Modular Theory

in Quantum Field Theory

(part 2: Bisognano–Wichmann theorem,

Unruh effect and entanglement entropy)

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Introduction

The theorem of Bisognano–Wichmann '75 gives an appealing geometric interpretation of the modular structure for free quantum fields on the Minkowski wedge (the context of the Unruh effect).

The geometric interpretation is rather exceptional, but the general structure remains and is made apparent by the Araki–Woods representation.

In a more general context, modular theory is key in making precise the concept of entanglement entropy in QFT.

References:

- Brunetti, R.; Guido, D.; Longo, R.: Modular Localization and Wigner Particles, Reviews in Mathematical Physics, Volume 14, Issue 07-08, pp. 759-785 (2002).
- C. Gérard, Microlocal Analysis of Quantum Fields on Curved Spacetimes
- S. Hollands, K. Sanders, Entanglement Measures and Their Properties in Quantum Field Theory

Bisognano–Wichmann theorem

Minkowski space $M := \mathbb{R}^d$, $d \geqslant 2$ with metric

$$\langle x, y \rangle = x^0 y^0 - \sum_{i=1}^{d-1} x^i y^i, \quad x, y \in \mathbb{R}^d.$$

Space-like means $\langle x, x \rangle < 0$, time-like $\langle x, x \rangle > 0$.

The group of symmetries is the Poincaré group $\mathcal{P}$. Of particular relevance is the proper part $\mathcal{P}_+$ of orientation-preserving elements, and $\mathcal{P}_+^\uparrow$—the subgroup of time-orientation preserving ones.

Wedges $W \subset \mathbb{R}^d$ are simply Poincaré transformed versions of

$$W_1 = \{x \in \mathbb{R}^d \,|\, x_1 > |x_0|\}$$

Here we take $W = W_1$, but everything we say can be generalized by applying Poincaré transformations.

Consider the following (rescaled) boosts preserving W:

$$\Lambda_W : \mathbb{R} \ni t \mapsto \Lambda_W(t) = \begin{pmatrix} \cosh(2\pi t) & -\sinh(2\pi t) & 0 \\ -\sinh(2\pi t) & \cosh(2\pi t) & 0 \\ 0 & 0 & \mathbf{1}_{\mathbb{R}^{d-2}} \end{pmatrix},$$

and $R \in \mathcal{P}_+$ the wedge reflection

$$R_W(x_0, x_1, \ldots, x_{d-1}) = (-x_0, -x_1, x_2, \ldots, x_{d-1}).$$

Now fix a strongly continuous unitary representation U of $\mathcal{P}_+$ on a Hilbert space $\mathcal{K}$ (actually, anti-unitary for $\mathcal{P}_+^{\downarrow} = \mathcal{P}_+ \setminus \mathcal{P}_+^{\uparrow}$). Let H_W be the self-adjoint generator of $U(\Lambda_W(t))$ and define

$$\Delta_W := \exp(H_W)$$
$$J_W := U(R_W).$$

Proposition

J_W is anti-unitary, $J_W^2 = \mathbf{1}$, and

$$J_W \Delta_W J_W^{-1} = \Delta_W^{-1}.$$

Define

$$S_W := J_W \Delta_W^{1/2} : \mathcal{K} \to \mathcal{K}.$$

Proposition

S_W is a densely defined, anti-linear, closed operator acting in $\mathcal{K}$ with $\mathrm{Ran}(S_W) = \mathrm{Dom}(S_W)$ and $S_W^2 \subset \mathbf{1}$.

Now define the real subspace

$$\mathcal{K}_W = \{h \in \mathrm{Dom}(S_W) \,|\, S_W h = h\}.$$

Recall that a $\mathbb{R}$-linear subspace $G \subset \mathcal{K}$ is called standard if

$$G \cap iG = \{0\}, \quad \overline{G + iG} = \mathcal{K}.$$

$$\mathcal{K}_W = \{h \in \mathrm{Dom}(S_W) \mid S_W h = h\}.$$

Proposition

$\mathcal{K}_W \subset \mathcal{K}$ is an $\mathbb{R}$-linear closed and standard subspace in $\mathcal{K}$, and S_W is the Tomita operator of $\mathcal{K}_W$, namely

$$\mathrm{Dom}(S_W) = \mathcal{K}_W + i\mathcal{K}_W, \quad S_W(h + ik) = h - ik, \quad h, k \in \mathcal{K}_W.$$

In particular $\Delta_W^{it}\mathcal{K}_W = \mathcal{K}_W$ and $J_W\mathcal{K}_W = \mathcal{K}'_W$, where

$$\mathcal{K}'_W := \{h \in \mathcal{K} \mid \mathrm{Im}\langle h|k\rangle = 0\, \forall k \in \mathcal{K}_W\}$$

is the symplectic complement of $\mathcal{K}_W$.

Note that to define J_W we needed a representation of $\mathcal{P}_+$. In practice this arises from a representation of $\mathcal{P}_+^\uparrow$ and a "PCT operator". Namely, given a representation U of $\mathcal{P}_+^\uparrow$ on $\mathcal{K}$, a reflection R and an anti-unitary involution C we get a unitary/anti-unitary representation of $\mathcal{P}_+$ on $\mathcal{K} \oplus \mathcal{K}$:

$$\tilde{U}(g) = \begin{pmatrix} U(g) & 0 \\ 0 & CU(RgR)C \end{pmatrix}, \quad g \in \mathcal{P}_+^\uparrow, \quad \tilde{U}(R) = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}.$$

The Bisognano–Wichmann theorem is the statement that if U is the irreducible representation of mass m and spin s (hence $\mathcal{K}$ is the one-particle Hilbert space for the free field) then

$$J_W = U(R_W)$$

$$\Delta_W^{it} = U(\Lambda_W(t))$$

are the modular conjugation and the modular group.

In practice in the free theory, $\mathcal{K}$ and its counterpart $\mathcal{K}_W$ "restricted to W" are obtained from the vacuum state ω_M on Minkowski space and its restriction ω_W to W.

The presented approach explains how the restriction $\mathcal{K}_W$ arises naturally from a purely representation-theoretical point of view.

Unruh effect

In practice, $\mathcal{K}$ and its counterpart $\mathcal{K}_W$ "restricted to W" are obtained from the vacuum state ω_M on Minkowski space and its restriction ω_W to W.

> Unruh effect: ω_W is a KMS state for the group of automorphisms of $\mathrm{CCR}(\mathcal{X}_W, \sigma)$ induced by boosts $\Lambda_W(t)$.

Can be understood as follows starting from W:

1. Symplectic space $(\mathcal{X}, \sigma)$ of Cauchy data of $(-\Box + m^2)u = 0$:

   $$\mathcal{X}_W = C_c^\infty(W)^{\oplus 2}, \quad \sigma((f_0, f_1), (g_0, g_1)) = \int_{x_1 > 0} f_0(x)g_1(x) - f_1(x)g_0(x)d^3x$$

2. $\Lambda_W(t)$ acts on solutions, hence on $(\mathcal{X}, \sigma)$. Well-defined KMS state ω_W on $\mathrm{CCR}(\mathcal{X}_W, \sigma)$, or directly, Araki–Woods representation as for Bose gas.
3. Kay doubling yields a pure state ω_M which we can compute to be the Minkowski vacuum. More generally: Hartle–Hawking state on e.g. Schwarzschild spacetime.
4. In Minkowski case, the GNS representation of ω_M is also constructed from irreducible representation U of $\mathcal{P}_+$ (lucky incident!), so Bisognano–Wichmann gives geometric interpretation to modular structure.

Historical sketch

- 1970s: black hole entropy in GR (Bekenstein, Hawking)
  - ⚠ But does it have an interpretation in terms of some microscopic laws? (Some attempts using string theory Strominger)
- 1980–90s: black hole entropy could arise from entanglement of quantum fields across the horizon ('t Hooft, Bombelli–Koul–Lee–Sorkin, Susskind, Srednicki,...)
- 1990s: entanglement entropy in QFT more generally, 2d conformal field theories (Wilczek et al.,)
- 2000s–...: in condensed matter physics, entanglement entropy as tool for phases of many-body systems (2d CFTs and t-dependence Calabrese–Cardy, area laws in gapped systems Hasting, entanglement spectrum to characterize fractional quantum states Li–Haldane, ...)
- 2000s–...: mainstream tool in HEP, especially holography and quantum gravity (Casini–Huerta, Ryu–Takayanagi, ...)
- recent rigorous developments using von Neumann algebras, modular theory, etc.

Entanglement entropy

Consider the bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$.

For instance, the pure state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\big)$$

is entangled. Denoting $\rho = |\Psi\rangle\langle\Psi|$ its reduced density matrix is

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\, \rho = \frac{1}{2}\left(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|\right)$$

which is <u>mixed</u>. This suggests that for instance the von Neumann entropy $S = -\,\mathrm{Tr}_{\mathcal{H}_A}\, \rho_A \ln \rho_A$ is a good candidate for an entanglement measure of ρ.

Area laws

Suppose $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ models a $d+1$-dimensional lattice of spacing ε.

For a randomly chosen state, one expects S of size $\log \dim \mathcal{H}_A$, hence the volume-law growth

$$S \sim (L/\varepsilon)^d.$$

However, physically interesting "low-energy" states . Guessing that entanglement is more or less "short-range", only nearby lattice sites close to ∂A should matter, hence an area law

$$S \sim (L/\varepsilon)^{d-1}.$$

In relativistic physics, it is interesting to take subsystems $\mathcal{H}_A$, $\mathcal{H}_B$ associated to causally separated spacetime regions.

A particularly striking observation is that formal computations of the von Neumann entropy S of some "ground state" when A corresponds to a black hole exterior and B to a "copy" give $S = \text{horizon area}/4\ell_{\mathsf{P}}^2$, i.e. the Bekenstein–Hawking entropy!

More generally, formulae for S have interesting geometric meaning (and are often derived with geometric methods, e.g. holography on anti-de Sitter spaces), the prime example being the Ryu–Takanagi conjecture.

However...

- ⚠ For mixed states, von Neumann entropy no longer a good entanglement measure (it can for instance return the same value for separable and maximally entangled states!)

- ⚠ Formal computations of von Neumann entropy in QFT are infinite (UV divergencies), and renormalized versions are not good entanglement measures.

  This is can be traced back to having infinite degrees of freedom: the operator algebras arising in QFT are type III, and separability of states is quite delicate.

So what are good entanglement measures in QFT context? How to compute them, and do they obey area laws? How do geometric terms arise?

Let $\mathfrak{A}$ be a C^*-algebra. Recall:

- a state ω is a functional ω s.t. $\omega(a^*a) \geqslant 0$ for all $a \in \mathfrak{A}$ and $\omega(\mathbf{1}) = 1$
- representations are $*$-homomorphisms $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$
- for $\rho \in \mathcal{B}(\mathcal{H})$ s.t. $\rho \geqslant 0$ and $\mathrm{Tr}_{\mathcal{H}}\, \rho = 1$, one gets a normal state

$$\omega_\rho(a) := \mathrm{Tr}_{\mathcal{H}}(\rho\pi(a)).$$

In our situations $\mathfrak{A}$ will be a von Neumann algebra represented on $\mathcal{H}$ (weakly closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$) in "standard form", meaning there exists a vector Ω which is cyclic (i.e. $\pi(\mathfrak{A})\Omega$ is dense) and separating (i.e. $a\Omega = 0$ implies $a = 0$).

Tomita–Takesaki theory in brief:

- define $S : \mathcal{H} \to \mathcal{H}$ (anti-linear) by $Sa\Omega = a^*\Omega$
- polar decomposition $S = J\Delta^{\frac{1}{2}}$ with J anti-linear, (anti)-unitary and $\Delta^{\frac{1}{2}}$ positive
- $\Delta\Omega = \Omega, \quad J\Omega = \Omega$
- $a \mapsto \sigma_t(a) = \Delta^{it} a \Delta^{-it}$ is a 1-parameter group of automorphisms
- $\omega_\sigma(a) := \langle \Omega | a\, \Omega \rangle$ is a KMS state for σ_t.

Entanglement entropy

While Bisognano–Wichman theorem and Unruh effect rely on very special setting, on curved space-times we work with more general quasi-free states ω_M and their restrictions to spacetime regions ω_A. We still know a great deal about the modular structures.

Modular theory plays in QFT a distinguished role because of its use for entanglement entropy. Background:

Given two von Neumann algebras $\mathfrak{A}_A$ and $\mathfrak{A}_B$, they generate another one: $\mathfrak{A}_A \vee \mathfrak{A}_B = (\mathfrak{A}'_A \cap \mathfrak{A}'_B)'$.

Definition

$\mathfrak{A}_A$ and $\mathfrak{A}_B$ are statistically independent iff $\mathfrak{A}_A \vee \mathfrak{A}_B \simeq \mathfrak{A}_A \otimes \mathfrak{A}_B$.

If $\mathfrak{A}_A$ and $\mathfrak{A}_B$ are finite dimensional and $\mathfrak{A}_A \cap \mathfrak{A}_B = \mathbb{C}\mathbf{1}$ then they are statistically independent. But not always true in infinite dimension!

In **Quantum Field Theory** the situation is typically as follows:

- spacetime (M, g), and von Neumann algebra $\mathfrak{A} = (\bigcup_O \mathfrak{A}(O))^{\mathrm{cpl}}$ (here $\mathfrak{A}(O)$ represent *abstract* field operators $\phi(t, x)$, smeared with test functions supported in $O \subset M$).
- they must satisfy:
  1. (Isotony) $\mathfrak{A}(O_1) \subset \mathfrak{A}(O_2)$ if $O_1 \subset O_2$
  2. (Causality) $[\mathfrak{A}(O_1), \mathfrak{A}(O_2)] = \{0\}$ if O_1, O_2 space-like related
- state ω chosen from physical principles (e.g. Minkowski vacuum from Poincaré invariance)
- the restriction of ω to $\mathfrak{A}(O)$ is <u>not</u> pure if $O \subsetneq M$
- if O_A and O_B spatially separated but touch each other, $\mathfrak{A}(O_A)$ and $\mathfrak{A}(O_B)$ are <u>not</u> statistically independent

  (however, if O_A and O_B spatially separated with non-zero distance then $\mathfrak{A}_A = \pi_\omega(\mathfrak{A}(O_A))''$ and $\mathfrak{A}_B = \pi_\omega(\mathfrak{A}(O_B))''$ are statistically independent)

(*Remark:* "Localisation" of states is a tricky concept! For instance, the Minkowski vacuum state satisfies the Reeh–Schlieder property:

$$\pi_\omega(\mathfrak{A}(O))\Omega \text{ is dense for any open } O \subset M.$$

Such states exist on any real analytic spacetime (Gérard–Wrochna '19))

Let $\mathfrak{A}_A$ and $\mathfrak{A}_B$ be commuting, statistically independent von Neumann algebras.

Definition

A normal state ω on $\mathfrak{A}_A \otimes \mathfrak{A}_B$ is separable if $\omega = \sum_j \varphi_j \otimes \psi_j$ (norm convergent sum) for positive normal functionals φ_j, ψ_j.

Separable states are always convex combinations of simple tensor products $\omega = \omega_A \otimes \omega_B$ with $\omega_A(a) = \langle \Phi | a \Phi \rangle$ and $\omega_B(a) = \langle \Psi | a \Psi \rangle$.

A normal state which is not separable is entangled.

What properties should a good entanglement measure $E(\omega)$ satisfy?

(e0) (symmetry) $E(\omega)$ is independent of the order of the systems A, B

(e1) (non-negativity) $E(\omega) \in [0, \infty]$ with $E(\omega) = 0$ iff ω is separable (and $E(\omega) = \infty$ when ω is not normal)

(e2) (continuity) For all sequences ω_i, ω_i' of normal states on increasing nets of type I factors $\mathfrak{N}_{A,i} \otimes \mathfrak{N}_{B,i} \simeq M_{n_{A,i}}(\mathbb{C}) \otimes M_{n_{B,i}}(\mathbb{C})$, if $\lim_{i\to\infty} \|\omega_i' - \omega_i\| = 0$ then

$$\lim_{i\to\infty} \frac{E(\omega_i') - E(\omega_i)}{\ln n_i} = 0$$

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A normal state which is not separable is entangled.

What properties should a good entanglement measure $E(\omega)$ satisfy?

(e3) (convexity) if $\omega = \sum_j \lambda_j \omega_j$ with $\lambda_j \geqslant 0$, $\sum_j \lambda_j = 1$, then

$$E(\omega) \leqslant \sum_j \lambda_j E(\omega_j)$$

A map $\mathcal{F} : \mathfrak{A}_1 \to \mathfrak{A}_2$ is completely positive if for all N,

$$\mathbf{1}_{M_N(\mathbb{C})} \otimes \mathcal{F} : M_N(\mathbb{C}) \otimes \mathfrak{A}_1 \to M_N(\mathbb{C}) \otimes \mathfrak{A}_2$$

maps positive elements to positive elements. (think of experimental manipulations independently to N copies of the system, or "quantum channels"). One says $\mathcal{F} : \mathfrak{A}_{\hat{A}} \otimes \mathfrak{A}_{\hat{B}} \to \mathfrak{A}_A \otimes \mathfrak{A}_B$ is local if

$$\mathcal{F}(a \otimes b) = \mathcal{F}_A(a) \otimes \mathcal{F}_B(b) \equiv (\mathcal{F}_A \otimes \mathcal{F}_B)(a \otimes b),$$

where $\mathcal{F}_A, \mathcal{F}_B$ are normal and completely positive.

Definition

A separable operation is a family $\mathcal{F}_j$ as above s.t. $\sum_j \mathcal{F}_j(\mathbf{1}) = 1$

We think of an operation as mapping a state ω to $\frac{1}{p_j}(\mathcal{F}_{A,j} \otimes \mathcal{F}_{B,j})^* \omega$ with probability $p_j := \omega((\mathcal{F}_{A,j} \otimes \mathcal{F}_{B,j})(\mathbf{1}))$.

(e4) (monotonicity) If $\mathcal{F}_j$ is a separable operation then

$$\textstyle\sum_j p_j E(\mathcal{F}_j^* \omega / p_j) \leqslant E(\omega),$$

where $\mathcal{F}_j^*(\omega)(a) = \omega(\mathcal{F}_j(a))$ and we sum over j s.t. $p_j := \omega(\mathcal{F}_j(\mathbf{1})) > 0$.

For a density matrix ρ on a Hilbert space $\mathcal{K}$, the von Neumann entropy is $-\operatorname{Tr}(\rho \ln \rho)$ (lack of information about system with state ρ, assuming we have access to all operations in $\mathcal{B}(\mathcal{H})$). A pure state $\rho = |\Psi\rangle\langle\Psi|$ has zero von Neumann entropy.

The relative entropy of ρ, ρ' is $H(\rho, \rho') = \operatorname{Tr}(\rho \ln \rho - \rho \ln \rho')$ (information gained when updating our belief about the state of the system from ρ' to ρ).

Only the latter generalizes well! Given faithful normal states ω, ω', Araki '73–'77 chooses vector representatives Ω, Ω' and defines $S_{\omega,\omega'} a|\Omega'\rangle = a^*\Omega$ and its polar decomposition $S_{\omega,\omega'} = J\Delta^{\frac{1}{2}}_{\omega,\omega'}$.

Definition

The relative entropy is $H(\omega, \omega') = \langle\Omega| \ln \Delta_{\omega,\omega'} \Omega\rangle$.

Remark: $H(\omega, \omega') = \infty$ for non normal states. This is the case for e.g. the Minkowski vacuum on $\mathfrak{A}_A \otimes \mathfrak{A}_B$ if A and B touch each other!

Definition

The relative entanglement entropy of ω on $\mathfrak{A}_A \otimes \mathfrak{A}_B$ is

$$E_{\mathrm{R}}(\omega) = \inf\{ H(\omega, \omega') \,|\, \omega' \text{ a separable state } \}$$

In the key type I example, $E_{\mathrm{R}}(\omega) = -\operatorname{Tr} \rho_A \ln \rho_A$ with $\rho_A = \operatorname{Tr}_{\mathcal{H}_B} \rho$ the reduced density matrix. Hollands–Sanders '15 show:

Theorem

The relative entanglement entropy E_{R} satisfies (e0)–(e4).

- Bounds for $E_R(\omega)$ in various situations Hollands–Sanders '18:
  1. For a real Klein–Gordon scalar QFT with field equation $(\Box - m^2)\phi = 0$ and $m > 0$, if (M, g) is static then the ground state ω_{vac} satisfies

     $$E_R(\omega_{vac}) \lesssim Ce^{-mr/2}$$

     for large mr, where $r = \text{dist}(A, B) > 0$.
  2. For Dirac fields and dim $M = 3$, if $A = M \cap \{y < 0\}$ and $B = M \cap \{y > r\}$ then

     $$E_R(\omega_{vac}) \lesssim C \left|\ln(mr)\right| \frac{|\partial A|}{r^{d-1}}.$$

  3. In axiomatic QFT in $d + 1$ under a nuclearity condition,

     $$E_R(\omega_{vac}) \lesssim Ce^{-(mr)^k} \quad E_R(\omega_{vac}) \lesssim Cr^{-\alpha+1}$$

     for the vacuum ω_{vac} and thermal states ω_β.

  (close to formal von Neumann entropy computations if $m > 0$, at least modulo $\ln(mr)$)

Outlook

The QFT framework based on quasi-free states gives direct access to the modular structures (cf. Longo '21).

QFT on curved space-times provides details on the quasi-free states of physical interest (e.g. Gérard–Wrochna '15).

Statistical mechanics provide the right intuitions and questions (entropy production, two-time measurements, etc.) in this context!

Entanglement entropy on AdS: no rigorous attempt on anti-de Sitter spacetimes yet (however, isomorphisms of bulk and boundary algebras proved by Dybalski–Wrochna '19).

Leaves many questions open, in particular geometric formulae, black hole spacetimes, AdS spacetimes, etc. What's behind holographic formulae for formal von Neumann entropy? Is there a better UV cutoff than $r = \mathrm{dist}(A, B)$*?*