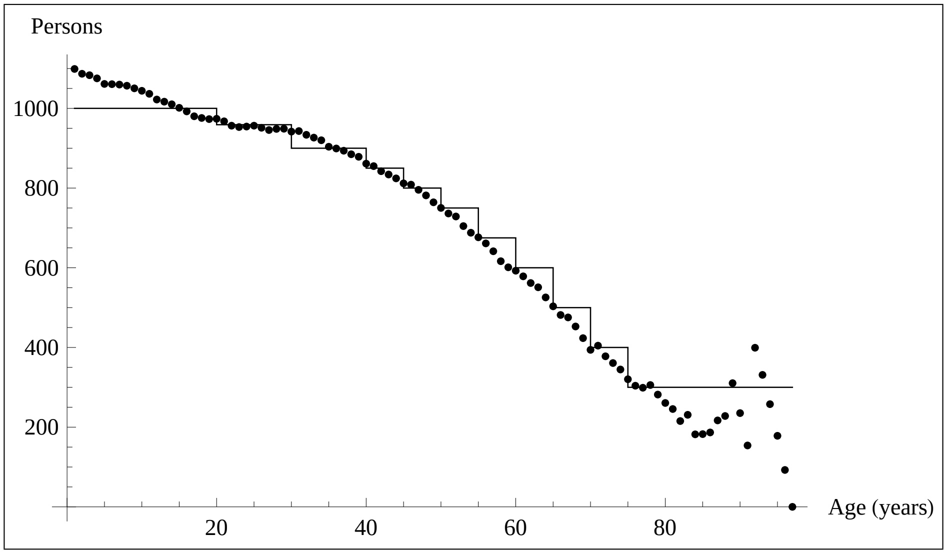

Figure 5.8: Graphs of the prices of 1672-1674 life annuities (line) and the prices calculated from Hudde's data (dots) assuming an interest rate of 8%, according to age of the annuitant

raise the traditionally low prices of annuities.

Therefore, although I cannot imagine that Hudde calculated these annuity prices from his own data, he might well have been involved as a politician in the decision to keep the prices low on purpose.

Chapter 6

A more detailed life table

6.1 The problem of finding accurate data

Now that the connection had been made between life tables and prices of life annuities, it became clear that aside from life expectancy being an interesting subject of investigation, considerable amounts of money could come to depend on the accuracy of a life table. Graunt and Petty's life table, although groundbreaking, was clearly not very precise.

This presented a problem. Ideally, you could simply keep track of a large and representative group of newborns, and record how long each of them lived. But to collect such data, you would have to wait until the last of the group had died - which would almost certainly not happen within your lifetime. And even then, the results would be outdated by the time they could be published.

Thus to create a life table within a reasonable time, researchers had to make the most of already available data. We have already seen that Johannes Hudde used registers of life annuities. A life table based on these data would therefore be a life table of annuitants, which would be very useful to predict the average lifespan of future annuitants and thus to calculate values of life annuities. But if one wanted to create a life table of the general population of a certain city or country, then Hudde's data would not be ideal, because annuitants need not be representative for this general population.

One reason for this is that newborn babies could hardly be annuitants. Even apart from the question whether it would be wise to buy such an annuity, it would simply be impossible to register a baby as an annuitant on the very moment he or she was born, and besides, the baby would not even have a name yet. Therefore, data obtained from life annuity registers would provide very little data on infant mortality. Secondly, it seems unlikely that somebody would take out a life annuity on a person who works in a very dangerous profession or who has a chronic illness. Presumably, one would pick a healthy person who would be likely to live a long life. Preferably, they would also come from a well-to-do family that could afford a good education, hopefully leading to a good job, and a more comfortable lifestyle than the average person had. The fact that a life annuity was expensive and therefore the owner would have to be reasonably wealthy, makes it likely that the owner would know people in these circles.

In conclusion, to create a life table that was representative for the general population, other kinds of data had to be found. Luckily, regions did exist in which for every death, the age of the deceased was noted. Together with the year of death, these data would be enough to serve as the basis for a life table.

The well-known British astronomer Edmond Halley was the first to use such data to create an accurate life table.

6.2 Edmond Halley (1656-1742)

Edmond Halley[1] was born in Haggerston (now part of London) as the son of a rich soap maker on October 29, 1656. He studied at Oxford University for three years, after which he travelled to the island of Saint Helena, in the South Atlantic Ocean, to observe the stars visible on the Southern Hemisphere. For this journey, Halley received funding from his father and support from, among others, King Charles II, who recommended him to the East India Company [18]. When Halley returned to England and published his calculations on the orbit of the Moon, he was awarded an MA degree and elected a Fellow of the Royal Society, while only 22 years old.

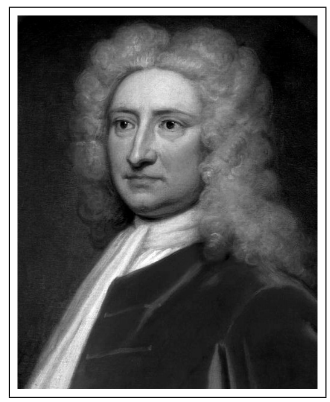

Figure 6.1: Portrait of Edmond Halley, by Sir Godfrey Kneller, ca. 1721

In 1684, Halley visited Isaac Newton to ask him for help in calculating orbits of celestial bodies. Newton's response impressed Halley so much that he stimulated Newton to write more work. This eventually resulted in Newton's famous book *'Philosophiae Naturalis Principia Mathematica'* (1687), which was edited, paid for and reviewed by Halley.

Halley came into contact with interesting data for a life table through the Royal Society. In the German city of Breslaw (now Wrocław, Poland), records were being kept of the number of births and of the age and sex of each person who died in the city. The pastor and scientist Caspar Neumann (1648-1715) sent some of these data to his acquaintance Gottfried Leibniz[2], who in turn informed the Royal Society. The Society then asked Halley, who had just promised to regularly fill a number of pages in the '*Philosophical Transactions*'[3], to analyze the data [29, 42]. The result was Halley's 1693 article '*An estimate of the degrees of mortality of mankind*' A.6.

[1] Also commonly spelled Edmund; however, Edmond was the spelling he himself used.

[2] We saw Leibniz earlier in this thesis: he knew Johannes Hudde and visited him in 1676. It may be that this contact sparked his interest in life tables.

[3] A regular publication of the Royal Society.

This article included a life table for Breslaw, which displayed the expected number of persons still alive after a certain number of years, out of 1000 one-year-old children. In the article, Halley also calculated the value of a life annuity depending on the age of the annuitant; he used an interest rate of 6% instead of 4%, but otherwise used the same method as Johan de Witt.

In the period 1698-1700, Halley observed the Earth's magnetic field around the Atlantic Ocean, as commander of the HMS Paramore. In 1704, he was appointed professor of geometry at Oxford, and soon after that, he published a book on comets that contained the calculation of the orbit of what is now known as Halley's Comet. In 1720, Halley became the second Astronomer Royal, a prestigious position which, among other things, made him director of the Royal Observatory in Greenwich.

Edmond Halley died in Greenwich on January 14, 1742 [12, 20]

6.3 Comparison of Halley's life table with Graunt and Petty's

In the opening paragraph of the article in which he presented his life table, *'An Estimate of the Degrees of Mortality of Mankind'*, Halley pointed out a number of flaws in Graunt and Petty's table and another table, which was based on data from Dublin.

> (...) first, as the number of the people was wanting; secondly, as the ages of the people dying was not mentioned; and lastly, as both London and Dublin, by reason of the great and casual accession of strangers who die there, as appeared by the great excess of the funerals above the births, rendered them unfit to be standards for this purpose; which requires, if it were possible, that the people we treat of should not at all be changed, but die where they were born, without any adventitious increase from abroad, or decay by migration.[4]

The last of these reasons was an important one. Consider, for example, a city where many immigrants come looking for work. It might well be that healthy adults make up a disproportionately large number of this group of immigrants. If these people eventually die in the city, this will result in a large number of deaths being recorded in the city for older people, and relatively few for children. From a resulting life table, it would then appear that the average person lived longer than was really the case.

The city of Breslaw, Halley argued, was more suitable than London or Dublin for the purpose of creating a life table, because it did not have a large flow of either immigrants or emigrants. Furthermore, detailed records were available: Caspar Neumann had sent the Royal Society the number of births and deaths in Breslaw over five years, namely from 1687 to 1691, as well as the age and sex of every person who

[4] From Halley's *'Estimate'* [30].

died within this period. With an average number of 1238 births per year, sufficient data were available to create a life table that would be far more reliable than Graunt and Petty's table. Halley's life table is reproduced in table A.6.

In figure 6.2, it is apparent how big the differences between Petty's and Halley's table really are. By multiplying all values in Petty's life table (table A.2) by 10, we obtain a table which gives the expected number of people who survive to a given age, out of 1000 (instead of 100) newborns, which makes a comparison with Halley's table possible.

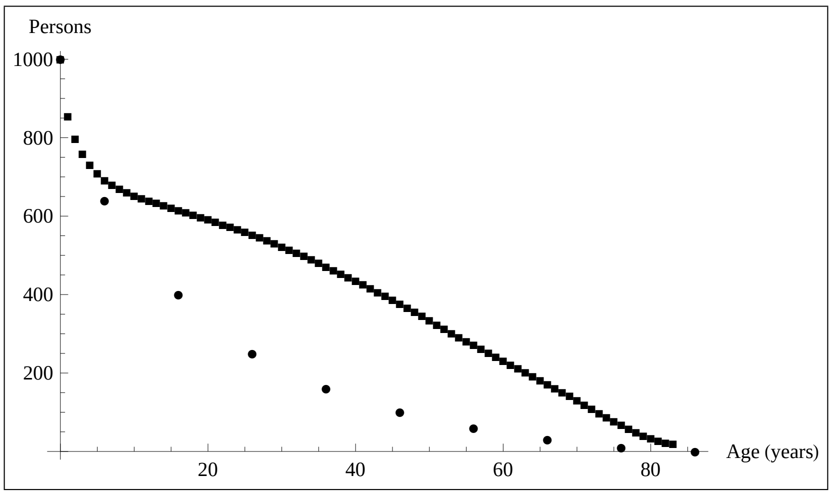

Figure 6.2: Combined graph of Edmond Halley's life table (squares) and the expected number of survivors to a given age, out of 1000 newborns, based on William Petty's life table (circles).

The differences between both tables are clearly very large. Even taking into account the fact that Petty's life table was much less precise than Halley's, it is apparent that the expected number of survivors in Petty's table is significantly lower than the one in Halley's table at all ages. This might simply be caused solely by the fact that Petty used estimates and probably guesses to fill his table. However, I think the differences between the tables are large enough to assume that life expectancy halfway through the seventeenth century in London was much lower than near the end of that century in Breslaw[5]. The fact that Petty's and Halley's table differ at all ages, suggests this difference is not caused by age-related factors like early aging or stillbirths, but more likely by the plague, which affected young and old people in roughly equal measure.

[5]For example, Petty apparently found it a reasonable estimate that only 62.5% of sixteen-year-olds would ever reach twenty-six, while Halley found this percentage to be 90%. I think that if the real percentage in Petty's time was close to 90%, he would have known that his estimate was off.

6.4 Ages in Halley's life table

It is perhaps remarkable that Halley's life table starts with 1000 people at age one. From a modern point of view, I personally would expect a life table to begin with age zero, so that the mortality of children under the age of one can be compared with the mortality at other ages.

However, the true meaning of the numbers in Edmond Halley's table becomes clear from the first calculation he makes with them. Printed next to his table was the calculation displayed in table A.7. In the accompanying text, Halley explained that to estimate the total number of inhabitants of Breslaw, he added up all the numbers in the table (and apparently added 107 persons above the age of 84 to obtain a nice rounded result), and the result was 34000. Indeed, the first entry in table A.7 corresponds to the sum of the first seven entries in table A.6, the second is the sum of the next seven entries in table A.6 and so on. So if 34000 is Halley's estimate of the number of people living in Breslaw, then the numbers in the life table must represent the average number of people of a given age living in Breslaw at any time. Therefore, the number of 1000 one-year-olds was not chosen arbitrarily: it apparently represented the number of one-year-olds living in Breslaw. Now the question arises where in this calculation the number of children under the age of one is taken into account. Halley wrote:

> Of these [1238 births] it appears (...) that 348 die yearly in the first year, and that but 890 arrive at a full year's age; (...)[6]

Assuming that the 348 deaths are distributed evenly over the first year of life, and that 1238 children are born each year, the average number of children under the age of one living in Breslaw at any given time is $1238 - \frac{348}{2} = 1064$. This number, rounded off, might well be the basis for the 1000 'one-year-olds' appearing in Halley's table. This would explain why the table does not begin with 'age zero': the 1000 would then represent the number of children up to one year old, the next entry would represent the number of children between one and two years old and so on. For that reason, from now on, whenever I write about the number of n-year-old people in Halley's data, I will mean the number displayed in his table next to the age $n + 1$. This is to make comparison with other life tables easier.

6.5 Adjustments

From table A.6, it quickly becomes apparent that the values are too regular to be the result of coincidence. For example, from age 53 to 63, the difference between any two consecutive values is exactly 10. This is because Halley wanted to remove any irregularities in the table, about which he wrote:

[6] From *'Estimate'* [30], p. 484.

(...) that [irregularity] seems rather to be owing to chance, as are also the other irregularities in the series of age, which would rectify themselves, were the number of years much more considerable, as 20 instead of 5.[7]

This, of course, made sense: Halley did not especially want to publish a life table of people who had died in Breslaw within that exact period of five years, but rather a much more general life table, which could be used to calculate life expectancies and the values of life annuities.

An easy way to analyze how he smoothed out irregularities in the data, is by considering the graph below, in which the expected number of deaths at age n is plotted against n. These numbers can be calculated by simply subtracting the number of people alive at age n from the number alive at age $n+1$.

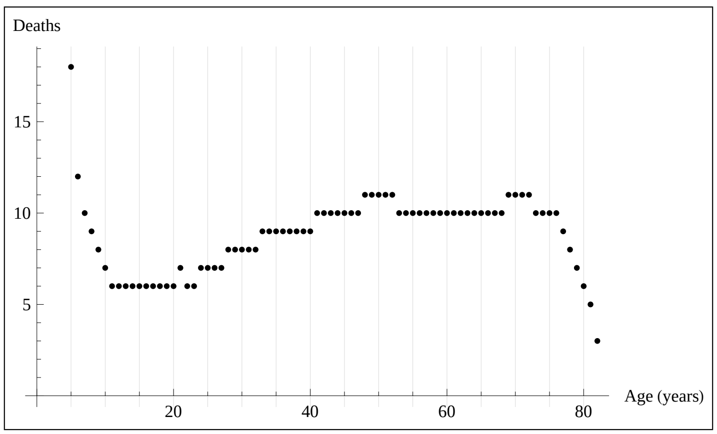

Figure 6.3: Graph of the expected number of deaths at a given age, out of 1000 children below the age of 1, based on Edmond Halley's life table (table A.6).

In a group of people of this size, it is quite unlikely that an exactly equal number of deaths occurred each year for several years in a row, yet this is what Halley's table suggests. Because of this, we can safely say that Halley adjusted the data in his table to artificially create these long periods of constant mortality rate[8]. The graph also shows an exactly linear trend in the age ranges from 7 to 11 and from 76 to 81, which appears to be another way in which Halley altered his data somewhat.

[7]From '*Estimate*' [30], p. 484. It is remarkable that Halley apparently realized as early as 1693 that uncertainty decreases as the number of trials increases. It is true that this had been discovered much earlier: Gerolamo Cardano published the theorem in 1565, although without a mathematical proof; a proof was first given by Jacob Bernoulli in his '*Ars conjectandi*' (1713) [42]. Still, many scientists held on to the belief that uncertainty would only increase if more observations were considered in a calculation; this view only really started to change around 1750[49]. Therefore, in this observation, Halley was ahead of his time.

[8]The *mortality rate* is the (absolute) number of deaths per year.

Chapter 7

Life and work of Nicolaas Struyck

7.1 Nicolaas Struyck (1686-1769)

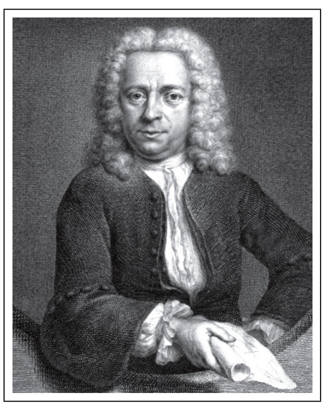

Figure 7.1: Portrait of Nicolaas Struyck, drawn by J.M. Quinkhart, engraved by J. Houbraken, 1738

Not very much is known about the life of Nicolaas Struyck[1] other than some entries in Amsterdam city records. He was born in Amsterdam in May of 1686[2]. His father, also called Nicolaas, was a goldsmith; the younger Nicolaas himself was registered in 1724 as 'matthesius' (mathematician) [3]. However, he was active in many other fields as well, including astronomy, geography and biology.

Struyck did not go to university; what kind of education he did have, is unknown.[3]

Struyck published his first work in 1716: *'Uytreekening der kansen in het speelen'* (*'Calculation of chances in play'*)[51]. In this book, Struyck finds the expected values for many different kinds of games of chance. He treats a number of explicit examples, but also gives formulas for some more general cases. He also makes some calculations about interest and annuities. For example, he answers the question after how many years the beneficiary of an annuity and another person saving his money at a certain interest rate, will have the same amount of money.

In 1740, Struyck published his magnum opus, titled *'Inleiding tot de algemeene geographie, benevens eenige sterrekundige en andere verhandelingen'* (*'Introduction to general geography, as well as some astronomical and other discourses'*)[52]. These 'other discourses' contain the work Struyck is now most famous for: life tables, calcu-

[1]Also spelled Struijk, Struijck, Struick; however, 'Struyck' is the spelling he himself used [52].

[2]He was baptized on May 21st, 1686 [2]. This seems to disagree with [44] (and several sources referring to it), in which his date of birth is stated as May 19th, 1687.

[3]Struyck had an uncle, Nicolaas Blancke, who was a schoolteacher[3]; Zuidervaart [63] guesses that he may have taught Nicolaas Struyck.

lations of life expectancy and the values of life annuities. However, in his own time, Struyck was more successful as an astronomer.

Struyck's work *'Viae cometarum'*, 1749, contained calculations of the orbits of several comets, some of them done by Struyck himself, others by friends and foreign colleagues. In 1753, Struyck published *'Vervolg van de beschryving der staartsterren, en nader ontdekkingen omtrent den staat van 't menschelyk geslagt'* (*'Continuation of the description of comets, and more discoveries about the state of the human race'*)[53]. This book was meant as a sequel to his *'Inleiding'*, containing among other things a classification of comets by the shape of their orbit and a detailed analysis of a population census Struyck had organized in a number of villages in North-Holland.

Nicolaas Struyck remained unmarried and died on May 15, 1769,[44] leaving behind a substantial inheritance, including several houses in Amsterdam [4]. In the record of his burial [5], it was noted that he was a Fellow of the Royal Society of London and a member of the *Hollandsche Maatschappij der Wetenschappen* (Holland Society of Sciences), as well as a correspondent of the *Académie Royale des Sciences* in Paris. Still, his influence remained mostly limited to the Netherlands, due to the fact that he published most of his work in Dutch [62].

7.2 Analysis of Nicolaas Struyck's life tables

The final three sections of Struyck's *'Inleiding tot de algemeene geographie'* are dedicated to demography. In order, they are:

- *'Gissingen over den Staat van 't Menschelyk geslagt'* (*'Conjectures about the state of the human race'*)
  This section consists largely of estimates of the populations of several countries and how these populations change over time.

- *'Uitrekening van de Lyfrenten'* (*'Calculation of life annuities'*)
  In this section, Struyck demonstrates how to calculate the values of life annuities from data on life expectancy.

- *'Aanhangsel op de Gissingen over den Staat van 't Menschelyk Geslagt, en de Uitrekening der Lyfrenten'* (an appendix to the previous two sections).

On pages 363 and 364 of his *'Aanhangsel'*, Struyck published the tables displayed in A.10 and A.11, which contain the number of survivors every five years out of a starting number of 794 men and 876 women of varying ages. He derived the data for these tables from records of the life annuities sold in Amsterdam between 1672 and 1674. Note that these were the life annuities for which Johannes Hudde may have set the prices; we will investigate in section 7.2.2 whether these prices had any effect on the numbers of registered annuitants and therefore on the quality of Struyck's source data.

Further on in his book, on page 377, Struyck presented tables A.8 and A.9, which give the yearly expected number of survivors out of an initial group of 710 men and 711 women. Struyck mentioned that he based these tables on the previous ones, but he did not explain how. This is what I will investigate in this section.

In his book, in the middle of Struyck's explanation of tables A.10 and A.11, he remarked the following.

> (...) dog om in de Lyfrenten te gebruiken, heb ik maar de volle half Jaaren gereekend, die dezelve geleefd hebben (...)[4]

This might mean that Struyck had more accurate data available than he included in his tables. Indeed, it seems likely that the authorities would have kept at least yearly data of their annuitants, because they would have to know whether they needed to pay out the annuity benefits in that year, or not. However, for now, I will assume that Struyck really did base his second pair of tables on the first pair, like he wrote.

[4]From [52], page 365. Roughly translated: "(...) though for use in life annuities, I have reckoned only the full half years that they have lived (...)"

7.2.1 How to read Struyck's initial tables

The triangular tables A.10 and A.11 look quite complicated. For that reason, it seems practical to include a paragraph on how to read these tables, before we continue.

Let us look at table A.10. Every column in this table denotes a five-year age category (0-4 years, 5-9 etcetera). The first row of data (starting with the number 100 in the first column) tells us that 100 male annuitants of ages 0-4 were registered; five years later, 95 of these remained, and so on. The second row (starting with 110 in the second column) tells us that 110 male annuitants of ages 5-9 were registered, of which 107 survived the next five years, etcetera. The third row contains no new data, but is equal to the sums of the numbers in the first two rows, leaving out the first column (so it starts in the second column with 95+110=205).

After that, every even-numbered row gives the number of annuitants of a certain age and how many survived each subsequent period of 5 years, while every odd-numbered row contains the sum of the previous two rows.

The reason for this is that Struyck used a similar method to the one used in section 5.6 in the calculation of a life table from Johannes Hudde's data. In that section, we used data on one-, two-, three- and four-year-old annuitants who had survived to age 5, to supplement data on five-year-old annuitants, because to find a life table that was as accurate as possible, we had to use as many of our data as possible. In Struyck's case, we can see in the table that 95 of the male annuitants of ages 0-4 survive the next five years. Now the trick is to pretend that another annuity was bought on these annuitants after these five years, making them, as it were, newly registered annuitants aged 5-9. We can then take these 95 annuitants of ages 5-9 together with the 110 annuitants of ages 5-9 from the second row, to obtain the total of 205 annuitants of age 5-9 displayed in the third row. If we now want to calculate the life expectancy of a boy between ages 5 and 9, we can use the third row instead of the second one to obtain more accurate results.

Table A.11, for female annuitants, can be read in the same way.

7.2.2 The source data

In the period 1672-1674, the city of Amsterdam sold life annuities in two negotiations: the first one lasted from July to September 1672, and the second from January 1673 to February 1674. Soon after the sale of the 1672 annuities began, it became clear that unexpectedly many of the registering annuitants were over 50 years of age. According to Commelin [19], the authorities feared that this would result in a net loss, so it was quickly decided that annuitants over the age of 50 would no longer be accepted; we will investigate the reasons behind this in the next paragraph. When the second negotiation started in January 1673, annuitants over 50 were again accepted, only this time the prices were different. The prices for annuitants younger than 50 stayed the same as in table 5.1, but for anyone over the age of 50, the price was set at 800 guilders instead of the lower prices seen in table 5.1 [19, 52].

Remarkably, we saw in section 5.6 that the city authorities were practically bound to make a net loss, whatever the age of the annuitants. In fact, the loss would be smaller for older annuitants[5]. The authorities may well have known this: burgomaster Hudde, at least, was intimately acquainted with Johan de Witt's '*Waerdye*', from which it follows easily. And even if they were unaware that annuities on older annuitants cost them less money than those on younger annuitants, I see no reason why they would assume the opposite. Therefore, I think that the real reason to discourage people over the age of 50 from registering as annuitants, was because the prices of the resulting annuities would be low in comparison with the yearly payout. This meant that relatively many annuities had to be sold to raise the required money for the city's defense, so that the city had to pay out a lot of money in the first few years, when most annuitants would still be alive. So although the net loss for the city might well, eventually, turn out to be a lot smaller if the annuitants were older, the short-term benefits would be smaller as well. Apparently, the short-term effects were deemed much more important than the long-term ones in a time of crisis.

Whatever the reason behind the measures, the result was that fewer annuitants over 50 were registered than would otherwise have been the case. In table A.10, the number of annuitants registered at ages 55-59 is only 20, while the number at ages 50-54 was 43. Likewise, in table A.11, the number of annuitants aged 55-59 is 28, while the number at ages 50-54 is 59.

It is remarkable that this sudden change occurs between age categories 50-54 and 55-59 instead of between 45-49 and 50-54. This may be partly because the restriction on older annuitants apparently only applied to annuitants *over* 50, so that 50-year-olds were still accepted throughout the first negotiation of life annuities. It also might be that buyers simply found 800 guilders still to be an acceptable price for a life annuity on a 50-year-old, but not on a 55-year-old.

However, because of Struyck's method of calculation, described in paragraph 7.2.1, it did not matter much how many older annuitants were registered. He could simply use the numbers of younger annuitants who had survived to a given age to supplement the number of annuitants registered at that age. Therefore, the relatively low numbers of older annuitants would not have much effect on the accuracy of Struyck's life tables.

7.2.3 A starting point

To create tables A.8 and A.9 from the data in tables A.10 and A.11, Struyck would have needed to pick a starting point: an age and a corresponding number of people, of whom he could then calculate the expected number of survivors every five years to obtain data for the rest of the table. He chose to start his tables with 710 five-year-old boys and 711 five-year-old girls. I found no reasonable way in which he could have obtained these numbers from his earlier tables. However, the numbers are remarkably close to those occurring in Edmond Halley's life table. Recall from

[5] We saw an illustration of this earlier, in graph 5.7.

the section on Halley's table that although the table starts with age 1, the number 1000 next to it is in fact the number of children *under* the age of one. Likewise, any number n in the 'Age' column of Halley's table in fact refers to the age period from $n-1$ to n. Therefore, although the number 710 appears at age 6 in Halley's table, it denotes the number of five-year-olds. This is exactly equal to the number of five-year-old boys in Struyck's table on men.

Struyck definitely knew Halley's work on life tables, as he mentioned it several times in his *'Aanhangsel'* and compared many of the results of his calculations with the ones Halley had found. It is therefore quite possible that Struyck chose the number of 710 five-year-old boys to match the one in Halley's table. The observation that the mortality rate among young boys was slightly higher than the one among young girls might then explain his choice to start his life table of females with 711 girls.

I found no clues to figure out how Struyck proceeded to create his life tables. The following, therefore, is my best guess of how he could have done it, followed by a comparison of the data I obtained with Struyck's life tables.

Having picked his starting points, Struyck would have had to create tables based on tables A.10 and A.11, in which for every five years, the number of survivors out of 710 five-year-old boys, respectively 711 five-year-old girls, would be recorded. These could then be interpolated to find his life tables.

From the calculations on life annuities Struyck made in his book, it is clear that his tables are to be read in such a way that the average age of people in the interval from 0 to 4 years in tables A.10 and A.11, is 2.5 years. Likewise, the average age of 5-year-olds in tables A.8 and A.9 is in fact 5.5 years. Consequently, the first two columns in table A.10 can be read as 'of every 100 boys aged 2.5, an expected 95 will survive to age 7.5'. Assuming for a moment that the mortality rate is roughly constant, this means that there remain approximately 97 boys aged 5.5 of every 100 boys aged 2.5. A starting point of 710 boys aged 5.5 therefore gives the same result as a starting point of $\frac{710}{97} \times 100 \approx 732.0$ boys aged 2.5. Likewise, a starting point of 711 girls aged 5.5 gives the same results as a starting point of $\frac{711}{74} \times 77 \approx 739.8$ girls aged 2.5.

Now it is easy to calculate the expected number of survivors out of 732 boys and 740 girls in the age interval 0-4, in periods of five years, from Struyck's tables A.10 and A.11. Graphs 7.2 and 7.3 compare my calculated data to Struyck's actual life tables.

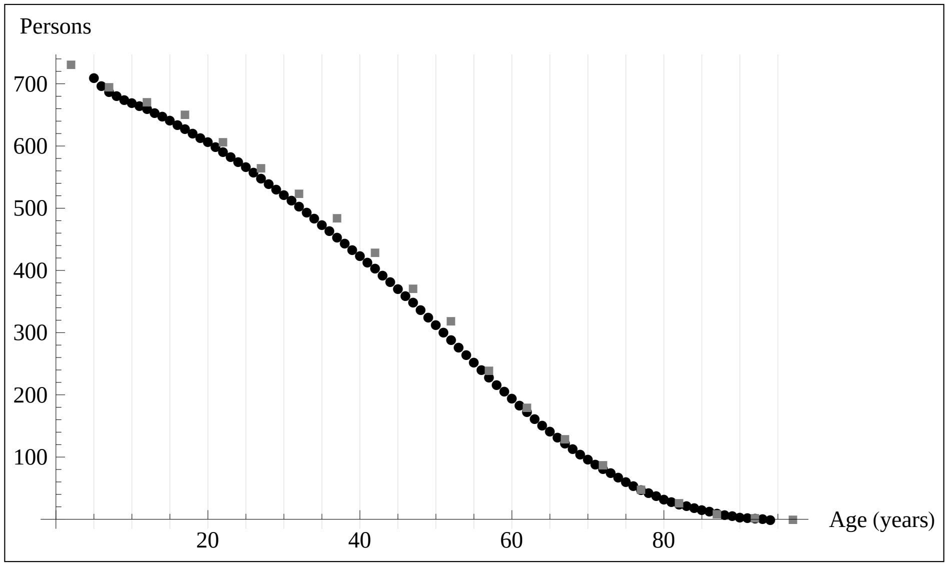

Figure 7.2: Combined graph of Struyck's life table for men (grey circles; table A.8) and the expected number of survivors out of 732 boys between the ages of 0 and 5, as calculated by me, based on Struyck's table A.10 (black squares).

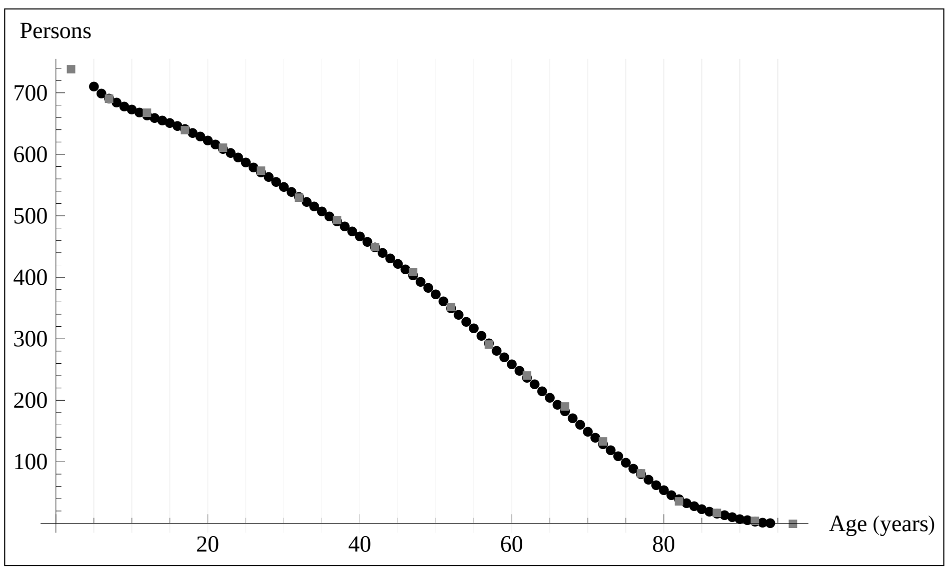

Figure 7.3: Combined graph of Struyck's life table for women (grey circles; table A.9) and the expected number of survivors out of 740 girls between the ages of 0 and 5, as calculated by me, based on Struyck's table A.11 (black squares).

The similarity in the second graph between my calculated data and the data from Struyck's life table, is striking. However, in the first graph, there are clearly visible differences. Struyck may have used a similar method to the one I used, whereby he somehow found a lower starting number of boys. However, it seems that this would not completely solve the problem, because the shapes of both graphs in figure 7.3 are different. More likely possibilities seem to be that either Struyck adjusted his data somewhat, like Halley, to obtain a slightly smoother graph, or that he indeed had more data available than he displayed in tables A.10 and A.11.

To decide which one of these options is more likely, we can look at how 'smooth' the data in tables A.8 and A.9 are.

7.2.4 Interpolation

In Struyck's first pair of tables, the data are recorded in five-year intervals. If Struyck constructed his second pair of tables on the basis of these data, he would therefore have needed to interpolate the data in some way.

We saw in the section on Edmond Halley that he adjusted his data to smooth out irregularities, and that this resulted in large parts of his table developing linearly. If Struyck interpolated his data as well, linear interpolation would be a straightforward way to do this. To investigate this possibility, let us consider two more graphs like figure 6.3, only this time based on Struyck's life tables.

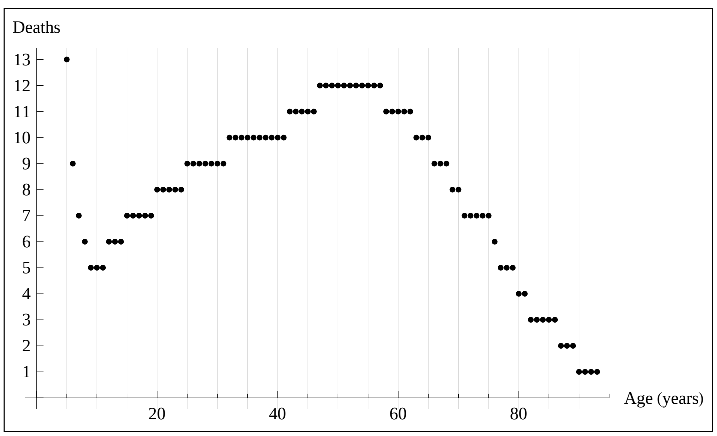

Figure 7.4: Graph of the expected absolute number of deaths at a given age (mortality rate), out of 710 five-year-old boys, based on Struyck's life table for men (table A.8).

Although the mortality rate is constant in many intervals in figures 7.4 and 7.5, it does not occur as much or as regularly as one might expect in a table that was interpolated from one entry for every five years to one entry for every year. For example, I would have expected the graph in figure 7.4 to be smoother between ages 70 and 80 than it is now: the mortality rate stays equal to 7 for a number of years, then drops quite suddenly to 5 and then stays at that level for several years. Additionally, I would have expected it to happen a lot more often that the mortality rate was constant for exactly 5 years, as a result of interpolation from 5-year data.

I think we can therefore safely assume that Struyck did not base his life tables on exactly these data; presumably, he had more data available than he shared in his book. Even though this means I cannot exactly replicate the calculations for Struyck's life tables, we can still look at his calculations on life annuities.

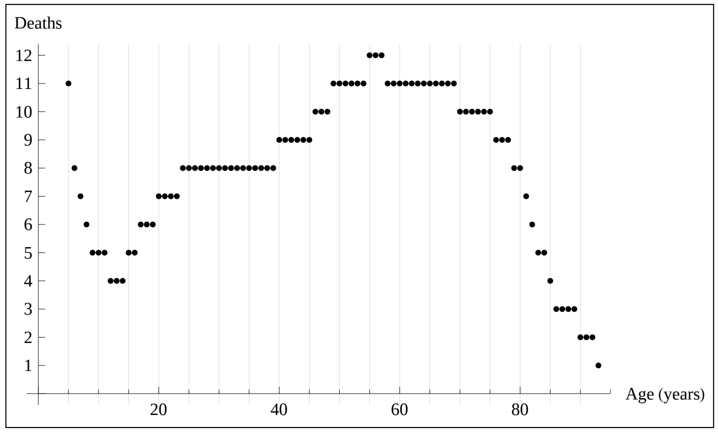

Figure 7.5: Graph of the expected absolute number of deaths at a given age (mortality rate), out of 711 five-year-old girls, based on Struyck's table for women (table A.9).

7.2.5 Values of life annuities

After presenting his life tables, Struyck used them for calculations of the values of life annuities, resulting in tables A.12 and A.13. As was the case with his life tables, Struyck did not explain his methods for these calculations very clearly. However, he very probably used the same method as De Witt and Halley. Struyck certainly knew De Witt's work, as he referred to it several times in his book. What follows now, therefore, is a reconstruction based upon De Witt's work and Struyck's bits and pieces of explanation.

Struyck considered the case of a life annuity which pays out 100 guilders yearly, with an interest rate of 2.5%. However, Struyck also took into account a tax on annuities, as a result of which the beneficiary would only receive 80 guilders yearly. Diligently, Struyck calculated the value of a life annuity separately for all four of his source tables[6]. I will first consider the second set of tables, A.8 and A.9.

First, Struyck would have calculated the total value of yearly payments of 80 guilders over $b - a$ years when the interest rate on savings is 2.5%. For this, he probably used the same method as De Witt did. Let v_i denote the total value of yearly payments of 80 guilders over i years, then De Witt's method results in the formula (in modern notation)

$$v_{b-a} = \sum_{k=1}^{b-a} 80 \cdot \left(\frac{100}{102.5}\right)^k \qquad (7.1)$$

[6]Struyck wrote that he did this "om de ongelykheden te vergoeden" ("to compensate for the inequalities"), i.e. to compensate for errors introduced by rounding off [52], page 367.

or, using the formula for the sum of the first $b-a$ terms of a geometric series[7],

$$v_{b-a} = 80 \cdot \left(\frac{1 - (\frac{100}{102.5})^{b-a+1}}{1 - \frac{100}{102.5}} - 1 \right) \tag{7.2}$$

Next, Struyck would have found the probability that an annuitant of age a will live to age b but not to age $b+1$. If we let n_i denote the number of people alive at age i, and $Q(j|i)$ (with $j \geq i$) the probability that an i-year-old will live to age j but not age $j+1$, then this probability can be found as

$$Q(b|a) = \frac{n_b - n_{b+1}}{n_a} \tag{7.3}$$

that is, the number of people who die between ages b and $b+1$, divided by the number of people alive at age a.

Now for an annuitant of age a, the total expected value V_a of a life annuity can be found as

$$V_a = \sum_{b \geq a} Q(b|a) \cdot v_{b-a} \tag{7.4}$$

We now have these expected values for all ages a. To be able to compare our outcomes with Struyck's, we need to find expected values for the age categories 5-9, 10-14 etcetera. For this, we need to find expected values for ages 7.5, 12.5, etcetera, since 7.5 is the average of age category 5-9 and so on. One possible way to do this, would be the following.

First we look up the numbers of people alive at age 7 and at age 8 (that is, n_7 and n_8), according to tables A.8 and A.9. Now the expected value at age 7.5 can be calculated as follows:

$$V_{7.5} = \frac{V_7 \cdot n_7 + V_8 \cdot n_8}{n_7 + n_8} \tag{7.5}$$

In this way, we have obtained a weighted average of the values at ages 7 and 8, according to the number of people alive at these ages. The results, for men and for women, are displayed in table 7.1.

The data do not agree exactly. Struyck probably used a similar calculation, although I did not find one for which the results were closer to Struyck's results. I tried several different ways to take averages, including averages over more than two data points, unweighted averages etcetera. Of course, it may well be that I missed some possibilities, or that I made a mistake in some calculation. However, it may also be that Struyck again used slightly different data than were included in his life tables, even though he explicitly stated that the results were based on his life tables.

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[7] Struyck probably used this method as well: in his '*Uytreekening*'[51], page 66, he states the formula $\frac{\overline{b^q-1}\times a}{b-1}$ for the sum of the finite geometric series $a + ab + ab^2 + \ldots + ab^{q-1}$. The line over $b^q - 1$ is Struyck's notation for what we would now denote with brackets.

Ages	Struyck Males	My results Males	Struyck Females	My results Females
5-9	ƒ 1823	ƒ 1815	ƒ 1931	ƒ 1926
10-14	1714	1712	1840	1839
15-19	1608	1608	1733	1729
20-24	1504	1504	1630	1630
25-29	1401	1400	1533	1537
30-34	1291	1292	1438	1437
35-39	1184	1185	1328	1325
40-44	1069	1068	1203	1204
45-49	955	955	1077	1080
50-54	840	853	964	963
55-59	756	758	851	852
60-64	661	665	733	732
65-69	575	574	616	611
70-74	481	477	493	491

Table 7.1: Comparison of prices of life annuities as calculated from tables A.8 and A.9 by Struyck and by me.

Now let's calculate the same values from Struyck's other pair of tables, the ones displayed in tables A.10 and A.11, to see if our results match Struyck's better this time.

In these tables, the ages are grouped into five-year categories, where the first category is 0-4 years, the second is 5-9 years and so on. To find the probability that an annuitant in the a-th age category will live to reach the b-th but not the $b+1$-st, we can proceed as follows. In the row of the table that starts in a column a, we can read off the total number of people in age category a, as well as the number of these people who were still alive in age categories b and $b+1$.

Now we will also need the probability that an annuitant in the a-th age category survives some number of years y and then dies within the next year. First let $Q(a|a)$ be the probability that an annuitant in age category a will not survive the next five years. In tables A.10 and A.11, this corresponds to the probability that a person from age category a is not counted as a survivor to category $a+1$. Likewise, let $Q(a+1|a)$ be the probability that the annuitant will survive the next five years, but not ten, etcetera. Then assuming the deaths in each age category are evenly distributed, we can take $\frac{1}{5}Q(a|a)$ to be the probability the annuitant will not survive the first year. Continuing in this way, we can calculate for all possible y, the probability that an annuitant from age category a will survive the next y, but not $y+1$ years.

The corresponding total values of the yearly payouts can now be calculated much like before, for each of the age categories. The results can be seen in table 7.2.

Again, the data do not match perfectly. Presumably Struyck used a somewhat different calculation, or perhaps he made use of other data. Still, I think we can safely say that we have found out what the data in his table mean, and roughly how they have been calculated.

Ages	Struyck Males	My results Males	Struyck Females	My results Females
5-9	*f* 1826	*f* 1818	*f* 1936	*f* 1932
10-14	1721	1721	1832	1831
15-19	1600	1595	1737	1736
20-24	1503	1504	1627	1628
25-29	1417	1417	1524	1526
30-34	1303	1302	1448	1447
35-39	1162	1163	1334	1334
40-44	1057	1057	1221	1222
45-49	944	946	1076	1077
50-54	809	815	969	969
55-59	754	758	884	883
60-64	671	675	753	752
65-69	577	576	613	615
70-74	469	475	493	494

Table 7.2: Comparison of prices of life annuities as calculated from tables A.10 and A.11 by Struyck and by me.

7.2.6 Further calculations on life annuities

Having thus calculated the sum of money that should be paid for a given yearly payout, Struyck also answered the reverse question: if the buyer pays 100 guilders for a life annuity, then how high should the yearly payout be?

For a female annuitant between the ages of 0 and 5, 1936 guilders is a fair price to pay for a yearly payout of 100 guilders. Therefore the yearly payout of such a life annuity should be $\frac{100}{1936}$ times the price paid for the annuity. The yearly payout for a life annuity costing 100 guilders should thus be $\frac{10\,000}{1936} \approx 5.165$ guilders; rounded to the nearest half of a stuiver, this amounts to 5 guilders and $3\frac{1}{2}$ stuivers. Repeating this calculation for the other age categories and for male annuitants, we find the values in column D of Struyck's tables A.12 and A.13.

If the 20% tax is taken into account, the net yearly payout is $0.8 \cdot \frac{10000}{1936} \approx 4.132$ guilders, which can be rounded to 4 guilders and $2\frac{1}{2}$ stuivers. In this way, we find the entries in column C of Struyck's tables A.12 and A.13.

The remaining column in each of these tables, denoted column E, contains lengths of time, expressed in years and months. For this column, Struyck did explain his calculation. His question was: how long would one have to receive a fixed yearly payout of 80 guilders, to get exactly the same total value as a given life annuity?

For a female annuitant in the age category 10-14, the calculation would go as follows. Let a life annuity on this annuitant be worth 1840 guilders, as in table A.13, column B, with a yearly payout of 80 guilders. Assuming the annuitant will still be alive in one year, the first payout will take place then. At the time of purchase, taking into account an interest of 2.5%, the current value of that payout is 80/1.025 guilders. Likewise, the current value of a payout that will take place x years after purchase, is $80/1.025^x$ guilders. Using the sum formula for a finite geometric series, the total value of all payouts in the first x years will be

$$80 \cdot (100/102.5) \cdot \frac{1-(102.5/100)^x}{1-(102.5/100)}. \tag{7.6}$$

Now we want this value to be equal to 1840 guilders; solving the equality for x yields $x \approx 34.65$. Rounded to the nearest month, this is 34 years and 8 months. This is exactly the value seen in column E of table A.13.

7.3 Relation between Struyck's life table and work by others

7.3.1 Johannes Hudde's data

Both Hudde and Struyck used data from life annuities sold in Amsterdam as a basis for their work. For this reason, it may be interesting to compare their tables directly.

In section 5.6, we demonstrated how to calculate a life table from Hudde's table of data. We chose a starting point of 1000 one-year-olds and rounded off every number to the nearest integer to obtain the life table displayed in A.5. Now to compare Hudde's data to Nicolaas Struyck's life tables, both will need to have the same starting point. Because Struyck's table for males starts with 710 boys of age 5, and his table for females with 711 girls of age 5, I will let the life table based on Hudde's data start with 710.5 five-year-olds. A graph comparing Struyck's life tables to the one calculated in this manner from Hudde's data, is given in figure 7.6.

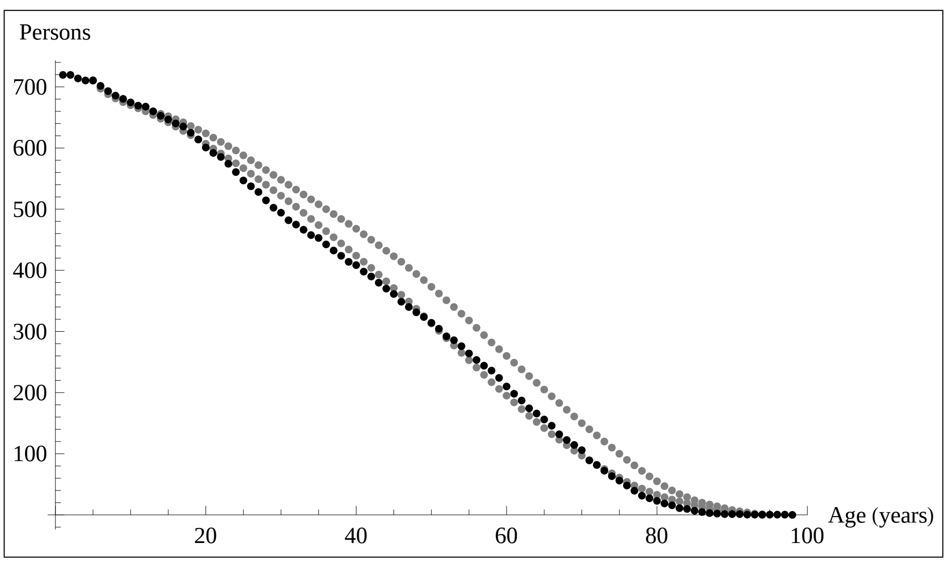

Figure 7.6: Graphs of Struyck's life tables for men and women (grey) and a life table calculated from Hudde's data (black)

From age 5 almost up to 20, the life table based on Hudde's work is very close to Struyck's. In the age interval from 20 to 30 however, there is a sharp decline in Hudde's graph compared to Struyck's. After age 30, Hudde's graph gradually converges back towards Struyck's. Apparently, there was a considerably higher mortality among twenty- to thirty-year-olds in the group of people that Hudde based his table on than in the group that Struyck considered, even though at other ages, the mortality rates agreed quite well. Hudde's data come from life annuities sold in Amsterdam between 1586 and 1590, while Struyck's data come from life annuities sold in Amsterdam between 1672 and 1674. Therefore, the difference between both data sets can almost certainly be explained by the different time periods in which they were collected. A possible explanation may be the Eighty Years' War, which lasted from 1568 until 1648 (with a truce in the period 1609-1621), and which presumably cost the lives of many young men during the time in which Hudde's data were collected.

It is also apparent from figure 7.6 that the graph based on Hudde's data fluctuates more sharply than the graphs based on Struyck's life tables. This is despite the fact that Hudde used data on 1495 annuitants, while Struyck only had data on 794 men and 876 women; it would thus be reasonable to expect that Hudde's graph would look smoother than both of Struyck's. As this is not the case, it seems like Struyck manually smoothed out the data in his life tables.

7.3.2 Halley's life table

Figure 7.3.2 shows a comparison between Struyck's life tables and Halley's table. The

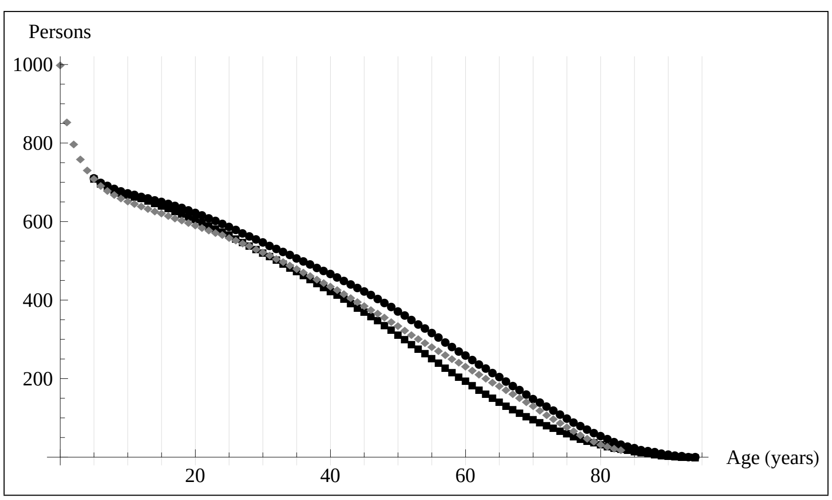

Figure 7.7: Combined graph of Struyck's life tables for males (black squares) and females (black circles) and Halley's life table (grey diamonds).

correspondence between all three tables at five years of age, where both Halley's table and Struyck's table for men give 710 and Struyck's table for women 711, is striking. It is also remarkable how similar the shapes of the graphs are: apparently the mortality rates in the general population of Breslaw between 1687 and 1691, were very similar to the mortality rates of life annuitants in Amsterdam registered between 1672 and 1674.

In the section on Edmond Halley's life table, we saw that he had data on the sexes as well as the ages of the people his tables were based on. Still, he did not separate men and women in his life table. Now that we have seen the differences Nicolaas Struyck found between the sexes, it might be interesting to investigate whether Halley could have found significantly distinct tables if he had tried to make separate tables for both sexes.

Halley based his table on data of 5869 deaths. Struyck used data on the lives of 794 male and 876 female annuitants, that is, a total of $794 + 876 = 1670$ people. Because Halley had many more data available than Struyck, the uncertainty in his calculations would have been much smaller. I think we can also safely assume that on average, women in Breslau lived longer than men: this phenomenon has been observed in practically all civilizations (see also section 7.4.3). Therefore, seeing as Struyck found significant differences between the mortality rates for men and women, Halley would very probably have found even more significant differences, if he had separated men and women in his table.

The reason he did not publish separate tables, might of course still be that the difference was too insignificant to make it worthwhile for Halley to do all his calculations twice, once for each table. However, it is also possible that the idea of distinguishing between the sexes simply did not occur to him.

7.4 The place of Nicolaas Struyck in the history of life tables and life annuities

7.4.1 Struyck's reputation in his own lifetime

It is not easy to decide how important Nicolaas Struyck has been for the development of the mathematics of life tables and life annuities. The name Nicolaas Struyck does not occur in every text on the history of life tables or life annuities, but neither is it rare to find. Struyck has been called 'the greatest Dutch mathematician of his time'[8], but then again, he did not have much competition for this title; the Golden Age and the era of Huygens, Hudde and De Witt had passed, and no great minds had taken their places.

During his own lifetime, he was recognized mostly for his work in astronomy. His work on life tables gained some attention in the Netherlands, but did not travel far internationally, because it was written in Dutch. After Struyck's death, his life tables were soon mostly forgotten. This, however, was not because they were not useful.

[8]'Nicolaas Struyck, van Amsterdam, was wel de belangrijkste Nederlandse wiskundige van zijn tijd.' From [50], page 174.

An important reason for the fact that Struyck's work did not receive its due credit, were the accusations of plagiary by fellow mathematician Willem Kersseboom.

7.4.2 Willem Kersseboom (1692-1771)

Willem Kersseboom was born on January 9, 1692[9], as the son of the mayor of Oudewater, a village near Woerden in the Netherlands. He studied law at the university of Leiden, but also had a keen interest in mathematics.

In 1724, Grand Pensionary Isaac van Hoornbeek instructed Kersseboom to review Johan de Witt's *'Waerdye'*. From that time on, the government regularly consulted Kersseboom about many financial issues, including plans for lotteries and life annuities.

From 1738 to 1742, Kersseboom published *'Verhandelingen tot een proeve om te weten de probabele meenigte des volks in de provintie van Hollandt en Westvrieslandt'* (*'Treatises on a test to know the probable number of people in the province of Hollandt en Westvrieslandt'*), in several parts. Apart from estimates of the number of people in a number of different cities and regions, this collection of treatises also contains both a life table and a table of values of life annuities.

In 1740, shortly after Struyck's book was published, Kersseboom wrote *'Eenige aanmerkingen op de Gissingen over den staat van het menschelyk geslagt, uitreekening van de lyfrenten en 't aanhangsel op beide'* (*'Some remarks about 'Gissingen over den staat van het menschelyk geslagt',' Uitreekening van de lyfrenten' and the appendix to both'*). In this eighteen-page treatise, he accused Struyck of plagiary rather viciously.

> Moet onze Schryver niet een uitmuntenden verrekyker gehad hebben, toen hy de voorsz twee Periodes op Paginis 333 en 335 van zyn Gissingen ter needer stelde, dat hy, zo net voor uit kon zien, wat ik meenen, en uit KING bybrengen zoude? Of dunkt het U, Myn Heer, en yder Edelmoedig Leezer, niet overtuigender, dat hy die zaaken in mijn voorsz. *Verhandeling tot een Proeve* eerst als wat nieuws geleezen hadde?[10]

Struyck never responded to these accusations [27], which might have led people to assume Kersseboom was right. In any case, it was Kersseboom's work that was most remembered. Even in 1876, the Dutch government still required all insurance companies to use Kersseboom's life table for the calculation of the values of life insurances. However, much later (around 1925), Van Haaften would convincingly

There are no known portraits of Willem Kersseboom.

[9] According to Claessens [17], who refers to baptism records of Oudewater. This disagrees with [28] and other sources, which state his year of birth as 1691.

[10] From [39], page 7. Freely translated: 'Must our Writer not have had an excellent telescope, when he wrote down the previously mentioned two Periods on pages 333 and 335 of his *Gissingen*, that he could predict, what I would think, and deduce from KING? Or do You not think, My Lord, and every Noble Reader, it would be more convincing, that he had read these things in my previously mentioned *Verhandeling tot een Proeve*?'

argue that Kersseboom's accusations were baseless [27]. Furthermore, it was stated in *'Bouwstoffen'* [44] that 'in general knowledge, development and civilization, he is below Nicolaas Struyck; although an excellent arithmetician, his algebraic knowledge does not seem to have been extensive'[11] and Schraa [45] writes that Kersseboom 'was much less meticulous'[12].

Willem Kersseboom died in The Hague in September 1771 [17, 28].

7.4.3 Different life expectancies for men and women

Struyck's recognition of the fact that men and women have different life expectancies, was quite important. For some time, it was assumed that it was a consequence of differences in lifestyle: men were exposed to more danger, they could have dangerous professions, and even young boys played rougher games than girls. But the differences in mortality rate were visible at all ages, and it gradually became clear that the same difference can be observed in practically all life tables of humans, no matter from which time, place or society.

A clear and rather beautiful illustration of this can be seen in figure 7.8, which shows that in modern times, life expectancy for females is higher than that for males in nearly every country in the world.

It is clear from the image that in many countries, the difference between male and female life expectancy is substantial, often several years. For that reason, one might expect insurance companies to charge different prices for life insurances for men and women. And indeed this is true, although it is currently being abolished in Europe. The European Court of Justice decided in 2011 that charging different insurance premiums for men and women constitutes discrimination. As a result of this, insurance companies will have to charge the same prices for both sexes starting December 21, 2012 [11].

[11] On page 122: 'In algemeene kennis, ontwikkeling en beschaving staat hij beneden Nicolaas Struyck; ofschoon een uitmuntend cijferaar, schijnt zijne algebraïsche kennis niet uitgebreid te zijn geweest.'

[12] On page 6: 'Willem Kersseboom schreef in dezelfde tijd als Struyck, maar was veel minder nauwkeurig.'

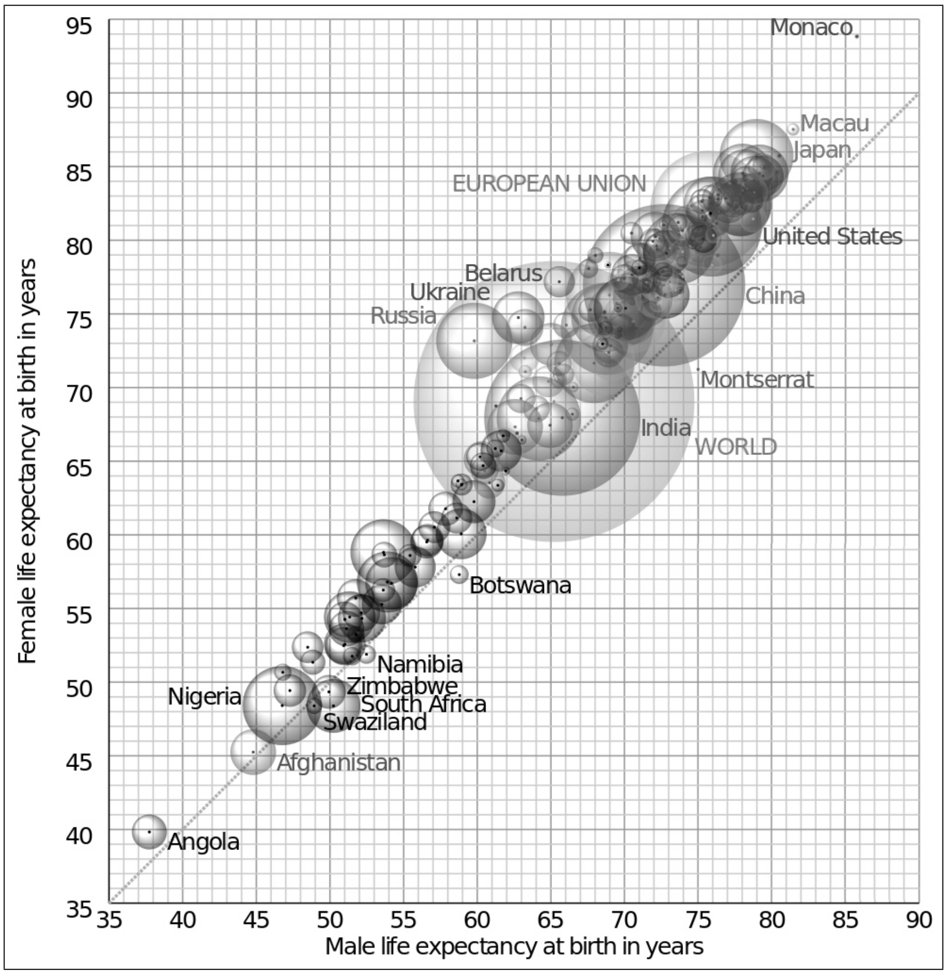

Figure 7.8: Comparison of male and female life expectancy for different countries. The apparent 3D volumes of the bubbles are linearly proportional to their population. Source data from CIA Factbook 2011. Image from Wikipedia.org.

Chapter 8

Analysis of human mortality

8.1 A change in mathematical thought

After the rapid developments in the seventeenth and eighteenth centuries, there were no groundbreaking inventions related to life tables and life annuities for some time. Correct and general methods for how to create a life table and how to calculate the value of a life annuity from it, were available; at best, one could make small improvements to them.

Meanwhile, the field of mathematics continued to evolve. The industrial revolution, around the end of the eighteenth and the start of the nineteenth century, brought with it a wave of new ways of thinking in many fields, from art to politics. In nineteenth century science, too, old ideas were challenged to make way for new ones. Another important change was the emancipation of mathematics as a science: instead of being considered mainly a tool for fields such as astronomy or mechanics, mathematics became a goal in itself. The main motivation for practicing mathematics shifted from meeting the demands of practical life, to doing mathematics for its own sake. Consequently, more research was done into analytical subjects, and subfields like mathematical logic emerged [50].

The next major discovery in the mathematics of life tables was a good example of these developments. Instead of focusing on how to use data from life tables for practical purposes, Benjamin Gompertz analyzed the data themselves. He discovered what is now known as Gompertz' law of mortality.

8.2 Benjamin Gompertz (1779-1865)

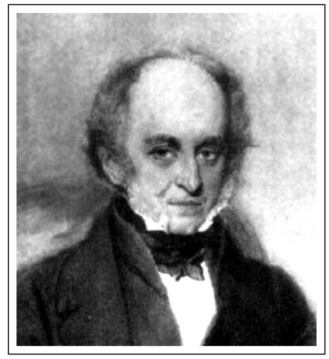

Figure 8.1: Portrait of Benjamin Gompertz, artist unknown (from [34]).

Benjamin Gompertz was born in London on March 5, 1779. He was a descendent of a distinguished Jewish family that originally came from the Netherlands, where his grandfather on the mother's side, Benjamin Cohen, had been a friend of the Stadtholder. Benjamin Gompertz' parents, however, lived in London, where his father was a successful diamond merchant.

Because he was Jewish, Gompertz was not allowed to study at a university. Instead, he taught himself mathematics from an early age. At only nineteen years of age, he became a regular contributor to the journal '*Gentleman's Mathematical Companion*'. In the period from 1812 to 1822, he won the annual prize this magazine offered for the best solution to a mathematical problem, every year, without exception.

Meanwhile, Gompertz worked at the Stock Exchange, in compliance with his father's wish. In 1810, he married Abigail Montefiore, with whom he had two daughters and one son.

From 1806, Gompertz frequently published his work in the Transactions of the Royal Society; he was elected a Fellow of this society in 1819, and from 1832 served on its council. He became a member of the Astronomical Society (now Royal Astronomical Society) soon after its foundation in 1820 and remained an active member for nearly ten years. During this time, he published on such subjects as aberration of light and corrections of certain astronomical instruments, but it is recorded in his obituary in the Memoirs of the Royal Astronomical Society that "he in no other sense ever became a practical astronomer", apparently preferring the theoretical, mathematical side of astronomy to the practical one.

Later in life, Gompertz turned to the subject of life insurance and human mortality, on which he produced the work he is now most famous for. After the death of his only son, Gompertz left the Stock Exchange and focused more fully on mathematics. When in 1824, Gompertz' brother-in-law Sir Moses Montefiore was one of the founders of the Alliance Assurance Company, Gompertz was appointed its first actuary and was admitted into the management. He first published his, now famous, law of mortality in a 1825 paper submitted to the Royal Society. Gompertz was recognized as a great mathematician in his own time, and was frequently consulted by the government on questions regarding mortality and life annuities.

He retired from the Alliance Assurance Company in 1847 due to ill health, but remained an active mathematician until his death on July 14, 1865 [9, 25, 34, 60].

8.3 Gompertz' Law

In Gompertz' paper, he wrote:

> It is possible that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or an increased inability to withstand destruction. If, for instance, there be a number of diseases to which the young and old were equally liable, and likewise which should be equally destructive whether the patient be young or old, (...) the intensity of mortality might then be said to be constant; (...) but if mankind be continually gaining seeds of indisposition, or in other words, an increased liability to death (...) it would follow that the number of living out of a given number of persons at a given age, at equal successive increments of age, would decrease in a greater ratio than the geometrical progression.[1]

One might expect from this quote that Gompertz' law of mortality would somehow take into account both age-independent and age-related causes of death. Interestingly, however, Gompertz did not return his attention to age-independent causes of death in the rest of his paper. Instead, he focused on the second of the causes he identified, proposing that 'a man's power to avoid death' might decrease by equal portions over equal infinitely small intervals of time. Gompertz therefore considered the *force of mortality* (or *mortality force*), that is, the probability that a person of a certain age will die within the next year. In formula form, he wrote the force of mortality as $a \cdot q^x$, where x stands for age and a and q are constants. This became known as the Gompertz law of mortality. In the paper, Gompertz proceeded to show that this law gave a good approximation to several different life tables. That is, he himself noted that it is applicable to a long range of ages, but not to young or very old people.

To illustrate how well a function of this form can approximate observed mortality force, I will use data from a life table compiled by the Dutch mathematician Rehuel Lobatto[2] for men in Amsterdam in the period 1816-1825, that is, in Gompertz' time. Figure 8.2 shows a logarithmic graph of the mortality force calculated from this life table, and an approximation given by Gompertz's law. Note how well these two correspond in the age range from roughly 50 to 85 years.

However, it turned out later that Gompertz' Law could be greatly improved if age-independent causes of death were taken into account. This could be done by simply adding a constant term to the exponential function. Perhaps surprisingly, Gompertz did not realize this himself. In fact, it was over forty years later that actuary William Makeham was the first to make this seemingly small, but significant discovery.

[1] From [25], pages 517-518.

[2] Rehuel Lobatto (June 6, 1797, Amsterdam - February 9, 1866, Delft) was a Dutch mathematician of Jewish descent. He published more than sixty papers on a wide variety of subjects, including three on probability theory and life insurance. Because of his religion, he was not allowed to teach mathematics; instead, he worked as a government official for most of his life. In 1842, he managed to obtain a position as professor of mathematics in Delft, which he kept until his death [48].

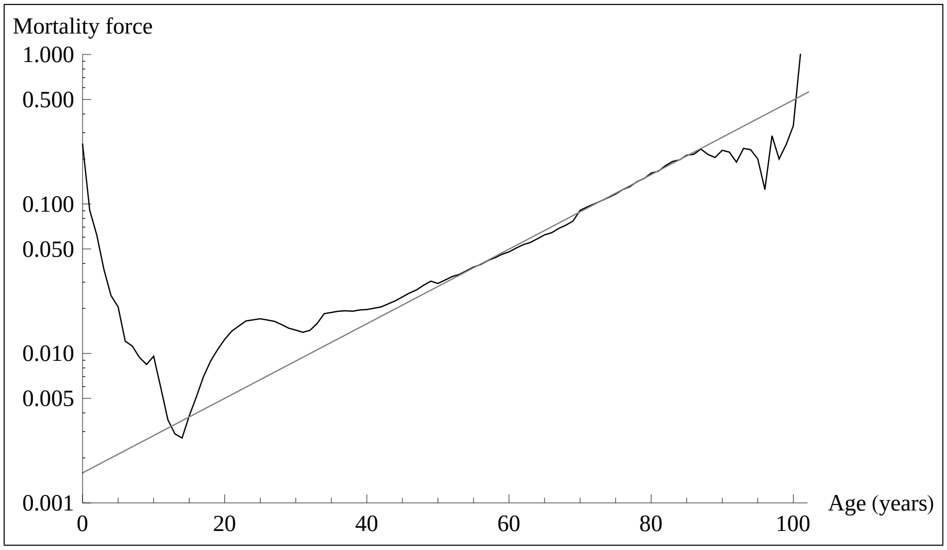

Figure 8.2: Combined logarithmic plot of the mortality force (probability of dying within the next year) for men in Amsterdam, according to age, in the period 1816-1825[48] (black line, data from [40]) and an approximation given by Gompertz' law of mortality (grey line; $y = a \cdot q^x$ with $a \approx 1.59 \cdot 10^{-3}, q \approx 1.06$).

8.4 William Makeham (1827-1891)

Very little appears to be known about the life of the British actuary William Makeham, but luckily I have been able to find some information in UK birth, death and marriage records[3].

William Matthew Makeham was born in London on September 11, 1827, into a well-to-do family[4]. He married Hephzibah Reed on August 28, 1853 and they had at least three daughters and four sons together[5]. He was an actuary by profession and was a fellow of the Institute of Actuaries[41]. The only other thing known about his employment is that in 1868, he worked for the Great Britain Mutual Life Assurance Society[23]. He died on November 17, 1891, at the nursing home in London where he and his wife had retired to.

[3] Of the two books I could find that state his years of birth and death, one [47] gives them as 1825-1865, but this seems unlikely because Makeham published an article in 1867 [41]. The other source [33] states that Makeham lived from 1826 to 1891. Using the website Ancestry.co.uk [10], I found records for a William Matthew Makeham (this middle name is mentioned in Makeham's article) who lived from 1827 to 1891 and whose profession is listed as 'Actuary'. It seems extremely likely that this is the right William Makeham.

[4] William's father John Partridge Makeham left a large sum of money to his widow when he died, while his occupation was listed simply as 'Gentleman'.

[5] There may have been more children, but I could find only seven birth records with the right parents' names and with distinct children's names.

8.5 Analysis of other life tables using the Gompertz and Gompertz-Makeham laws

In an 1867 article [41], Makeham suggested a simple modification to Gompertz' law, namely to add a constant (and therefore age-independent) term to Gompertz' formula for human mortality. The new formula, now known as the Gompertz-Makeham law of mortality, became $c + a \cdot q^x$. Figure 8.3 shows how much of an improvement this was to Gompertz' law: the approximation given by the Gompertz-Makeham law corresponds well to the observed data in a much longer age range than the Gompertz law, namely from around 30 to 85 years.

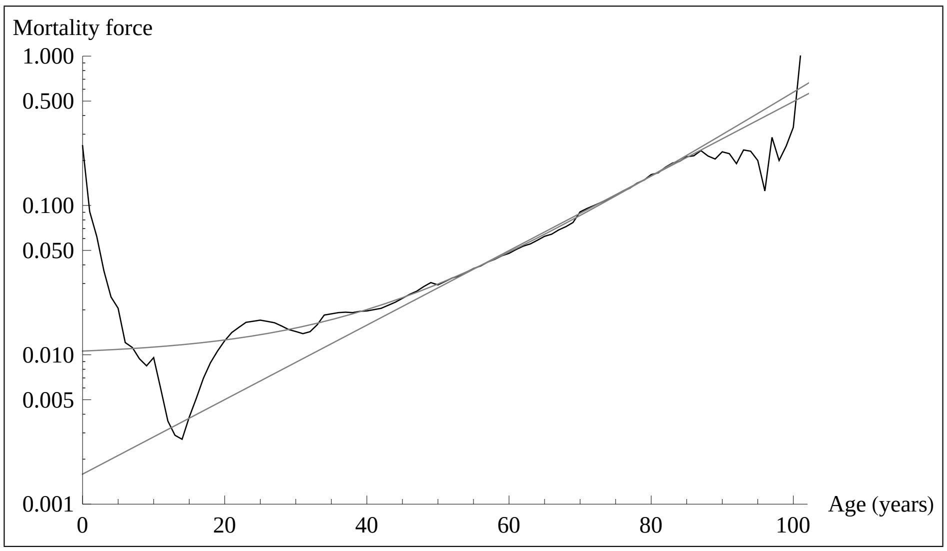

Figure 8.3: The same graph as figure 8.2, this time with an approximation given by the Gompertz-Makeham law of mortality (curved grey line; $y = c + a \cdot q^x$ with $c \approx 9.89 \cdot 10^{-3}, a \approx 7.02 \cdot 10^{-4}, q \approx 1.07$).

We can use the fact that the Gompertz-Makeham law takes into account age-independent causes of death, while the Gompertz law does not, to easily estimate the prevalence of age-independent causes of death in a population of which a life table was compiled. For example, if the Gompertz formula approximates the mortality force from a certain life table equally well as the Gompertz-Makeham formula, then apparently, the age-independent term can be taken to be close to 0. This suggests that in the population on which this life table was based, age-independent causes of death were rare. On the other hand, if the Gompertz-Makeham formula gives a significantly better approximation than the Gompertz formula, then age-independent causes of death may have been much more prevalent. This could indicate, for example, an epidemic of an infectious disease or a natural disaster.

Figure 8.4 shows the mortality force calculated from a recent life table, together with an approximation given by the Gompertz law.

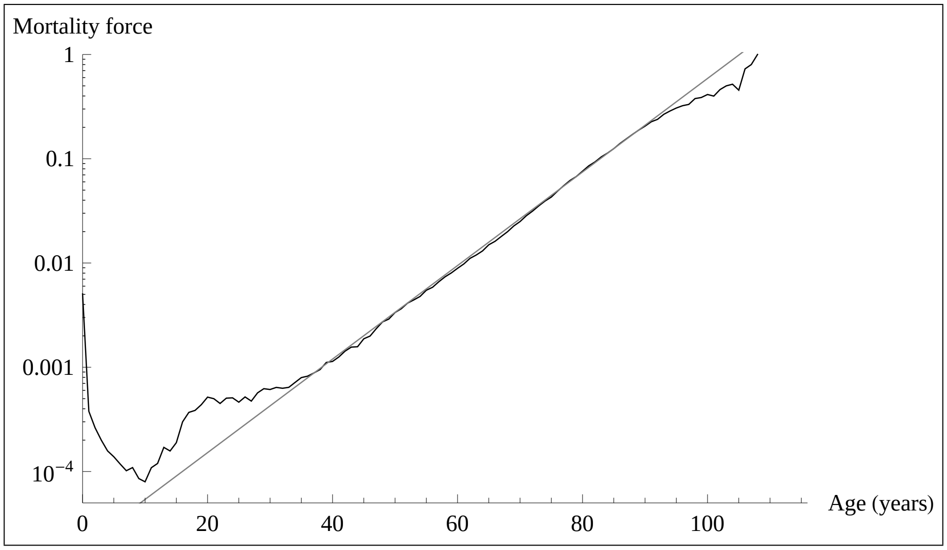

Figure 8.4: Combined logarithmic plot of the mortality force for men in the Netherlands, according to age, in the period 2003-2008 (black line, from [6]) and an approximation given by the Gompertz law of mortality (grey line; $y = a \cdot q^x$ with $a \approx 1.92 \cdot 10^{-5}, q \approx 1.11$).

Note how good this approximation is, even without Makeham's improvement to Gompertz' law. Apparently, the mortality in the Netherlands due to causes of death that are unrelated to age, was much smaller between 2003 and 2008 than between 1816 and 1825. This makes sense, since for example, improvements in health care have meant that deadly infectious diseases like the plague have been as good as eradicated. We can, of course, also try to fit Gompertz' formula to Nicolaas Struyck's data. The results are visible in figures 8.5 and 8.6.

Both approximations in figure 8.5 are so close together as to be practically indistinguishable except for very low ages. In figure 8.6, the graphs are not quite that close together, but still the difference is relatively small. It appears that the incidence of age-independent deaths according to Struyck's data, may have been very low. However, it could also well be that the data were simply not accurate enough to see these kinds of effects.

I think the truth lies somewhere in the middle. The annuitants were probably quite wealthy, so they suffered less from the agonies of poverty than the average person, which would have made them less likely to, for example, die of an infectious disease. At the same time, comparing these graphs to figure 8.4 above, there are some very clear differences. The modern data hardly show fluctuations, except at places where the absolute mortality was low, namely at very low and very high ages. Struyck's data, on the other hand, show significant 'jumps' in many places, indicating that the data are probably not precise enough to distinguish between age-related and age-unrelated causes of death with any certainty.

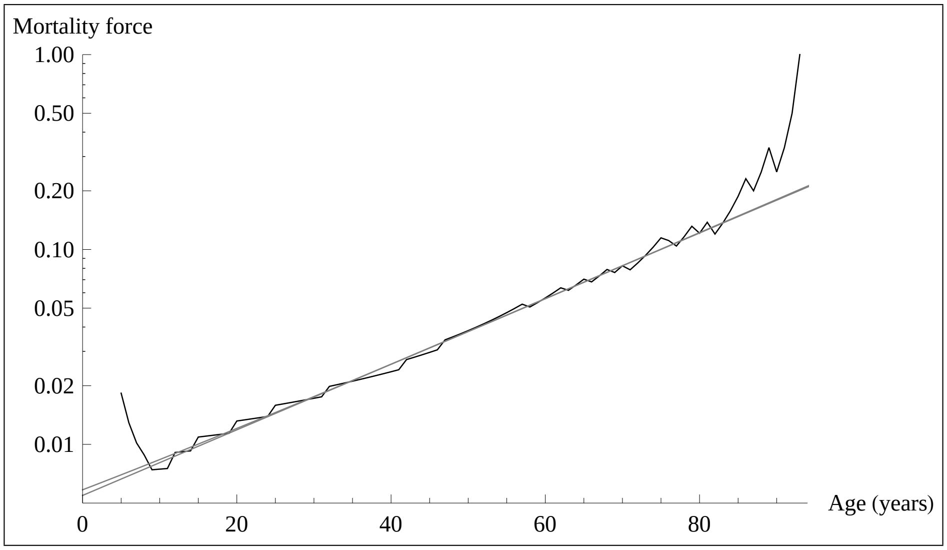

Figure 8.5: Combined logarithmic plot of the mortality force calculated from Struyck's life table for men (black line) and approximations given by the Gompertz law (lower grey line; $y = a \cdot q^x$ with $a \approx 5.47 \cdot 10^{-3}, q \approx 1.04$) and the Gompertz-Makeham law (upper grey line; $y = c + a \cdot q^x$ with $c \approx 6.48 \cdot 10^{-4}, a \approx 5.20 \cdot 10^{-3}, q \approx 1.04$).

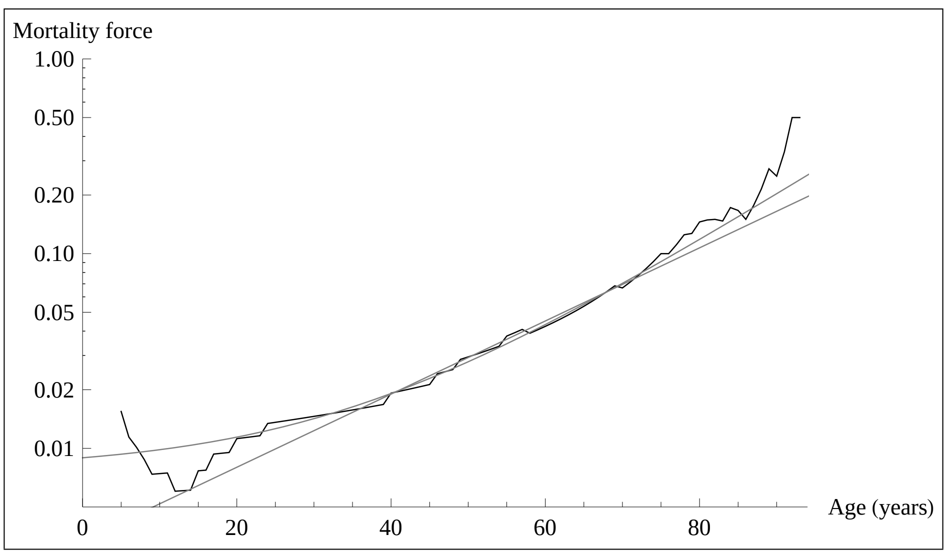

Figure 8.6: Combined logarithmic plot of the mortality force calculated from Struyck's life table for women (black line) and approximations given by the Gompertz law (lower grey line; $y = a \cdot q^x$ with $a \approx 3.37 \cdot 10^{-3}, q \approx 1.04$) and the Gompertz-Makeham law (upper grey line; $y = c + a \cdot q^x$ with $c \approx 7.78 \cdot 10^{-3}, a \approx 1.16 \cdot 10^{-3}, q \approx 1.06$).

Chapter 9

Modern analysis of human mortality: reliability theory

9.1 Introduction

Before concluding this thesis, I would like to add a chapter on reliability theory: one of the most recent major developments in mathematics related to mortality. The theory provides a relatively simple model for human mortality, so in a way, it can be seen as the most recent continuation of the work of Gompertz and Makeham.

Reliability theory has been developed mainly by Dr. Leonid Gavrilov and Dr. Natalia Gavrilova, currently at the university of Chicago, who published the influential book '*The Biology of Life Span: A Quantitative Approach*' (1986)[1] and who continue their work on the subject to this day.

9.2 Imperfections of the Gompertz-Makeham law

We have seen the Gompertz-Makeham law of mortality in an earlier section of this thesis. According to this law, the mortality force[2] can be approximated by a function of the form $c + a \cdot q^x$ for some $a, q > 0$. This approximation is usually applicable from a certain minimum age onwards; for low ages, the mortality force can still vary quite wildly.

While this law does indeed, in general, give quite a good approximation to the observed mortality force, it is still far from perfect. A main difference between the observed mortality force and the approximation given by the Gompertz-Makeham law, is late-life mortality deceleration, which is one of the two subjects we will look at in this section. The other is the empirical observation that there should be some

[1] Originally in Russian; English translation published in 1991.

[2] Recall that the mortality force for some age n is defined as the probability an n-year-old will die before age $n + 1$.

relation between the values of a and q: the so-called compensation law of mortality. We will now take a closer look at these two phenomena.

9.2.1 Late-life mortality deceleration

Figure 8.4, displayed in an earlier section, showed a logarithmic plot of the mortality force for men in the Netherlands between 2003 and 2008 and an approximation given by the Gompertz law. For ages 40 to 90, this approximation appears to be very good. However, from around age 90 upwards, the mortality force seems to level off slightly: it grows significantly slower than the Gompertz and the Gompertz-Makeham law predict. This is not just coincidence: the same phenomenon can be observed in life tables from around the world and from different time periods, as long as they contain enough data for these high ages. This surprising phenomenon is called late-life mortality deceleration.

9.2.2 Compensation law of mortality

The phenomenon of mortality convergence has only been discovered relatively recently. It can be observed from a comparison between a number of different graphs of mortality force. Figure 9.1 combines logarithmic graphs of the mortality force for men in the Netherlands for several different five-year periods. In these graphs, some fluctuations are visible for ages 90 to 100. This is caused by the fact that not very many people live to these high ages and therefore not many data are available. However, it is still clear that the graphs generally seem to converge almost to a point somewhere around age 95. This mortality convergence is observed for all life tables for humans, male or female, from all historical periods and across all civilizations. It is also observed for life tables for other animal species, although the location of the point to which the graphs seem to converge, generally differs between different species.

Mortality convergence suggests that in populations with higher overall mortality, the mortality force increases more slowly than in populations with lower overall mortality, suggesting some sort of relation between the constants a and q from the Gompertz-Makeham law. This phenomenon is known as the compensation law of mortality.

9.3 Reliability theory

A new possible model for mortality, which tries to explain the aforementioned phenomena, is reliability theory. It considers a certain type of machine, which is made up of m blocks of n elements in such a way that the machine stops working if any one of its m blocks does not work, while a block only stops working when *all* of its n elements have stopped working.

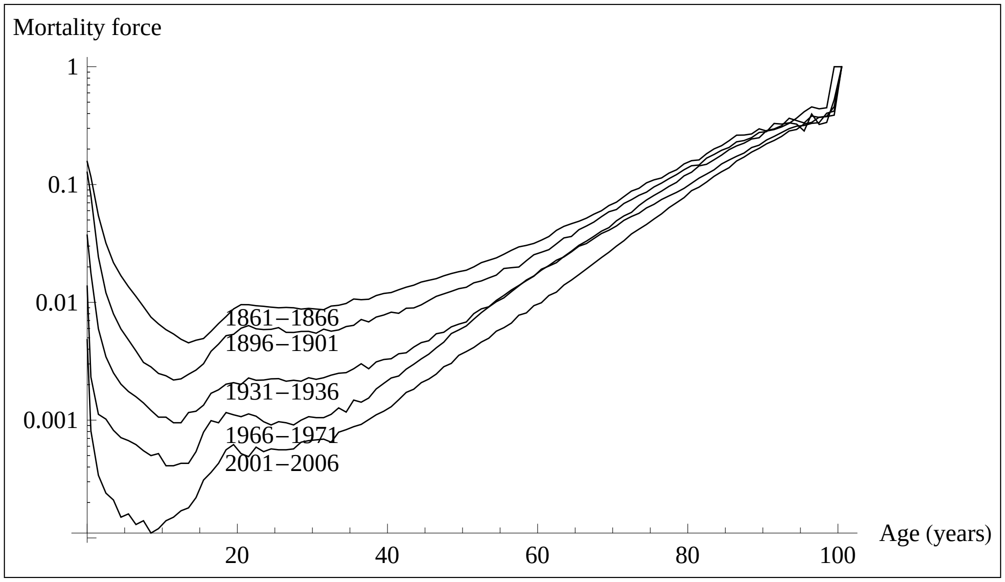

Figure 9.1: Combined logarithmic plot of the mortality force for men, based on life tables for men in the Netherlands for the periods 1861-1866, 1896-1901, 1931-1936, 1966-1971 and 2001-2006, from [16].

A possible translation of this model to human beings could be to compare the machine to a human body, in which a block could represent a vital organ, and the elements in a block may represent the cells in an organ. In this (evidently greatly simplified) model, the person would die if any of his or her vital organs failed, but an organ would only fail if all of its cells had stopped functioning.

If a system does not age, then the failure rate (the probability that the system will fail within the next unit of time[3]), denoted $\lambda(t)$, is constant over time; say $\lambda(t) = k$. Now consider a system made up of blocks of n elements, such that the block only fails when all n of its elements fail. Let all elements be initially functional, and let each element have constant failure rate k. Then the cumulative distribution function for failure of one block is $F_b(n,k,t) = P(X \leq t) = (1-e^{-kt})^n$, so the reliability function[4] of a block is $S_b(n,k,t) = 1 - F_b(n,k,t) = 1-(1-e^{-kt})^n$. The failure rate of a block is then

$$\mu_b(n,k,t) = \frac{-\frac{\mathrm{d}}{\mathrm{d}t}S_b(n,k,t)}{S_b(n,k,t)} \tag{9.1}$$

$$= \frac{nke^{-kt}(1-e^{-kt})^{n-1}}{1-(1-e^{-kt})^n} \tag{9.2}$$

The failure rate of a whole system made up of m of these blocks, where the system

[3]Note that the failure rate for machines is equivalent to mortality force for humans.

[4]The reliability function gives the probability that a system will still be functioning at a time $t > 0$, given that it was functioning at time $t = 0$.

fails when any of the blocks fail, is then simply m times this failure rate:

$$\mu_s(t) = \frac{mnke^{-kt}(1-e^{-kt})^{n-1}}{1-(1-e^{-kt})^n} \tag{9.3}$$

Now let us look at how this function develops for low ages. If $t \to 0$, then $e^{-kt} \to 1$. Therefore, if $t \to 0$,

$$\mu_s(t) \to \frac{mnk(1-e^{-kt})^{n-1}}{1} \tag{9.4}$$

$$= mnk(1-e^{-kt})^{n-1} \tag{9.5}$$

Now observe that using Taylor series, we get

$$e^t = 1 + t + \mathcal{O}(t^2) \tag{9.6}$$

$$\Rightarrow e^{-kt} = 1 - kt + \mathcal{O}(t^2) \tag{9.7}$$

$$\Rightarrow 1 - e^{-kt} = kt + \mathcal{O}(t^2) \tag{9.8}$$

$$\Rightarrow (1-e^{-kt})^{n-1} = k^{n-1}t^{n-1} + \mathcal{O}(t^n) \tag{9.9}$$

Therefore, for $t \to 0$,

$$\mu_s(t) \to mnk(1-e^{-kt})^{n-1} \tag{9.10}$$

$$= mnk \cdot k^{n-1}t^{n-1} + \mathcal{O}(t^n) \tag{9.11}$$

$$\approx mnk^n t^{n-1} \tag{9.12}$$

Apparently, for low values of t, $\mu_s(t)$ approximates a power function instead of the exponential function given by Gompertz' law. Gavrilov and Gavrilova [24] write that for this reason, this model for human mortality was rejected for a long time. They then propose the following simple amendment to the model to eliminate this problem.

Until now, we have assumed that all elements of the system are initially functional. However, in a living organism, it is readily imaginable that not every little part may be functional at birth. Returning to the analogy between elements and cells we treated earlier, a small error may occur during cell multiplication, causing a genetic mutation that may cause the cell in question not to function as it should. In general, this nonfunctional cell will not create a health problem for the organism, because there will most probably be many other cells in the same organ that are functioning properly, so the organ will generally not fail. Therefore, in the model, not all elements of the system need to be initially functional for the system to be initially functional.

To incorporate this new possibility, Gavrilov and Gavrilova propose that the number of initially functional elements in a block may be described by the Poisson distribution with some small adjustments. The Poisson distribution gives the probability that an event will occur exactly i times within some time interval, if it is expected to occur, on average, λ times within this interval[5]. The equation for the Poisson distribution is $P_\lambda(i) := \frac{\lambda^i e^{-\lambda}}{i!}$, for all $i \in \mathbb{Z}_{\geq 0}$.

In the case of this model, we will not use the Poisson distribution to give the probability that a certain event will occur a number of times within some time interval; instead, we will use it to give the number of initially functioning elements in a block, given that there are expected to be λ such elements.

To make this distribution applicable to the number of initially functional elements out of the n elements in a block, we need to ensure that the number of initially functioning elements never exceeds n. Also, the number of initially functional elements in a block can never be 0, otherwise the system would not be initially functional. These two conditions lead to the definition of a distribution P'_λ with $P'_\lambda(i) = 0$ if $i > n$ or $i = 0$, and $P'_\lambda(i) = cP_\lambda(i)$ otherwise, for $c := \frac{1}{1-P_\lambda(0)-\sum_{i>n} P_\lambda(i)}$. The factor c ensures that $\sum_{i\geq 0} P'_\lambda(i) = 1$.

We now obtain

$$\mu_s(t) = \sum_{i=1}^{n} \left(\frac{mike^{-kt}(1-e^{-kt})^{i-1}}{1-(1-e^{-kt})^i} \cdot c \cdot \frac{\lambda^i e^{-\lambda}}{i!} \right) \tag{9.13}$$

$$= mkce^{-kt}e^{-\lambda} \sum_{i=1}^{n} \frac{\lambda^i i(1-e^{-kt})^{i-1}}{i!(1-(1-e^{-kt})^i)} \tag{9.14}$$

$$= mk\lambda ce^{-kt}e^{-\lambda} \sum_{i=1}^{n} \frac{\lambda^{i-1}(1-e^{-kt})^{i-1}}{(i-1)!(1-(1-e^{-kt})^i)} \tag{9.15}$$

Using the same approximations as earlier, for $t \to 0$, we find

$$\mu_s(t) \approx mk\lambda ce^{-\lambda} \sum_{i=1}^{n} \frac{(\lambda kt)^{i-1}}{(i-1)!} \tag{9.16}$$

$$= mk\lambda ce^{-\lambda} \cdot \left(e^{\lambda kt} - \sum_{i=n+1}^{\infty} \frac{(\lambda kt)^{i-1}}{(i-1)!} \right) \tag{9.17}$$

$$\approx mk\lambda ce^{-\lambda}e^{\lambda kt} \tag{9.18}$$

Therefore, for low ages, $\mu_s(t)$ approximates an exponential function, just as we'd expect from the Gompertz law.

Next, we can look at $\mu_s(t)$ for high ages.

[5]Note that this constant λ is not related to the function $\lambda(t) = k$ seen earlier in this chapter.

9.3.1 Late-life mortality deceleration and mortality convergence

In this model, although the individual elements do not age (they have constant failure rate), a block of elements does age, because its elements fail one by one and so its failure rate changes.

If a machine reaches a high 'age', then its blocks will probably not contain many working elements any more. That is, although each block must still have at least one working element, it is very probable that the number of working elements in such a block is very low. Therefore, there is quite a high probability that a block will fail the next time an element in that block fails. Consequently, the failure rate of a block approaches the failure rate k of one element, so the failure rate of the entire system approaches mk.

To prove this result, first define $z(t) := 1 - e^{-kt}$ and remember that by definition of c, it holds that $\sum_{i=1}^{n} \frac{c\lambda^i e^{-\lambda}}{i!} = 1$. Now note that

$$\lim_{t\to\infty} \mu_s(t) = \lim_{t\to\infty} \sum_{i=1}^{n} \left(\frac{mkie^{-kt}(1-e^{-kt})^{i-1}}{1-(1-e^{-kt})^i} \cdot \frac{c\lambda^i e^{-\lambda}}{i!} \right) \tag{9.19}$$

$$= \lim_{z\to 1} \sum_{i=1}^{n} \left(\frac{mki(1-z)z^{i-1}}{1-z^i} \cdot \frac{c\lambda^i e^{-\lambda}}{i!} \right) \tag{9.20}$$

$$= \lim_{z\to 1} \sum_{i=1}^{n} \left(\frac{mk(1-z)(i \cdot z^{i-1})}{1-z^i} \cdot \frac{c\lambda^i e^{-\lambda}}{i!} \right) \tag{9.21}$$

$$= \lim_{z\to 1} \sum_{i=1}^{n} \left(\frac{mk(1-z)(z^{i-1} + z^{i-2} + \ldots + 1)}{1-z^i} \cdot \frac{c\lambda^i e^{-\lambda}}{i!} \right) \tag{9.22}$$

$$= \lim_{z\to 1} \sum_{i=1}^{n} \left(\frac{mk(1-z^i)}{1-z^i} \cdot \frac{c\lambda^i e^{-\lambda}}{i!} \right) \tag{9.23}$$

$$= \sum_{i=1}^{n} mk \cdot \frac{c\lambda^i e^{-\lambda}}{i!} \tag{9.24}$$

$$= mk \tag{9.25}$$

Therefore, for high ages, the failure rate in this model will converge to a constant value. This is consistent with the observed phenomenon of late-life mortality deceleration.

Now consider a number of different machines, all made up of the same number of blocks m and with the same failure rate k for each element, but with varying numbers of elements n per block and varying values of λ. As we just proved, at high ages, the failure rate of each of these machines will eventually converge to mk, independent of n and λ. This could explain mortality convergence.

9.4 Mortality convergence in Struyck's life tables

Figures 9.2 and 9.3 show combined logarithmic plots of mortality force in the same way as figure 9.1, only this time overlayed with life tables for Amsterdam in the period 1816-1825, and the mortality force calculated from Nicolaas Struyck's data, for respectively men and women.

Although Struyck's data are less precise than the data from more recent life tables, the graphs do seem to show mortality convergence at higher ages.

It is striking that the mortality force from Struyck's life tables is very close to the mortality force calculated from early nineteenth century life tables. Unless mortality rates hardly changed throughout the eighteenth century, which seems unlikely, Struyck's data show a remarkably low mortality force. There are several possible explanations for this.

Firstly, the data Struyck used to base his tables on, came from registers of life annuities. For obvious reasons, annuities were generally bought for healthy annuitants, who were expected to live longer than average. Also, the buyer of an annuity would be a person with a large sum of money to invest, so quite possibly, the annuitants would come from a richer background than the average person. Then again, we saw earlier that Struyck's data are very close to the life table Halley created based on all deaths in Breslaw in a certain period. If we assume that living conditions in Breslaw were comparable to those in Amsterdam, then the fact that Struyck's data came from annuitants may not have made a big difference.

Another possible explanation is that Struyck's data came from Amsterdam, while the data for the more recent life tables came from all deaths in the Netherlands. Maybe the living conditions in cities, where there were hospitals close by and where many well-to-do families lived, were much better than those in rural areas.

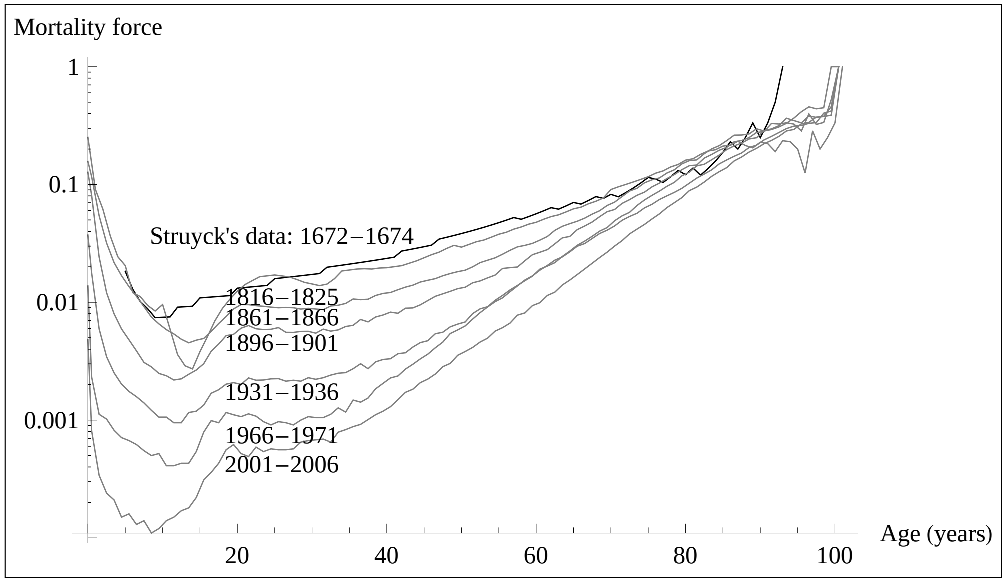

Figure 9.2: Combined logarithmic plot of the mortality force for men, based on Struyck's life table, a life table for men in Amsterdam in 1816-1825 [40] and more recent life tables for men in the Netherlands [16].

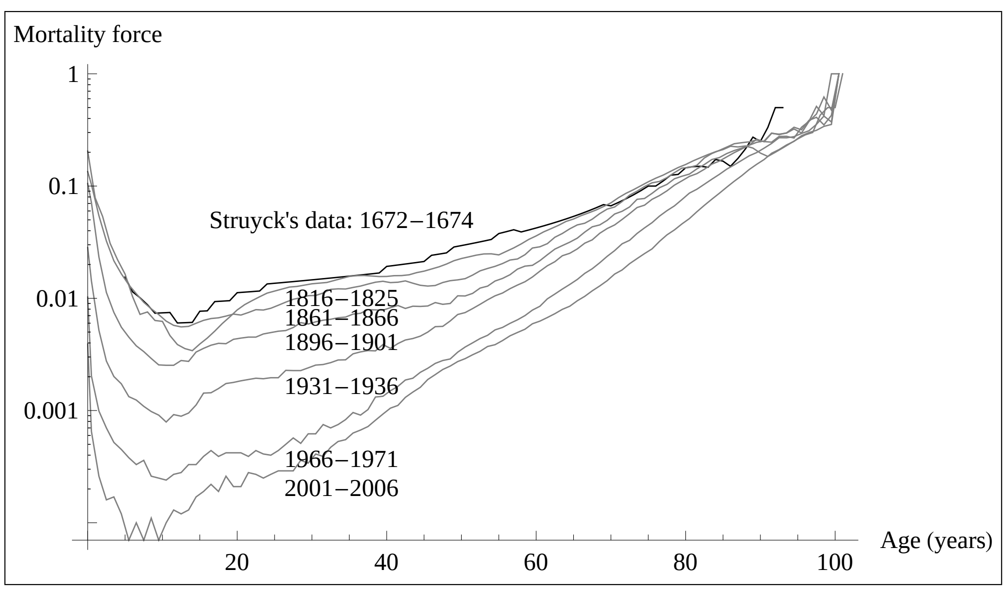

Figure 9.3: Combined logarithmic plot of the mortality force for women, based on Struyck's life table, a life table for women in Amsterdam in 1816-1825 [40] and more recent life tables for women in the Netherlands [16].

Chapter 10

Conclusion

The invention of the life table seems to have been mostly a coincidence, as is often the case with great inventions. John Graunt, although clearly a competent and enthusiastic mathematician, had little or no formal mathematical education; nor did he have any particularly strong reason to do research into mortality. He appears to have simply been at the right place and the right time with enough creativity and scientific curiosity to take a closer look at the Bills of Mortality.

William Petty does not seem to have been an extremely gifted mathematician, but the idea to draw a table of mortality rates at different ages from data in which only the cause of death (not the age at death) was recorded, is certainly impressive. That the table is not very elaborate or, for that matter, accurate, may then easily be forgiven.

It does not seem like either Petty or Graunt fully understood the significance of their discovery. Otherwise, they might have given the table a more prominent place in Graunt's *'Observations'*, or proposed some uses for it other than to estimate the number of men available to fight.

The Huygens brothers came from a very different background than either Graunt or Petty. Christiaan Huygens was already an accomplished scientist when his correspondence with his brother Lodewijk about Petty's life table unfolded. Also, importantly, he had ample experience with the very young field of (what we would now call) probability theory, having written an influential book on the subject. Presumably, Lodewijk Huygens had also come into contact with this field, because his invention of life expectancy required quite a new way of thinking about life tables.

In the '*Observations*', Graunt or Petty[1] wrote that "men do not die in exact Proportions, nor in Fractions[2]". Without wanting to give too much weight to one remark, I gather from this that they thought a life table could only be accurate to a certain extent. They realized a life table could have some predictive value, or was

[1] As we saw in section 3.3, it is probable that Graunt wrote most of the '*Observations*', but that Petty contributed the life table. Therefore, either of them could have written this passage.

[2] From '*Observations*' [26], page 62.

at least quite generally applicable, as witnessed by their estimation of the number of 'fighting men' in London. But it seems to me like calculating something as concrete as a life expectancy would have required an additional leap of thought that Graunt and Petty were not about to take.

The Huygens brothers, however, were already familiar with the notion of an expectation value in the context of games of chance. In fact, Christiaan Huygens himself had introduced the concept; his '*De ratiociniis in ludo aleae*' contained methods to calculate it in different situations. Assuming Lodewijk Huygens was familiar with this work, this would certainly have given him an advantage in applying probability theory to life tables. It may provide one reason why it was him instead of Graunt, Petty or another reader of Graunt's '*Observations*' who invented the concept of life expectancy.

What does surprise me, is the fact that neither of the Huygens brothers ever published their findings. As I pointed out earlier, the brothers knew the relevance of life tables for life annuities[3], and surely realized life expectancy could be useful in many other fields, not to mention interesting for anyone to read.

Still, it seems that they may at least have spread word of their work to Johannes Hudde and Johan de Witt.

In their positions as politicians, Hudde and De Witt belonged to the organizations that sold life annuities. Both had had a thorough mathematical education and showed interest in financial topics, so they were probably well acquainted with prices of life annuities and their dependence on age. They might even have realized that the issuers of the annuities were making substantial losses before they ever started their work on life tables; this would certainly have provided a strong motivation for their work.

In any case, Johan de Witt apparently quickly saw the possible application of a life table to calculate annuity prices, as he published his '*Waerdye van lyf-rente naer proportie van los-renten*' within two years of the Huygens brothers' correspondence. He very probably knew of the life table in John Graunt's book, whether through Huygens or otherwise, yet he did not use it. One possible reason for this could be that he decided the table was not suitable to base annuity prices on, either because it lacked accuracy or because it was based on data from England. However, I think that if De Witt had wanted to make an accurate estimate of fair prices for annuities, he would have used this table, because it was the only one available and because it would still have been more accurate than his own estimates of mortality. For that reason, I think that De Witt had some other source of mortality data available, and he deliberately chose his estimate to be not just a little, but a lot more pessimistic than the real data. Clearly, the message he was trying to get across, was that the prices of annuities were much too low, but he did not want to show how high they really needed to be. This may have been to make his point as clear and indisputable as possible. However, the growing hostility against him may also have had something

[3]Christiaan remarked this in a letter dated November 28, 1669 [36].

to do with it: 1671 was not the best time for Johan de Witt to deliver an unpopular message.

In any case, after De Witt's death, while Johannes Hudde was burgomaster of Amsterdam, life annuities were sold there for very low prices. With his interest in finance, and seeing as raising money for defense was so important for the city, Hudde must have known about the sale of these annuities in advance. He very probably advised on the prices, as Commelin [19] suggests: after all, Hudde had sent his table to Huygens a year before the sale of the annuities began, demonstrating that he was already interested in the subject. He also definitely knew De Witt's '*Waerdye*' well, and agreed with its calculations, as he had checked these before the work was published. Therefore he must have known the prices for the new annuities were much too low, and agreed to them anyway. It seems to me, then, that this was a political decision. Whether the reason was just that the city needed money very quickly, no matter what the eventual cost, or if it played a role that De Witt had just been lynched after proposing higher prices, we cannot know.

In Edmond Halley's '*Estimate of the Degrees of Mortality of Mankind*', after printing his life table, he gives an extensive list of seven possible uses of this table. They range from estimating the number of men of fighting age, and finding the median remaining lifetime of a person of given age, to calculating the value of a life annuity. None of these were new inventions, although Halley might not have known of the work of some of his predecessors, in which case he would have had to re-invent some concepts. In my view, the most important contributions Halley made to the mathematics of life tables and life annuities, then, are his life table and introducing his readers to the subject.

Halley's life table was a great attempt at finding a life table that was as generally applicable as possible. It might be by coincidence that Halley received the data on births and deaths in Breslaw, but it is greatly to his credit that he understood the usefulness of these data. He seems to have been the first to realize that such data can be greatly influenced by immigration and emigration, and that to obtain an accurate life table, one should ideally base it on a city or region without significant migration of either kind. It is also valuable in itself that his table was not based on the lives of annuitants, but on the general population of Breslaw.

Edmond Halley was a famous scientist even in his own time. His work will probably have been read by many. Therefore, the fact that he published his '*Estimate*', in which his main point seems to have been to demonstrate how useful a life table was, might well in itself have given a major boost to the mathematics of life tables. He ended the '*Estimate*' with the words:

> (...) it is desired that in imitation hereof the curious in other cities would attempt something of the same nature, than which nothing perhaps can be more useful.

Investigating how many of 'the curious in other cities' followed this advice, perhaps falls outside the scope of this thesis. In any case, it inspired at least one person: Nicolaas Struyck, who referred to Halley's '*Estimate*' many times in his own work.

Like Halley, and arguably Christiaan Huygens, Nicolaas Struyck was principally an astronomer. This could, of course, be a coincidence, or maybe Struyck came into contact with Halley's work on life tables while studying his astronomical work. Still, it makes sense that astronomers would work on life tables. Astronomers regularly worked with large amounts of data and tried to extract general rules from them. For example, from many observations of comets, Halley was able to deduce that many of them were in fact recurring appearances of what is now known as Halley's Comet, and then to calculate the comet's orbit. In the case of life tables, the challenge was to collect as much reliable information as possible on births and deaths, and to somehow use this to make general observations on the mortality rate at different ages. It may be that of all scientists, astronomers were simply best equipped to do this.

The similarities between Struyck's and Halley's work do not end there: Struyck seems to have been an admirer of Halley's work on life tables. However, he did not simply want to emulate Halley's findings, but rather improve on them. Struyck seems to have been much more precise with his data, taking care to do his calculations in several different ways whenever possible, "om de ongelykheden te vergoeden[4]". Taking this into account, it seems like Struyck's ultimate goal in creating his life table, was quite different from Halley's. Halley adjusted his data in many places to get a 'smoother' end result, apparently to create a life table that was as general as possible. Struyck, on the other hand, stayed much more faithful to his source data. He seems to have wanted to give an accurate account of the number of people who died at each age out of the annuitants from the period 1672-1674, without polluting these data too much with his own assumptions. Although there is certainly something to say for Halley's method as well, I think this is an admirable goal. Struyck's resulting life table may have shown some more 'irregularities' than Halley's, but these were at least based on real data. They therefore gave some indication of the uncertainty in these kinds of calculation, which could be useful in its own right. If somebody then wanted to smooth out these irregularities, they could always do so themselves. But from Halley's table, although its graph may look more regular, it is hard to make out how much is based on fact and how much on assumptions.

Struyck's idea to separate men and women in his life tables, was a remarkable one. Assuming that records of annuities were not kept in separate lists for men and women, Struyck would have had to separate his data by hand, and then do all his calculations twice, once for men and once for women. I wonder whether he had conjectured in advance that both sexes would have different mortality rates, or was simply surprised by this fact after he had completed his calculations. In any case, the differences in mortality rate were significant. The fact that these different life expectancies for men and women were later found to be present in practically all life tables for humans

[4]From [52], page 367. Roughly translated: "to compensate for the inequalities"

ever compiled, arguably made this Struyck's most important discovery.

Between the time of Struyck and that of Benjamin Gompertz, much had changed in mathematics. It was a very different time than when Johannes Hudde had written about "vruchteloose questien, die niet een olykoeck waert zyn[5]". Questions without immediate practical applications were becoming more important. Also, statistics became more accepted as a subfield of mathematics. This was remarkable because mathematicians, who normally worked with certainties and proofs, now involved themselves in the science of uncertainty.

Thus the observation that mortality force according to age could be approximated by an exponential function, was not meant to make calculations easier or anything of that nature. Instead, it was an attempt to determine, as Gompertz put it, "the law of human mortality[6]". His work, then, as well as that of Makeham, Gavrilov, Gavrilova and other researchers into this subject, may even be seen as steps towards answering the age-old question of why people die, and what can be done to extend life, by first answering the question of *how* people die.

In a sense, this may have been the ultimate goal of earlier researchers as well. I would like to end with a quote by Nicolaas Struyck that seems to indicate this. Struyck, a devoutly religious man, wrote in his '*Gissingen over de staat van het menschelyk geslagt*':

> Dat ik dit onderzoek Gissingen noem, is, om de onzekerheid; daar is nog weinig van bekend; men heeft geen Waarneemingen genoeg van alle Gewesten tot voorbeelden: het schynt my toe, dat men hier nog wonderlyke zaaken in ontdekken zal, die ons zullen opleiden, om een gedeelte van de overgroote Wysheid te zien, die de Schepper van 't Heel-Al heeft believen te gebruiken tot onderhouding van 't Menschelyk Geslagt;[7]

Indeed, many wonderful things have been discovered since, and hopefully, many more are still to come.

[5]From a letter from Hudde to his teacher Frans van Schooten dated December 1, 1657, with a copy to Christiaan Huygens, from [36], page 101. Roughly translated: "useless questions, not worth a fig"

[6]From the title of [25].

[7]From '*Gissingen over de staat van het menschelyk geslagt*' ('*Conjectures about the state of the human race*') [52], page 321. Roughly translated: "The reason I call this treatise 'Conjectures', is, because of the uncertainty; little is yet known about that; there are not enough Observations from all Regions as examples: it seems to me, that wonderful things will still be discovered about this, which will educate us, to see a part of the overwhelming Wisdom, that the Creator of the Universe has chosen to use for perpetuation of the Human Race; (...)"

Appendix A

Tables

A.1 John Graunt & William Petty

Viz. of 100 there dies within the first six years	36
The next ten years, or *Decad*	24
The second *Decad*	15
The third *Decad*	9
The fourth	6
The next	4
The next	3
The next	2
The next	1

Table A.1: The expected number of deaths between the ages 0-6, 6-16, 16-26 etc. out of 100 newborns, from [26], page 62.

Of the said 100 conceived there remains alive	
At six years end	64
At Sixteen years end	40
At Twenty six	25
At Tirty six	16
At Fourty six	10
At Fifty six	6
At Sixty six	3
At Seventy six	1
At Eighty	0

Table A.2: Graunt and Petty's life table, from [26], page 62.

A.2 Lodewijk Huygens

	years	months
[For every newborn]	18	2
For every child of 6 years	20	10
For every person of 16 years	20	3
For those of 26	19	4
For those of 36	17	6
For those of 46	15	0
For those of 56	12[a]	8
For those of 66	8	4
For those of 76	5	0
For those of 86	0	0

Table A.3: Table of the remaining life expectancy for people of different ages, from [36], page 518.

[a] I expect this is a writing error, and it should say 11.

A.3 Johannes Hudde

1	2	3	4	5	6	7	8	9	10
1	1	2	1	0	1	1	3	3	3
6	4	2	2	3	2	1	4	3	4
6	8	2	2	5	3	2	4	3	7
13	11	3	2	5	3	2	5	4	7
15	12	4	2	7	3	5	5	4	9
15	13	7	3	9	4	5	6	11	9
15	13	10	3	10	4	6	6	12	10
16	13	12	4	10	5	7	10	13	12
22	15	13	5	10	6	9	11	14	13
23	15	14	12	11	6	10	11	15	13
23	16	14	12	12	7	10	12	15	13
24	16	15	13	13	7	11	13	15	13
24	16	16	13	13	11	12	14	17	14
25	17	16	13	14	12	12	14	18	14
25	20	17	14	14	13	14	15	19	15
25	22	17	15	14	13	15	16	20	15
25	23	17	15	14	13	15	16	24	16
26	25	18	16	15	13	15	16	25	16
31	26	19	16	15	13	15	19	27	17
31	26	21	17	15	14	15	19	27	17
31	27	22	17	15	15	16	19	28	17
31	29	22	18	16	17	16	20	29	18
32	30	23	19	16	17	16	22	29	19
33	31	24	19	16	18	17	22	34	19
38	33	24	20	16	18	17	22	36	20
40	33	24	20	17	18	18	22	37	21
41	34	26	22	17	18	18	23	37	25
41	35	26	22	18	19	20	24	38	26
44	35	27	23	18	21	20	26	39	30
44	36	27	24	18	26	20	27	40	31
46	36	27	26	19	27	21	29	41	33
46	38	28	26	19	27	22	29	45	33
47	38	28	27	20	29	23	29	46	34
48	38	29	28	20	30	24	30	46	37
50	44	29	31	23	30	26	30	47	37
52	47	29	32	24	31	26	30	47	38
52	50	30	32	25	31	26	31	47	39
56	51	33	33	26	32	28	31	49	39
57	52	34	33	28	34	29	32	49	40
58	53	34	33	28	34	29	33	49	41
58	54	34	34	29	35	29	34	49	44
65	56	34	34	29	36	30	35	50	45
65	56	35	35	30	36	31	37	51	46
66	57	35	35	37	38	32	37	51	46
66	57	35	36	37	38	33	38	51	47
68	59	35	36	37	38	33	38	51	48
68	60	37	37	38	39	35	38	52	48
68	61	37	37	38	39	36	38	53	48
69	62	37	37	38	40	38	39	54	48
70	63	38	38	40	40	38	39	54	49
71	64	38	38	40	40	39	40	55	49
71	64	38	39	42	42	40	41	57	50
71	65	39	39	43	43	41	41	59	50
72	66	39	39	45	44	45	43	61	51
72	68	39	40	45	44	46	43	63	53
73	72	41	40	47	46	53	44	64	53
74	72	42	41	50	47	56	44	64	54
75	73	43	41	51	48	56	45	64	54
77	74	46	42	51	48	59	46	65	59
79	74	47	43	51	50	59	48	66	60

11	12	13	14	15	16	17	18	19	20
7	3	2	0	3	4	6	4	0	5
7	3	5	0	3	4	12	9	4	8
7	5	9	3	3	9	12	10	6	10
7	7	9	3	13	10	12	14	9	16
12	10	9	4	13	10	13	14	14	16
12	12	10	5	14	10	14	15	16	23
13	12	14	6	14	11	16	17	22	23
14	12	14	8	14	14	19	17	23	27
15	13	15	8	15	15	23	18	26	27
16	15	15	9	15	23	24	18	26	28
17	15	16	9	18	26	25	25	31	30
19	16	17	10	20	26	30	31	31	30
19	16	18	12	20	27	31	35	32	35
20	18	18	12	20	27	31	35	34	37
22	19	19	12	21	27	32	36	34	38
27	21	20	14	22	31	34	36	35	39
28	22	21	15	27	35	37	36	36	41
30	23	21	16	30	36	37	37	36	41
31	24	22	16	30	37	38	37	36	44
32	24	25	18	35	37	39	44	37	44
33	28	26	18	35	38	40	48	40	45
34	29	29	20	36	38	40	48	42	47
35	31	31	21	39	39	41	49	54	49
35	32	32	22	39	39	45	49	55	50
36	35	32	24	39	41	48	50	62	50
37	35	32	26	40	42	49	51	63	50
39	36	36	27	43	47	51	59		50
40	36	36	29	44	47	52	59		51
40	38	38	30	45	49	52	60		53
42	39	38	33	46	51	53	73		54
43	39	38	35	46	51	53			59
44	39	39	35	46	56	58			65
46	39	41	37	49	57	60			
48	40	42	42	50	60	65			
49	42	49	42	52	61				
50	43	49	45	53	62				
50	46	49	46	54					
50	46	52	48	54					
51	47	52	49	55					
52	47	53	49	55					
54	48	53	50	55					
54	49	53	50	55					
55	51	54	52	56					
56	52	55	52	59					
58	52	57	53	60					
59	53	57	57	60					
59	56	59	57	62					
60	56	60	58	62					
61	57	60	59	63					
62	58	60	60	64					
62	58	61	60	65					
62	59	63	62	67					
63	63	67	63						
70	64	68	68						
73	64		71						
	70		83						
	72								
	76								

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	50
0	1	2	4	2	0	8	4	11	5	13	8	18	15	23	4	19	9	7	17	3	6	9	7	13	3	7	11	3
2	3	7	7	2	2	26	10	12	12	19	12	18	19	25	8	24	12	13	18	4	6	14	10	14	24	23	13	8
6	4	15	9	3	6	29	11	16	14	19	13	19	38	37	8	29	12	24	19	16	20		15	17	36	23	18	8
6	6	18	12	3	14	34	12	20	16	24	16	20	42	42	10		22	25	19				17	17		27	18	20
13	6	26	13	5	17	35	14	26	19	28	26	20	46		14		24	26	23								23	26
14	7	27	16	10	18	37	25	42	21	39	45	23			15		39	26	44								28	30
16	8	28	17	10	25	39	31		22			29			21			36										
22	13	31	20	13	33	42	35		34			32			24													
24	18	38	21	13	34	48	35		36			33			29													
27	23	39	25	14	36		36		41			43			46													
28	24	39	26	14	36		36		49			52																
36	24	43	27	19	38		37																					
37	28	43	28	20	42		42																					
37	29	46	29	30	56		47																					
37	31	49	33	31																								
48	31		35	34																								
49	38		35	37																								
50	45		36	46																								
52	51		36	47																								
57	52		36	52																								
58	53		36																									
	54		38																									
	56		38																									
	58		39																									
	61		40																									
			45																									
			46																									
			48																									
			50																									
			52																									
			53																									
			62																									

(Continued)

1	2	3	4	5	6	7	8	9	10
80	75	48	43	51	53	59	49	69	65
	79	49	44	52	53	59	51	70	65
	80	50	47	53	53	60	51	78	66
	84	51	50	55	54	61	52	78	66
		51	50	56	54	61	52		69
		52	50	59	55	62	52		70
		52	51	59	55	62	53		71
		53	51	59	56	63	54		72
		53	53	60	58	63	54		74
		54	53	60	59	65	54		
		54	54	61	60	65	54		
		56	55	61	60	67	55		
		56	56	61	60	69	55		
		57	57	62	61	72	57		
		58	57	62	62	77	57		
		58	58	62	62	78	59		
		59	58	62	62	84	60		
		62	60	62	64		60		
		62	61	63	64		62		
		63	61	63	64		62		
		66	63	65	65		62		
		66	64	66	66		64		
		67	66	66	67		64		
		69	67	66	67		70		
		72	67	67	70		71		
		72	70	67	71		75		
		72	76	67	71		77		
		72	79	68	71				
		72	84	68	72				
		73		69	73				
		73		70	73				
		78		72	74				
		83		72	74				
				72	78				
				73	78				
				77	80				

Table A.4: Table based on annuities sold in the Netherlands between 1586 and 1590, from [44]. The table gives the age of each annuitant when the annuity was bought (at the top of each column) and the number of years the annuitant survived after that. The smaller table appears like this in Hudde's work as well; it contains additional data that would otherwise have made the table too tall to fit on the page.

Age	Pers.	Age	Pers.	Age	Pers.	Age	Pers.	Age	Pers.	Age	Pers.	Age	Pers.
1	1000	15	899	29	698	43	528	57	339	71	124	85	9
2	1000	16	890	30	687	44	514	58	328	72	113	86	6
3	992	17	883	31	670	45	502	59	311	73	100	87	4
4	987	18	869	32	660	46	485	60	292	74	88	88	3
5	987	19	853	33	648	47	472	61	275	75	78	89	2
6	975	20	835	34	636	48	460	62	260	76	67	90	2
7	963	21	823	35	629	49	450	63	242	77	55	91	2
8	953	22	814	36	615	50	437	64	230	78	44	92	1
9	946	23	798	37	601	51	423	65	217	79	38	93	1
10	937	24	779	38	589	52	406	66	203	80	32	94	1
11	930	25	760	39	575	53	397	67	183	81	26	95	1
12	928	26	747	40	568	54	383	68	170	82	22	96	1
13	917	27	734	41	553	55	367	69	159	83	15	97	1
14	907	28	715	42	542	56	352	70	147	84	14	98	0

Table A.5: Life table calculated by me from Hudde's table of data (seen in the previous table). For each age, this table gives the expected number of persons (Pers.) to survive to that age, out of an initial group of 1000 one-year-olds. Calculated using the method outlined in section 5.6.

A.4 Edmond Halley

Age. Curt.	Per-sons.	Age. Curt.	Per-sons.	Age. Curt.	Per-sons.	Age. Curt.	Per-sons.	Age. Curt.	Per-sons.	Age. Curt.	Per-sons.
1	1000	15	628	29	539	43	417	57	272	71	131
2	855	16	622	30	531	44	407	58	262	72	120
3	798	17	616	31	523	45	397	59	252	73	109
4	760	18	610	32	515	46	387	60	242	74	98
5	732	19	604	33	507	47	377	61	232	75	88
6	710	20	598	34	499	48	367	62	222	76	78
7	692	21	592	35	490	49	357	63	212	77	68
8	680	22	586	36	481	50	346	64	202	78	58
9	670	23	579	37	472	51	335	65	192	79	49
10	661	24	573	38	463	52	324	66	182	80	41
11	653	25	567	39	454	53	313	67	172	81	34
12	646	26	560	40	445	54	302	68	162	82	28
13	640	27	553	41	436	55	292	69	152	83	23
14	634	28	546	42	427	56	282	70	142	84	20

Table A.6: Life table, from [30], page 485.

Age.	Persons.
7	5547
14	4584
21	4270
28	3964
35	3604
42	3178
49	2709
56	2194
63	1694
70	1204
77	692
84	253
100	107
Total	34000

Table A.7: Estimation of the population of Breslaw, from [30], page 485.

A.5 Nicolaas Struyck

Jaar.	Perz	Jaar.	Perz	Jaar.	Perz	Jaar.	Perz	Jaar.	Perz	Jaar.	Perz
5	710	20	607	35	474	50	313	65	142	80	33
6	697	21	599	36	464	51	301	66	132	81	29
7	688	22	591	37	454	52	289	67	123	82	25
8	681	23	583	38	444	53	277	68	114	83	22
9	675	24	575	39	434	54	265	69	105	84	19
10	670	25	567	40	424	55	253	70	97	85	16
11	665	26	558	41	414	56	241	71	89	86	13
12	660	27	549	42	404	57	229	72	82	87	10
13	654	28	540	43	393	58	217	73	75	88	8
14	648	29	531	44	382	59	206	74	68	89	6
15	642	30	522	45	371	60	195	75	61	90	4
16	635	31	513	46	360	61	184	76	54	91	3
17	628	32	504	47	349	62	173	77	48	92	2
18	621	33	494	48	337	63	162	78	43	93	1
19	614	34	484	49	325	64	152	79	38	94	

Table A.8: Life table of males, from [52], page 377.

Jaar.	Perz	Jaar.	Perz	Jaar.	Perz	Jaar.	Perz	Jaar.	Perz	Jaar.	Perz
5	711	20	624	35	508	50	373	65	205	80	55
6	700	21	617	36	500	51	362	66	194	81	47
7	692	22	610	37	492	52	351	67	183	82	40
8	685	23	603	38	484	53	340	68	172	83	34
9	679	24	596	39	476	54	329	69	161	84	29
10	674	25	588	40	468	55	318	70	150	85	24
11	669	26	580	41	459	56	306	71	140	86	20
12	664	27	572	42	450	57	294	72	130	87	17
13	660	28	564	43	441	58	282	73	120	88	14
14	656	29	556	44	432	59	271	74	110	89	11
15	652	30	548	45	423	60	260	75	100	90	8
16	647	31	540	46	414	61	249	76	90	91	6
17	642	32	532	47	404	62	238	77	81	92	4
18	636	33	524	48	394	63	227	78	72	93	2
19	630	34	516	49	384	64	216	79	63	94	1

Table A.9: Life table of females, from [52], page 377.

van 0 tot 4	van 5 tot 9	van 10 tot 14	van 15 tot 19	van 20 tot 24	van 25 tot 29	van 30 tot 34	van 35 tot 39	van 40 tot 44	van 45 tot 49	van 50 tot 54	van 55 tot 59	van 60 tot 64	van 65 tot 69	van 70 tot 74	van 75 tot 79	van 80 tot 84	van 85 tot 89	van 90 tot 94	van 95 tot 99
100	95	91	87	78	68	64	58	50	41	36	27	18	15	8	4	2			
	110	107	106	98	95	89	80	65	62	52	33	22	16	12	6	3	1		
	205	198	193	176	163	153	138	115	103	88	60	40	31	20	10	5	1		
		108	104	97	90	84	79	73	59	50	41	28	16	11	5	3	1		
		306	297	273	253	237	217	188	162	138	101	68	47	31	15	8	2		
			68	67	63	56	51	49	44	33	21	13	8	6	2	1			
			365	340	316	293	268	237	206	171	122	81	55	37	17	9	2		
				65	61	56	52	44	37	31	24	18	14	9	4	2			
				405	377	347	320	281	243	202	146	99	69	46	21	11	2		
					50	49	46	43	34	30	27	20	15	11	6	3	1		
					427	396	366	324	277	232	173	119	84	57	27	14	3		
						48	45	40	34	30	23	17	11	9	6	4	2	[b]	
						444	411	364	311	262	196	136	95	66	33	18	5	1	
							26	23	18	18	13	13	11	5	3	2			
							437	387	329	280	209	149	106	71	36	20	5	1	
								53	52	44	31	21	16	11	5	4	3	1	
								440	381	324	240	170	122	82	41	24	8	2	
									52	48	36	31	21	12	4	2	1		
									433	372	276	201	143	94	45	26	9	2	
										43	36	29	20	15	10	7	4	1	
										415	312	230	163	109	55	33	13	3	
											20	19	12	9	6	2			
											332	249	175	118	61	35	13	3	
												16	16	10	6	2			
												265	191	128	67	37	13	3	
													8	7	4	3	2	1	[b]
													199	135	71	40	15	4	1
														20	15	7	2	1	
														155	86	47	17	5	1
															7	5	1	1	
															93	52	18	6	1

Table A.10: 'The life of 794 male persons', from Nicolaas Struyck's '*Inleiding*' [52], page 363. For a guide on how to read this table, see paragraph 7.2.1.

[b] I would expect numbers 1 in these locations, but they are omitted in Struyck's table.

van 0 tot 4	van 5 tot 9	van 10 tot 14	van 15 tot 19	van 20 tot 24	van 25 tot 29	van 30 tot 34	van 35 tot 39	van 40 tot 44	van 45 tot 49	van 50 tot 54	van 55 tot 59	van 60 tot 64	van 65 tot 69	van 70 tot 74	van 75 tot 79	van 80 tot 84	van 85 tot 89	van 90 tot 94	van 95 tot 99
77	72	69	65	60	55	50	47	44	42	38	27	23	15	10	5	2	1	1	
	110	107	103	100	95	92	86	76	67	57	47	37	28	16	15	7	3		
	182	176	168	160	150	142	133	120	109	95	74	60	43	26	20	9	4	1	
		111	107	103	95	89	84	79	62	55	44	35	26	18	12	5	2	1	
		287	275	263	245	231	217	199	171	150	118	95	69	44	32	14	6	2	
			85	81	76	68	64	60	57	43	41	33	29	21	10	2	1		
			360	344	321	299	281	259	228	193	159	128	98	65	42	16	7	2	
				62	60	52	49	43	40	33	29	26	22	15	12	5	2		
				406	381	351	330	302	268	226	188	154	120	80	54	21	9	2	
					49	46	39	37	36	33	21	18	12	10	3	2	1		
					430	397	369	339	304	259	209	172	132	90	57	23	10	2	
						74	70	63	59	49	43	37	33	23	12	4	3		
						471	439	402	363	308	252	209	165	113	69	27	13	2	
							69	61	56	52	46	39	31	21	13	8	4	1	
							508	463	419	360	298	248	196	134	82	35	17	3	
								59	56	46	40	31	23	17	15	6	2	1	
								522	475	406	338	279	219	151	97	41	19	4	
									54	49	34	28	21	14	11	6	2	1	
									529	455	372	307	240	165	108	47	21	5	
										59	54	41	32	23	12	4	3	1	
										514	426	348	272	188	120	51	24	6	1
											28	28	21	19	13	6	3	2	
											454	376	293	207	133	57	27	8	1
												22	21	12	4	3	2		
												398	314	219	137	60	29	8	1
													9	9	3	2	1		
													323	228	140	62	30	8	1
														3	2				
														231	142	62	30	8	1
															5	4	3	1	
															147	66	33	9	1

Table A.11: ‘The life of 876 female persons’, from Nicolaas Struyck’s ‘*Inleiding*’ [52], page 364. For a guide on how to read this table, see paragraph 7.2.1.

	A In Classen.[a]	B Door de Tafel.[b]	C Zonder Lasten.[c]	D Met Lasten.[d]	E Gelijke Uitkeering.[e]
Van 5 tot 9 Jaaren	*f* 1856[f]	*f* 1823	*f* 4:8	*f* 5:10	34:2
—10 - 14—	1721	1714	4:13$\frac{1}{2}$	5:17	31:1
—15 - 19—	1600	1608	4:19$\frac{1}{2}$	6:4$\frac{1}{2}$	28:3
—20 - 24—	1503	1504	5:6$\frac{1}{2}$	6:13	25:9
—25 - 29—	1417	1401	5:14	7:2$\frac{1}{2}$	23:4
—30 - 34—	1303	1291	6:4	7:15	20:11
—35 - 39—	1162	1184	6:15	8:9	18:8
—40 - 44—	1057	1069	7:9$\frac{1}{2}$	9:7	16:6
—45 - 49—	944	955	8:7$\frac{1}{2}$	10:9$\frac{1}{2}$	14:4
—50 - 54—	809	840	9:10$\frac{1}{2}$	11:18	12:4
—55 - 59—	754	756	10:11$\frac{1}{2}$	13:4$\frac{1}{2}$	10:11
—60 - 64—	671	661	12:2	15:2$\frac{1}{2}$	9:5
—65 - 69—	577	575	13:18	17:8	8:-
—70 - 74—	469	481	16:12$\frac{1}{2}$	20:16	6:7

Table A.12: Values of life annuities on males, from [52], page 368.

[a] The value of a life annuity of 100 guilders yearly, in guilders, calculated from table A.10.

[b] The same, except this time calculated from table A.8.

[c] Conversely, the yearly payout one should receive for every 100 guilders invested in a life annuity, expressed in guilders:stuivers, where 20 stuivers = 1 guilder, and based on column A. A tax of 20% on payouts from life annuities is taken into account.

[d] The same, but without taking tax into account.

[e] The length of time one should receive the fixed yearly payout for the annuity to have exactly the calculated value, expressed in years:months.

[f] I assume this is a typo, and that it should say 1826.

	A In Classen.[a]	B Door de Tafel.[b]	C Zonder Lasten.[c]	D Met Lasten.[d]	E Gelijke Uitkeering.[e]
Van 5 tot 9 Jaaren	*f* 1936	*f* 1931	*f* 4:3	*f* 5:4	37:6
—10 - 14—	1832	1840	4:7	5:9	34:8
—15 - 19—	1737	1733	4:12$\frac{1}{2}$	5:15$\frac{1}{2}$	31:7
—20 - 24—	1627	1630	4:18$\frac{1}{2}$	6:3	28:10
—25 - 29—	1524	1533	5:4$\frac{1}{2}$	6:10$\frac{1}{2}$	26:5
—30 - 34—	1448	1438	5:11$\frac{1}{2}$	6:19	24:2
—35 - 39—	1334	1328	6:0$\frac{1}{2}$	7:10$\frac{1}{2}$	21:8
—40 - 44—	1221	1203	6:13	8:6	19:1
—45 - 49—	1076	1077	7:8$\frac{1}{2}$	9:6	16:6
—50 - 54—	969	964	8:6	10:7$\frac{1}{2}$	14:6
—55 - 59—	884	851	9:8	11:15	12:5
—60 - 64—	753	733	10:18$\frac{1}{2}$	13:13	10:6
—65 - 69—	613	616	13:-	16:4	8:8
—70 - 74—	493	493	16:5	20:6	6:9

Table A.13: Values of life annuities on females, from [52], page 366.

[a] The value of a life annuity of 100 guilders yearly, in guilders, calculated from table A.11.

[b] The same, except this time calculated from table A.9.

[c] Conversely, the yearly payout one should receive for every 100 guilders invested in a life annuity, expressed in guilders:stuivers, where 20 stuivers = 1 guilder, and based on column A. A tax of 20% on payouts from life annuities is taken into account.

[d] The same, but without taking tax into account.

[e] The length of time one should receive the fixed yearly payout for the annuity to have exactly the calculated value, expressed in years:months.

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