Magic square (help␣magic): magic(10) ↵

Hilbert matrix (help␣hilb) : hilb(6) ↵

help␣ones : ones(7) ↵

help␣zeros : zeros(20) ↵

identity matrix (help␣eye) : eye(9) ↵

transpose of a matrix A : A' ↵
(A^T) ex. magic(4)

help␣diag : B=rand(4)

random matrix (help␣rand) : rand(11) ↵
format␣long ↵
rand(11) ↵

square root of a matrix (help␣sqrtm) : sqrtm(magic(5)) ↵

exponential of a matrix (help␣expm) : expm(magic(4)) ↵

take power 2 $\overset{?!}{\Rightarrow}$ magic(5) ?!
8

inverse of a matrix (help␣inv):

C=rand(4) ↵
inv(C) ↵
inv(inv(C)) ↵ compare!

Define by hand:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightsquigarrow$$

```
>> A = [1 2 3; 4 5 6; 7 8 9]
```

with or without ;

Linear system: $\begin{cases} x_1 + 2x_2 = 17 \\ 2x_1 - x_2 = -1 \end{cases} \iff A\vec{x} = \vec{b}$

$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 17 \\ -1 \end{pmatrix}$

in Matlab:

```
>> A = [1 2; 2 -1] (;)
>> b = [17 -1] (;)   (or? [17; -1] ?)
```

try first `A * b` $\Rightarrow$ error message

then `A * b'` $\Rightarrow$ ~~8~~

(`'` = transpose)

how to solve $A\vec{x} = \vec{b}$?

1) `x = inv(A) * b' (;)`

2) `x = A \ b' (;)`

check the differences!

```
>> length(b) (;)
>> size(A) (;)
```

`b = diag(ones(50))` $\to$ `length(b) = 50`

`A = hilb(100)` $\to$ `size(A) = 100 100`

---

determinant of a matrix (help␣det): det (magic(6)) (@)

det (magic(10)) (@)

det (magic(100)) (@)

→ "0"

condition number of a matrix (help␣cond): cond(magic (10)) (@)

cond (magic (100)) (@)

"↗ goes up"

norm of a matrix (help␣norm): norm (magic(3)) (@)

→ different norms ←

eigenvalues of a matrix (help␣eig): eig (magic(5)) (@)

[V,D] = eig (magic(5)) (@)

matrix of eigenvectors

matrix on the diag: eig. values

V (@)

D (@)

diag (D) (@)

rank of a matrix (help␣rank): rank (hilb(5)) (@)

→ 5

rank (ones(10)) (@)

→ 1

rank (zeros(3)) (@)

→ 0

"combine" matrices → new matrix

M = [ones(3)␣zeros(3); hilb(3)␣rand(3)] (@)

try!

→ 100×100 system

→ random matrix A

→ vector $\vec{b}$: only "1's".

solve with Matlab $A\vec{x} = \vec{b}$ $\begin{cases} \text{inv} \\ \text{"}\backslash\text{"} \end{cases}$

compare results

does $A\vec{x} - \vec{b}$ give the "zero-vector" ?

norm(A * x − b) = ?

$$C = \begin{pmatrix} 6 & 5 \\ \pi & \sin(1) \end{pmatrix} \qquad D = \begin{pmatrix} \tan(3) & -\frac{1}{\sqrt{8}} \\ 5 & \end{pmatrix}$$

calculate $CD - DC \Rightarrow ? \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$?

determinant : >> det(C), det(D), det(CD − DC), det(CD)−det(DC)

<u>trace</u> of a matrix (help, trace) : >> hilb(3) (?) >> trace(hilb(3)) (?)

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

difference between A * B and A .* B ?

↑

$\boxed{\text{Method of Jacobi}}$ $\quad A\vec{x}=\vec{b}$

$D \overset{d}{=}$ matrix of diagonal elements of A

assume these elements are $\neq 0$

$$A\vec{x}=\vec{b} \iff D^{-1}A\vec{x} = D^{-1}\vec{b} \iff D^{-1}A\vec{x}-\vec{x} = D^{-1}\vec{b}-\vec{x}$$

$$\iff \vec{x}-D^{-1}A\vec{x} = \vec{x}-D^{-1}\vec{b}$$

$$\iff (I-D^{-1}A)\vec{x} = \vec{x}-D^{-1}\vec{b}$$

$$\iff \vec{x} = (I-D^{-1}A)\vec{x}+D^{-1}\vec{b}$$

iterative process: $\vec{x}_{j+1} = (I-D^{-1}A)\vec{x}_j + D^{-1}\vec{b} \quad j=0,1,2,\ldots$

$$\Longrightarrow \begin{cases} \vec{x}_j = (I-D^{-1}A)^j\vec{x}_0 + D^{-1}\displaystyle\sum_{k=0}^{j}(I-D^{-1}A)^k\vec{b} \\ j=1,2,\ldots \end{cases}$$

"geometric series": $\displaystyle\lim_{j\to\infty} D^{-1}\sum_{k=0}^{j}(I-D^{-1}A)^k\vec{b} = D^{-1}(I-(I-D^{-1}A))^{-1}\vec{b}$

$$= D^{-1}(D^{-1}A)^{-1}\vec{b}$$

$$= A^{-1}\vec{b}$$

suppose: $|\lambda(I-D^{-1}A)|<1$

then $\displaystyle\lim_{j\to\infty}(I-D^{-1}A)^j\vec{x}_0 = \vec{0}$

independent from $\vec{x}_0$

$\Longrightarrow$ method of Jacobi converges to $A^{-1}\vec{b}$ (the solution of the original linear system)

In general

"stationary" method
(G does not depend on j)

$$A\vec{x} = \vec{b} \iff \vec{x} = G\vec{x} + \vec{c} \quad \boxed{**}$$

$\left(\text{example: } \begin{array}{l} G = I - D^{-1}A \\ \vec{c} = D^{-1}\vec{b} \end{array} \right)$

$$\boxed{*} \begin{cases} \vec{x}_{j+1} = G\vec{x}_j + \vec{c}, \; j = 0, 1, \ldots \\ \vec{x}_0 = \ldots \end{cases}$$

<u>theorem:</u>

$\boxed{*}$ converges to the unique solution of $\boxed{**}$ $\forall \vec{x}_0$ if $\rho(G) < 1$

$(\max |\lambda(G)| < 1)$

shift index $j \to j-1$

$$\vec{x}_j = G\vec{x}_{j-1} + \vec{c}$$

$$\iff \vec{x}_j - \vec{x} = G(\vec{x}_{j-1} - \vec{x}) \quad (\vec{c} \text{ drops})$$

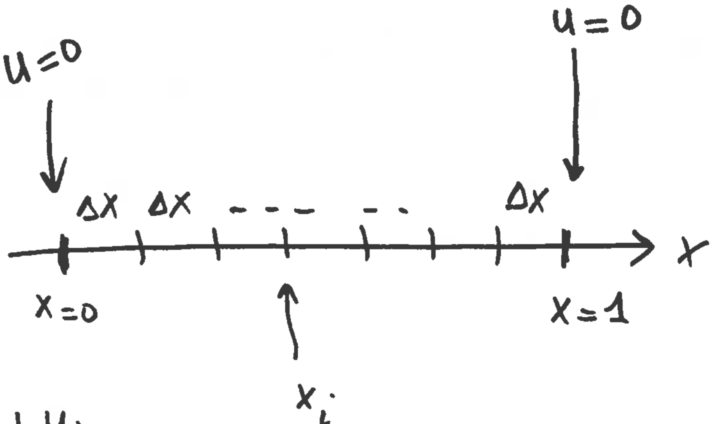
$$\begin{aligned} \Rightarrow \|\vec{x}_j - \vec{x}\| &\leq \|G\| . \|\vec{x}_{j-1} - \vec{x}\| \\ &= \|G\| . \|G(\vec{x}_{j-2} - \vec{x})\| \\ &\leq \|G\|^2 . \|\vec{x}_{j-2} - \vec{x}\| \\ &\vdots \\ &\leq \|G\|^j . \|\vec{x}_0 - \vec{x}\| \end{aligned}$$

$$\rho(G) < 1 \Rightarrow \lim_{j \to \infty} \|G\|^j = 0$$

$$\Rightarrow \lim_{j \to \infty} \|\vec{x}_j - \vec{x}\| = 0$$

Assignment 2b

D_2 - matrix



$$-u''\Big|_{x=x_i} \approx \frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta x)^2}$$

$i = \dots, \dots$

find

$$\vec{u} = \begin{pmatrix} \vdots \\ u_{i-1} \\ u_i \\ u_{i+1} \\ \vdots \end{pmatrix} \quad ?$$

length?

$$\vec{b} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

length?

?

$$\text{"}+\text{"}\, D_2 = \frac{1}{(\Delta x)^2} \begin{pmatrix} \ddots & & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

−2 on diagonal

1's on upper and lower diagonal

Assignment 2b (d)

Also Useful for lectures 4, 5, 6 and final project

Consider the linear system of ODEs: $\dot{\vec{y}} = A\vec{y}$ ← vector

time-derivative

↑ matrix (given)

with initial vector: $\vec{y}(0) = \vec{y}_0$ ← given

Two basic methods for

Euler-Forward: $\dfrac{\vec{y}^{n+1} - \vec{y}^n}{\Delta t} = A\vec{y}^n$, $n = 0, 1, 2, \dots, N-1$ ← choose

choose stepsize

$\vec{y}^0 = \vec{y}_0$

$\Rightarrow \begin{cases} \vec{y}^0 = \vec{y}_0 \\ n = 0, 1, \dots, N-1 \\ \vec{y}^{n+1} = (I + \Delta t A)\vec{y}^n \end{cases}$

"for"

Euler-Backward: $\dfrac{\vec{y}^{n+1} - \vec{y}^n}{\Delta t} = A\vec{y}^{n+1}$, $n = 0, 1, 2, \dots, N-1$

$$\Rightarrow \vec{y}^{n+1} = (I - \Delta t A)^{-1}\vec{y}^n$$

$\begin{cases} \vec{y}^0 = \vec{y}_0 \\ \underbrace{(I - \Delta t A)}_{\text{define } B \text{ (matrix)}}\vec{y}^{n+1} = \vec{y}^n \end{cases} \rightsquigarrow B\vec{y}^{n+1} = \vec{y}^n \rightarrow \begin{cases} n = 0, 1, \dots, N-1 \\ \vec{y}^{n+1} = B\backslash\vec{y}^n \end{cases}$

Matlab

"Exact" $\vec{y}(t) = e^{tA}\vec{y}_0$ ← expm in matlab

[[IMEX: $\dot{\vec{y}} = A\vec{y} + B\vec{y} \rightsquigarrow \dfrac{\vec{y}^{n+1} - \vec{y}^n}{\Delta t} = A\vec{y}^{n+1} + B\vec{y}^n \Rightarrow (I - \Delta t A)\vec{y}^{n+1} = (I + \Delta t B)\vec{y}^n$

loop $n = 0, 1, 2, \dots, N-1$]]

Condition number of a matrix A

extra info

want to solve $A\vec{x} = \vec{b}$

but "in computer": $(A + \varepsilon F)\vec{x}(\varepsilon) = \vec{b} + \varepsilon \vec{f}$

small perturbations ("rounding" .....)

$0 < \varepsilon \ll 1$

take $\frac{d}{d\varepsilon}$: $\quad A\frac{d\vec{x}}{d\varepsilon} + F\vec{x} + \varepsilon F\frac{d\vec{x}}{d\varepsilon} = \vec{0} + \vec{f} \qquad (\vec{x} = \vec{x}(\varepsilon))$

"at" $\varepsilon = 0$: $A\frac{d\vec{x}}{d\varepsilon}(0) + F\vec{x}(0) = \vec{f}$

$\iff$ (if A^{-1} exists) $\quad \frac{d\vec{x}}{d\varepsilon}(0) = A^{-1}(\vec{f} - F\vec{x}(0)) = A^{-1}(\vec{f} - F\vec{x}) \qquad (*)$

Taylorseries for $\vec{x}(\varepsilon)$ "around" $\varepsilon = 0$:

$$\vec{x}(\varepsilon) = \vec{x} + \varepsilon\frac{d\vec{x}}{d\varepsilon}(0) + O(\varepsilon^2)$$

for relative error:

$$\frac{\|\vec{x}(\varepsilon) - \vec{x}\|}{\|\vec{x}\|} = \frac{\|\varepsilon\frac{d\vec{x}}{d\varepsilon}(0) + O(\varepsilon^2)\|}{\|\vec{x}\|}$$

$$\overset{(*)}{=} \frac{\|\varepsilon A^{-1}(\vec{f} - F\vec{x}) + O(\varepsilon^2)\|}{\|\vec{x}\|}$$

$$\leq \varepsilon\|A^{-1}\|\left\{\frac{\|\vec{f}\|}{\|\vec{x}\|} + \frac{\|F\vec{x}\|}{\|\vec{x}\|}\right\} + O(\varepsilon^2)$$

$$\leq \varepsilon\|A^{-1}\|\left\{\frac{\|\vec{f}\|}{\|\vec{x}\|} + \|F\|\right\} + O(\varepsilon^2) \qquad (**)$$

matrix-norm & $\|F\vec{x}\| \leq \|F\| \cdot \|\vec{x}\|$

Define condition number of A : $K(A) = \|A\| \cdot \|A^{-1}\| \in [1, \infty)$

$K(A)$ depends on the used norm

Note: $A\vec{x} = \vec{b} \Leftrightarrow \|A\vec{x}\| = \|\vec{b}\|$

$$\Downarrow$$

$$\|\vec{b}\| = \|A\vec{x}\| \leq \|A\| . \|\vec{x}\|$$

$$\Rightarrow \frac{\|\vec{f}\|}{\|\vec{x}\|} \leq \frac{\|\vec{f}\| . \|A\|}{\|\vec{b}\|}$$

Define relative error in A and $\vec{b}$, respectively, ρ_A and ρ_b, then

with $\rho_A = \varepsilon \frac{\|F\|}{\|A\|}$ and $\rho_b = \varepsilon \frac{\|\vec{f}\|}{\|\vec{b}\|}$

(**) is equivalent with: $\frac{\|\vec{x}(\varepsilon) - \vec{x}\|}{\|\vec{x}\|} \leq K(A) . (\rho_A + \rho_b) + O(\varepsilon^2)$

conclusion: if $K(A)$ large, then the relative error in $\vec{x}$ __can__ become large

("A is ill-conditioned")

[If A is an orthogonal matrix, then A is perfectly-conditioned:

$K_2(A) = 1$

$$K_2(A) = \|A\|_2 . \|A^{-1}\|_2 = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$$

__2-norm__]

Determinant of A versus condition number of A

$\boxed{A\vec{x} = \vec{b}}$

example 1 $\quad B_N = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ & 1 & -1 & \cdots & -1 \\ & & \ddots & & -1 \\ & 0 & & \ddots & -1 \\ & & & & 1 \end{pmatrix}$ $\Rightarrow \det B_N = 1^N = 1 \; (\forall N)$ — ρ!!

ill-conditioned

and $K_\infty(B_N) = \|B_N\|_\infty . \|B_N^{-1}\|_\infty$

$$= \max_i \sum_{j=1}^{N} |a_{ij}| . \|B_N^{-1}\|_\infty$$

$$= N . 2^{N-1} \longrightarrow \infty \text{ for } N \to \infty$$

example 2 $\quad D_N = \begin{pmatrix} 10^{-1} & & \Theta \\ & 10^{-1} & \\ \Theta & & \ddots \\ & & & 10^{-1} \end{pmatrix}$

$$\det D_N = (10^{-1})^N$$
$$= 10^{-N} \to 0 \text{ for } N \to \infty \quad \text{"singular"}$$

and $\kappa_p(D_N) = 1$ "perfect" condition number

$$\left\{ \begin{aligned} \kappa_p(D_N) = \|D_N\|_p \cdot \|D_N^{-1}\|_p &= \sup_{\|\vec{x}\|_p=1} \|D_N \vec{x}\|_p \cdot \sup_{\|\vec{x}\|_p=1} \|D_N^{-1} \vec{x}\|_p \\ &= 10^{-1} \sup_{\|\vec{x}\|_p=1} \|I\vec{x}\|_p \cdot \sup_{\|\vec{x}\|_p=1} \|D_N^{-1}\vec{x}\|_p \\ &= 10^{-1} \sup_{\|\vec{x}\|_p=1} \|\vec{x}\|_p \cdot 10 \cdot \sup_{\|\vec{x}\|_p=1} \|\vec{x}\|_p \\ &= 10^{-1} \cdot 10 = 1 \end{aligned} \right\}$$

$\left\{ \text{in example 1: } B_N^{-1} = \begin{pmatrix} & 2^{N-1} \\ & \vdots \\ \Theta & 2^{N-j} \\ & \vdots \\ & 2 \\ & 1 \end{pmatrix} \text{ and } \max_i \sum_{j=1}^{N} |a_{ij}| = 2^{N-1} \right\}$