Schedule

Day 3, Today :
$$\begin{cases} f(x)=0 & \Rightarrow x \approx \cdots \\ f(x,y)=0,\ g(x,y)=0 & \Rightarrow (x,y) \approx \cdots \\ f(z)=0 & \Rightarrow z \approx \cdots \end{cases}$$

Day 4, Monday May 4 :

- matrices $D_1, D_2, \cdots$
- heat equation
- nonlinear PDE; IMEX-method

Day 5, Thursday May 7:

- $X^2 - A = 0 \Rightarrow X \approx \cdots$
- $\sqrt{A}$

Day 6, Monday May 11:

- matrix D_3; $-\sqrt{-D_2}$
- fractional derivative
- space-fractional heat equation
- blow-up DE; Sundman transformation

Thursday May 14: No Class ("Hemelvaartsdag")

Monday May 18: $13^{15} - 15^{00} / 17^{00}$ Questions can be asked about Project 1

Friday May 22: Deadline Report 1

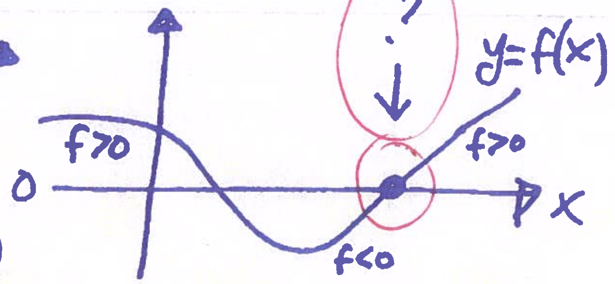
$\leq$ 23:59

- individual
- 1 pdf via e-mail
- Matlab-codes in attachment

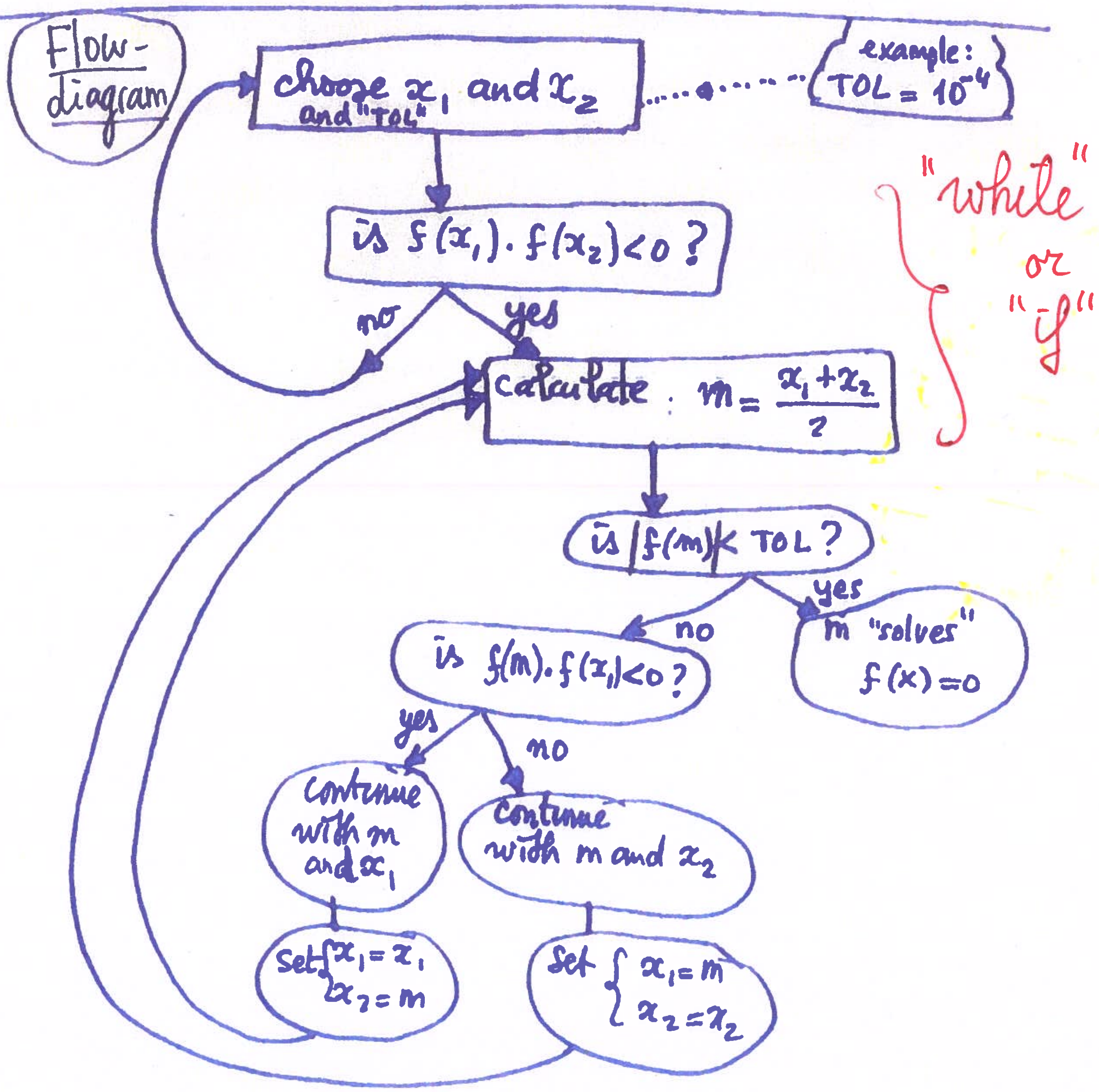
② Solving nonlinear equations: $f(x)=0$

Bisection method

$f(x)$ in f.m



{makes use of intermediate value theorem}



Consider the sequence $x_0, x_1, x_2, \ldots$

Does this sequence converge?

Example
$$\begin{cases} x_0 = 2 \\ x_{i+1} = \underbrace{\frac{1}{2}\left(x_i + \frac{2}{x_i}\right)}_{\text{a nonlinear recursion}}, \quad i = 0,1,2,3,\ldots \end{cases}$$

("successive substitution")

Calculate:
$$x_1 = \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{1}{2} \cdot 3 = \frac{3}{2} = 1.5$$
$$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{1}{2} \cdot \frac{17}{6} = \frac{17}{12} \approx 1.41667$$
$$x_3 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \cdots \approx 1.4142\ldots$$

etcetera

Note:
$$\underbrace{\lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} x_i = \lim_{i\to\infty} x_{i-1} = \cdots}_{\text{call this limit } x \text{ (if it exists)}}$$

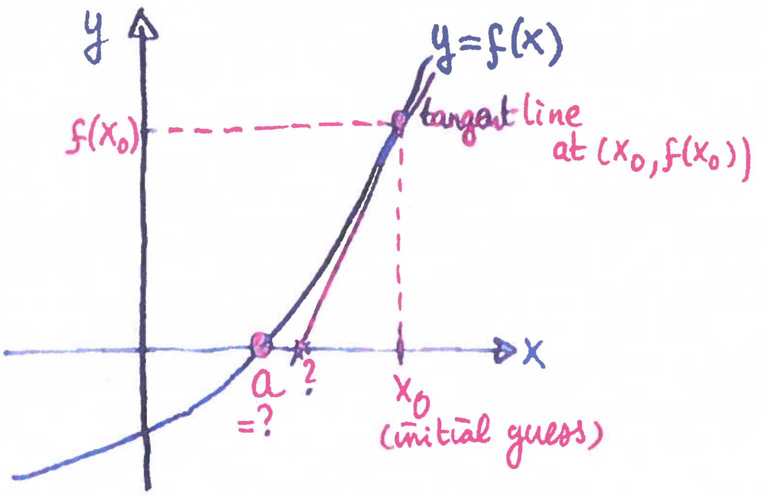
$$\Rightarrow \lim_{i\to\infty}: \quad x = \frac{1}{2}\left(x + \frac{2}{x}\right) \Leftrightarrow 2x = x + \frac{2}{x}$$
$$\Leftrightarrow 2x^2 = x^2 + 2$$
$$\Leftrightarrow \boxed{x^2 - 2 = 0}$$
$$\Leftrightarrow x_{1,2} = \pm\sqrt{2}$$

Check:

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right)$$

$$\Leftrightarrow x_{i+1} = x_i - \frac{x_i^2 - 2}{2x_i} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad f(x) = x^2 - 2$$

A derivation using tangent lines



The tangent line at $(x_0, f(x_0))$: $y = Ax + B$ with $A = f'(x_0)$ $\Rightarrow y = f'(x_0)x + B$

this straight line goes through $(x_0, f(x_0))$: $f(x_0) = f'(x_0)x + B$

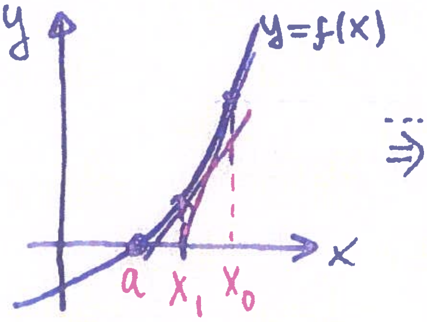
$$\Rightarrow B = f(x_0) - f'(x_0)x_0$$

$$\Rightarrow y = f'(x_0)x + f(x_0) - f'(x_0)x_0$$

The zero of this line: $0 = f'(x_0)x + f(x_0) - f'(x_0)x_0$

$$\Rightarrow x = \frac{f'(x_0)x_0 - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

call this zero x : x_1 and repeat the process



$$\Rightarrow \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

etcetera $x_3 = --$

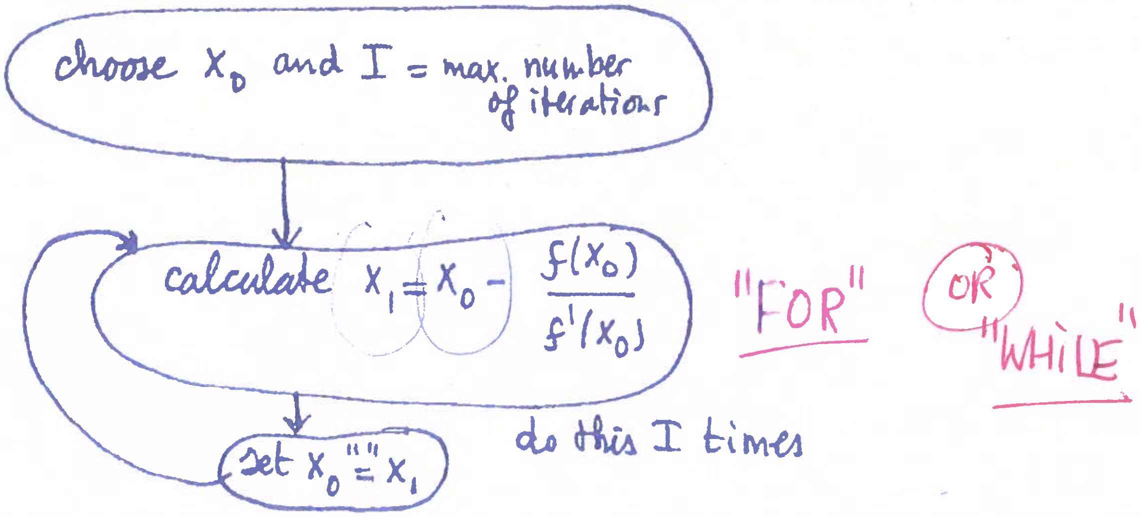
$x_4 = --$

Newton-Raphson

- starting value (initial guess): $x_0 = ?$
- how/when to stop the process (iteration) ?

Newton's method in Matlab

- define $f(x)$ in separate file f.m (or using "@")
- similarly $f'(x)$ in fp.m

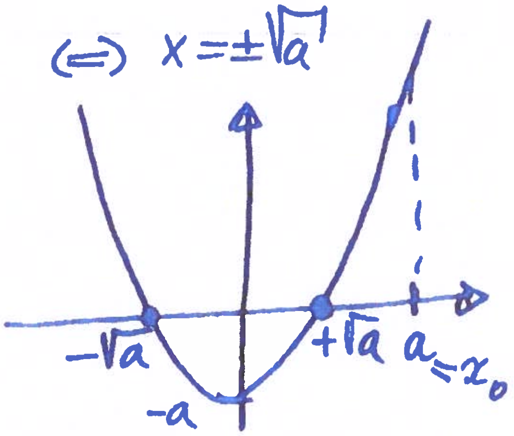


instead of I, you could make use of a <u>while loop</u>, where you test, whether $|x_1 - x_0| < TOL$ ← defined at the beginning

⑥

$f(x) = x^2 - a = 0 \iff x = \pm\sqrt{a}$

$(a>0)$



Newton-Raphson: $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$✱ \quad = x_k - \frac{x_k^2 - a}{2x_k} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$$

$k = 0, 1, 2, \ldots$ ("$k \to \infty$")

initial guess: $x_0 = a > \sqrt{a}$

Continuous Newton: $\underline{\text{ODE}}$ $\begin{cases} \dot{x} = -\dfrac{x^2 - a}{2x} \\ x(0) = a \end{cases}$

$\Longrightarrow x(t) = \ldots$

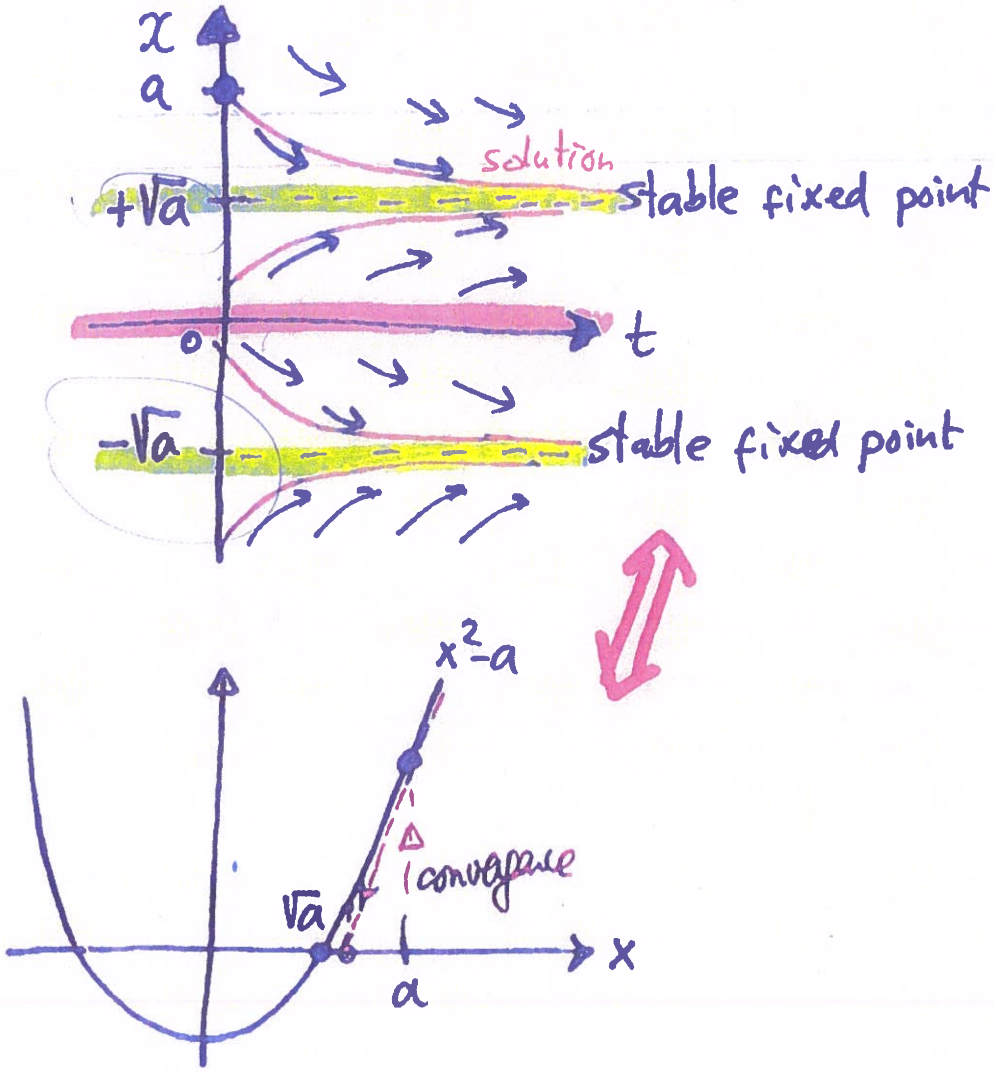
take limit "$t \to \infty$"

index

apply Euler-Forward: $\begin{cases} x^{n+1} = x^n + \Delta t\left(-\dfrac{(x^n)^2 - a}{2x^n}\right) \\ x^0 = a \end{cases}$, $n = 0, 1, 2, \ldots$

"take" $\Delta t = 1$: $\begin{cases} x^{n+1} = x^n - \dfrac{(x^n)^2 - a}{2x^n} \\ x^0 = a \end{cases}$, $n = 0, 1, 2, \ldots$ $=$ ✱

⑦



$x_0 > 0 \Rightarrow \lim\limits_{t \to \infty} x(t) = \sqrt{2}$

$x_0 = 0$ ↯

$x_0 < 0 \Rightarrow \lim\limits_{t \to \infty} x(t) = -\sqrt{2}$

⑧

Alternative

for the equation $f(x) = x^2 - a = 0$:

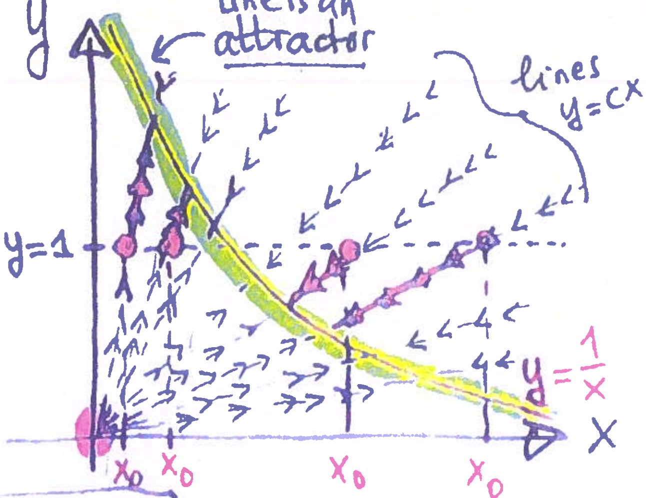
$$k=0,1,2,\dots \quad \begin{cases} x_{k+1} = \frac{1}{2}\left(x_k + \frac{1}{y_k}\right), & x_0 = a > 0 \\ y_{k+1} = \frac{1}{2}\left(y_k + \frac{1}{x_k}\right), & y_0 = 1 \end{cases}$$

"Denman-Beavers" method

continuous Denman Beavers: $\begin{cases} \dot{x} = -\frac{1}{2}x + \frac{1}{2y}, & x(0) = a > 0 \\ \dot{y} = -\frac{1}{2}y + \frac{1}{2x}, & y(0) = 1 \end{cases}$

Euler-Forward with step-size $\Delta t = 1$

$\boxed{\text{dynamical system}}$



$\boxed{\text{phase-plane}}$

$$\lim_{t\to\infty} (x(t), y(t)) = \left(\sqrt{a}, \frac{1}{\sqrt{a}}\right)$$

⑨ Newton-Raphson : $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

$f(x) = 0$

$\Downarrow$ can be shown (if x_0 is chosen appropriately)

error $= x_i - a$
decreases quadratically

$$x_{i+1} - a \approx c.(x_i - a)^2$$

<u>extension</u>:

define $\varphi(x_i) \overset{d}{=} x_i - \dfrac{f(x_i)}{f'(x_i)}$

and $x_{i+1} = \Phi(x_i) \overset{d}{=} \dfrac{x_i \cdot \varphi(\varphi(x_i)) - (\varphi(x_i))^2}{\varphi(\varphi(x_i)) - 2\varphi(x_i) + x_i}$

$\Rightarrow$ higher-order convergence
(check with Matlab for $x^2 - 2 = 0$, which order of convergence do you find?)

A (very) high-order (see exercises) can be obtained with the <u>Gerlach-scheme</u>:

$$\begin{cases} x_0 = 9/2 \\ x_{i+1} = x_i - \dfrac{\cdots\cdots\cdots}{\cdots\cdots\cdots} \end{cases}$$

$i = 0, 1, 2, \cdots$

which order of convergence?

Two dimensions

- The <u>gradient</u> of a <u>scalar</u> function $f(x,y): \underset{\text{"vector"}}{\mathbb{R}^2} \to \underset{\text{number}}{\mathbb{R}}$

$$\text{grad}(f) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \frac{\partial f}{\partial x} \cdot \underset{\begin{pmatrix}1\\0\end{pmatrix}}{\underset{\parallel}{\vec{i}}} + \frac{\partial f}{\partial y} \cdot \underset{\begin{pmatrix}0\\1\end{pmatrix}}{\underset{\parallel}{\vec{j}}}$$

$$= \underset{\text{'habla" or "del"}}{\nabla} f \qquad \boxed{\text{scalar} \to \text{vector}}$$

example: $f(x,y) = 6xy - y^2$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(6xy - y^2) = 6y \\ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(6xy - y^2) = 6x - 2y \end{cases}$$

$$\Rightarrow \text{grad}(f) = \begin{pmatrix} 6y \\ 6x-2y \end{pmatrix}$$

* the <u>Jacobian</u> (Jacobi matrix) of a vectorfunction $\vec{f}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}, \text{ a } 2 \times 2 \text{ matrix}$$

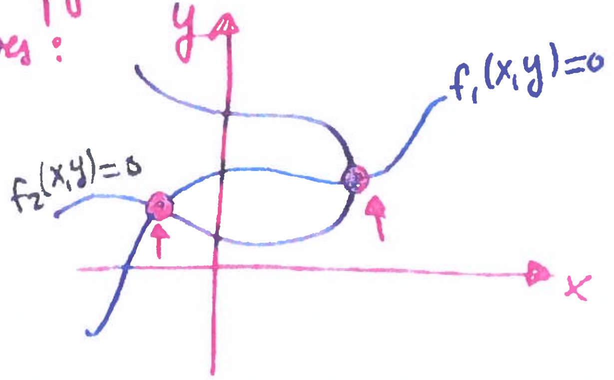
Given are two functions $f_1(x,y)$ and $f_2(x,y)$

Find $(x,y) \in \mathbb{R}^2$ such that $\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases}$ $\circledast$

$f_1(x,y)=0$ is a curve in the x-y plane

$f_2(x,y)=0$ is another curve in the x-y plane

the (x,y) that satisfy $\circledast$ are the points of intersection of the two curves:



⑫

$$\begin{cases} f_1(x,y) \approx f_1(a,b) + \frac{\partial f_1}{\partial x}(a,b)(x-a) + \frac{\partial f_1}{\partial y}(a,b)(y-b) \\ f_2(x,y) \approx f_2(a,b) + \frac{\partial f_2}{\partial x}(a,b)(x-a) + \frac{\partial f_2}{\partial y}(a,b)(y-b) \end{cases}$$

2 D-Taylor expansion for f_1 and f_2 near the point $(a,b) \in \mathbb{R}^2$

"set" $\approx \rightarrow =$ and $=0$ $\Rightarrow$ solve the two linear equations in the two unknowns x and y

$$\Rightarrow \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}}_{\text{Jacobian } J}\Bigg|_{(a,b)} \begin{pmatrix} x-a \\ y-b \end{pmatrix} = - \begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} - \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}^{-1}}_{= J^{-1}}\Bigg|_{(a,b)} \begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix}$$

repeat this process as in 1D:

$$\begin{cases} \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} - J^{-1}\Big|_{(x_i,y_i)} \begin{pmatrix} f_1(x_i,y_i) \\ f_2(x_i,y_i) \end{pmatrix} \quad i=0,1,2,3,\ldots \\ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ starting values} \end{cases}$$

Newton-fractals

Apply Newton-Raphson to $f(z)=0$, $z \in \underset{\uparrow}{\mathbb{C}}$

$$\Rightarrow \begin{cases} z_{i+1} = z_i - \dfrac{f(z_i)}{f'(z_i)} , & i=0,1,2,\dots \\ z_0 = - \in \mathbb{C} \end{cases}$$

Example: $f(z) = z^3 - 1$, $f'(z) = 3z^2$

There are three complex solutions z:

$$\begin{cases} z_1 = 1 \\ z_2 = \frac{1}{2}(-1+\sqrt{-3}) \\ z_3 = \frac{1}{2}(-1-\sqrt{-3}) \end{cases}$$

Run NR "until convergence" and check to which of the three roots (solutions) the method converges.

Assign the color "1" , if it converges to z_1,
the color "2" , " " " " z_2,
and color "3" , " " " " z_3.

$\Rightarrow$ (- - ?! - -)

Exercise 3f ("eigenvalues")

eigenvalue/vector problem:

$$\underset{\substack{\uparrow\\ n\times n\ \text{matrix}\\ (\text{given})}}{A}\,\vec{x} = \underset{\substack{\uparrow\\ \text{eigenvalue}}}{\lambda}\,\overset{\in\mathbb{R}^n}{\vec{x}} \leftarrow \substack{\text{eigenvector}\\ (\text{not}\\ \text{unique})}$$

Take $\vec{x}$ such that $\|\vec{x}\| = 1$ (a unique eigenvector of length 1)

$$\underset{\substack{\|\\ \vec{x}\cdot\vec{x}\\ (\text{innerproduct})}}{\|\vec{x}\|} = 1 \qquad \Longleftrightarrow \qquad \vec{x}\cdot\vec{x} - 1 = 0$$

find $\vec{x}$ and λ

$$\Longrightarrow \begin{cases} A\vec{x} - \lambda\vec{x} = \vec{0} \\ \vec{x}\cdot\vec{x} - 1 = 0 \end{cases} \Longleftrightarrow \quad \vec{f}(\vec{z}) = \vec{0} \quad \text{with } \vec{z} \overset{\text{def.}}{=} \begin{pmatrix} \vec{x} \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+1}$$

A nonlinear system of equations

$$\underset{\text{Newton-Raphson}}{\Longrightarrow} \begin{cases} \vec{z}_{k+1} = \vec{z}_k - J_k^{-1}\,\vec{f}(\vec{z}_k), \quad k=0,1,\dots \\ \vec{z}_0 = \dots \ (\text{initial guess}) \end{cases}$$

J_k^{-1} ← inverse of Jacobian matrix

The case $n=2$:

$$\begin{cases} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x_1^2 + x_2^2 - 1 = 0 \end{cases}$$

$$\underset{\substack{a_{11},a_{12},a_{21},a_{22}\\ \text{given}}}{\Longleftrightarrow} \begin{cases} a_{11}x_1 + a_{12}x_2 - \lambda x_1 = 0 \\ a_{21}x_1 + a_{22}x_2 - \lambda x_2 = 0 \\ x_1^2 + x_2^2 - 1 = 0 \end{cases} \quad \dots\dots \underset{NR}{\Longrightarrow} \begin{cases} x_1 \approx \dots \\ x_2 \approx \dots \\ \lambda \approx \dots \end{cases}$$

Try this with a few matrices (Matlab)

and compare your numerical solution with "eig" in Matlab