Introduction Scientific Computing

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Lecture 6

Basic thoughts ("motivation")

$$\boxed{\text{positive integers} : 1, 2, 3, \ldots \;\Rightarrow\; \text{negative integers} : \ldots, -3, -2, -1}$$

$$\boxed{?!? \;\rightsquigarrow\; \sqrt{2}}$$

Basic thoughts ("motivation")

$$\boxed{\text{positive integers}: 1, 2, 3, \ldots \;\Rightarrow\; \text{negative integers}: \ldots, -3, -2, -1}$$

$$\boxed{?!? \;\rightsquigarrow\; \sqrt{2}}$$

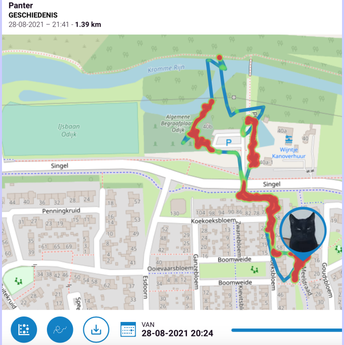
$$\boxed{\frac{\partial}{\partial x}, \;\frac{\partial^2}{\partial x^2}, \;\frac{\partial^3}{\partial x^3}, \ldots \;\Rightarrow\; \int, \;\int\int, \;\int\int\int, \ldots}$$

$$\boxed{?!? \;\rightsquigarrow\; \mathcal{D}^{\alpha}}$$

PANTER [1]

PANTER [2]

Panter
GESCHIEDENIS
31-12-2020 – 21:05 - **772.55 m**
Koekoeksbloem
Meeldraad
Goudsbloem
VAN
31-12-2020 20:11



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28-08-2021 – 21:41 - **1.39 km**
Kromme Rijn
IJsbaan Odijk
Algemene Begraafplaats Odijk
Wijntje Kanoverhuur
Singel
Penningkruid
Koekoeksbloem
Boomweide
Ooievaarsbloem
Ganzebloem
Esdoorn
Meeldraad
Goudsbloem
VAN
28-08-2021 20:24

Fractional-order PDEs: applications [1]

- Geo-hydrology (non-Fickian laws)
- Finance (Lévy-flights, non-Markovian models)
- Non-Brownian motions
- Super- and Sub-diffusion (anomalous transport)
- Visco-elasticity, Rheology, ...
- Electro-physiology of the heart

Applications [2]

Article

Fractional Diffusion Models for the Atmosphere of Mars

Salvador Jiménez [1], David Usero [2], Luis Vázquez [2] and Maria Pilar Velasco [1,*]

1 Department of Applied Mathematics to the Information and Communications Technologies, Universidad Politécnica de Madrid, 28040 Madrid, Spain; s.jimenez@upm.es
2 Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, 28040 Madrid, Spain; umdavid@mat.ucm.es (D.U.); lvazquez@fdi.ucm.es (L.V.)
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Received: 5 December 2017; Accepted: 24 December 2017; Published: 28 December 2017

Abstract: The dust aerosols floating in the atmosphere of Mars cause an attenuation of the solar radiation traversing the atmosphere that cannot be modeled through the use of classical diffusion processes. However, the definition of a type of fractional diffusion equation offers a more accurate model for this dynamic and the second order moment of this equation allows one to establish a connection between the fractional equation and the Ångstrom law that models the attenuation of the solar radiation. In this work we consider both one and three dimensional wavelength-fractional diffusion equations, and we obtain the analytical solutions and numerical methods using two different approaches of the fractional derivative.

Applications [3]

Mathematical Models and Methods in Applied Sciences
Vol. 28, No. 9 (2018) 1857–1880
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DOI: 10.1142/S0218202518400080

Crime modeling with truncated Lévy flights for residential burglary models

Applications [4]

M.M. Meerschaert, C. Tadjeran / Journal of Computational and Applied Mathematics 172 (2004) 65–77 75

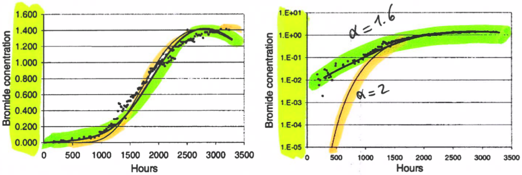
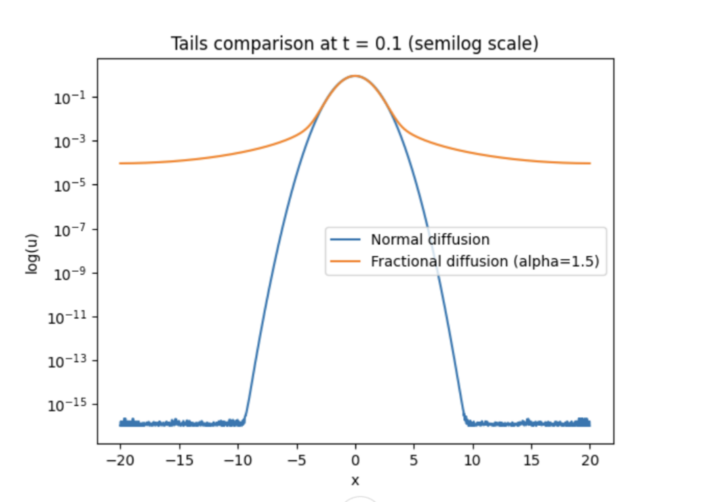
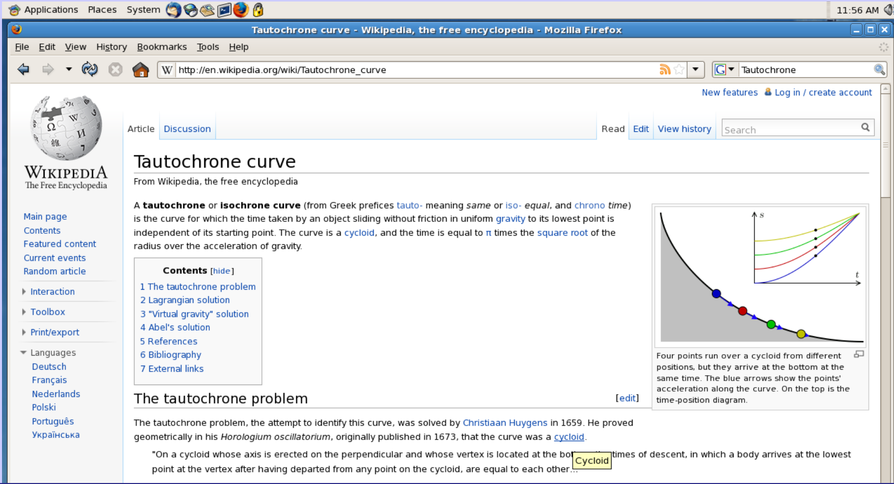


Fig. 1. Concentration of bromide tracer (gm/m^3) at extraction well. Fractional radial flow model (25) with $v_0=4.0$, $d_0=2.4$ and $\alpha = 1.6$ (thick line) captures early breakthrough better than classical radial flow model with $v_0 = 3.5$, $d_0 = 5.0$ and $\alpha = 2$ (thin line).

Applications [5]



Applications [6]: the tautochrone curve



Applications Places System 11:56 AM

Tautochrone curve - Wikipedia, the free encyclopedia - Mozilla Firefox

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http://en.wikipedia.org/wiki/Tautochrone_curve

Tautochrone

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Tautochrone curve

From Wikipedia, the free encyclopedia

A **tautochrone** or **isochrone curve** (from Greek prefices tauto- meaning *same* or iso- *equal*, and chrono *time*) is the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point. The curve is a cycloid, and the time is equal to π times the square root of the radius over the acceleration of gravity.

Four points run over a cycloid from different positions, but they arrive at the bottom at the same time. The blue arrows show the points' acceleration along the curve. On the top is the time-position diagram.

Contents [hide]

The tautochrone problem [edit]

The tautochrone problem, the attempt to identify this curve, was solved by Christiaan Huygens in 1659. He proved geometrically in his *Horologium oscillatorium*, originally published in 1673, that the curve was a cycloid.

"On a cycloid whose axis is erected on the perpendicular and whose vertex is located at the bot[Cycloid] times of descent, in which a body arrives at the lowest point at the vertex after having departed from any point on the cycloid, are equal to each other...

The tautochrone: Abel's mechanical problem

Abel, 1823 (integral equation)

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{y(s)}{(x-s)^{1-\alpha}}\, ds \;= h(x), \quad 0 < \alpha < 1$$

$\Rightarrow$

$$y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{h(s)}{(x-s)^{\alpha}}\, ds \;= \mathcal{D}_{RL}^{\alpha} h(x)$$

$\rightsquigarrow$ see later

Fourier's definition of a fractional derivative

Fourier (1822):

$$f(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} f(\gamma) \cos(p(x - \gamma)) \; dp \; d\gamma$$

$\Rightarrow$

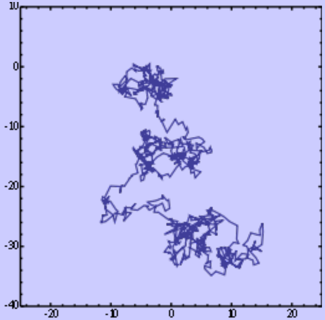
$$\frac{d^n f}{dx^n}(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} f(\gamma) p^n \cos(p(x - \gamma) + \frac{n\pi}{2}) \; dp \; d\gamma, \quad n \in \mathbb{N}$$

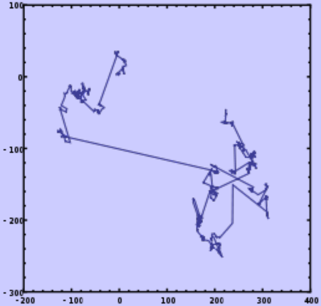
$\rightsquigarrow$

$$\frac{d^\alpha f}{dx^\alpha}(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} f(\gamma) p^\alpha \cos(p(x - \gamma) + \frac{\alpha\pi}{2}) \; dp \; d\gamma, \quad \alpha \in \mathbb{R}$$

Brownian motions vs Lévy flights [1]

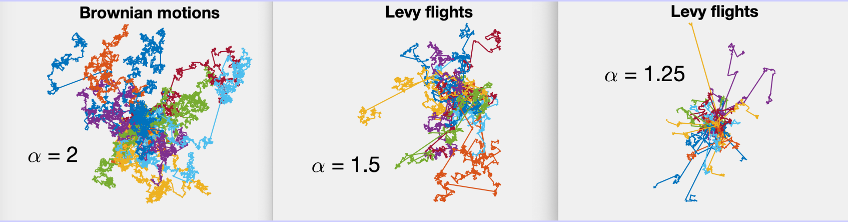
One "particle":





Brownian motions vs Lévy flights [2]

Many "particles":



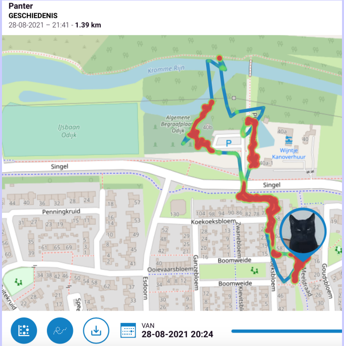
$$\lim_{N\to\infty} \Rightarrow \frac{\partial^2}{\partial x^2} = \Delta$$

vs

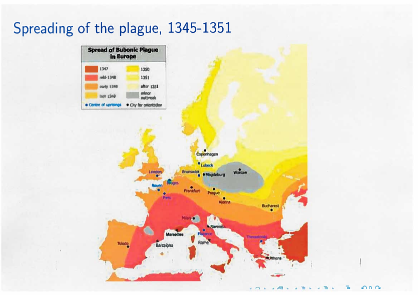
$$\lim_{N\to\infty} \Rightarrow -(-\frac{\partial^2}{\partial x^2})^{\alpha/2} = -(-\Delta)^{\alpha/2}$$

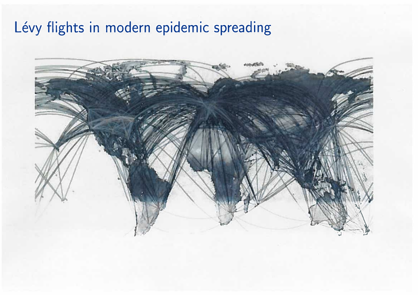
Brownian motions vs Lévy flights [3]

$\alpha \approx 2?$ vs $\alpha \approx 1.5?$

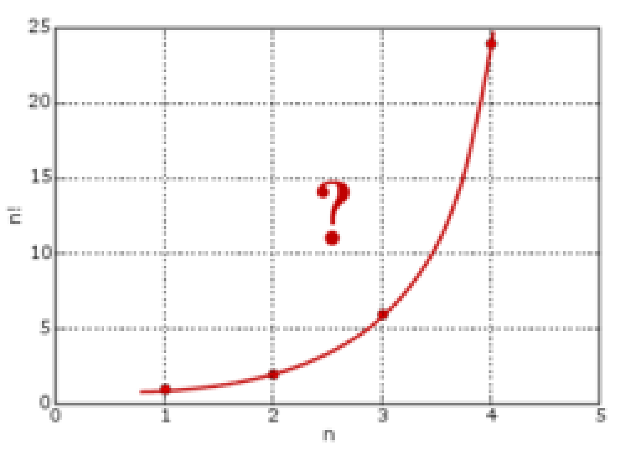


Reaction-diffusion vs *fractional* reaction-diffusion





The Gamma-function $\Gamma(x)$ [1]



The Gamma-function $\Gamma(x)$ [2]

Euler 1730, Legendre 1809 $\Gamma(x)$, Gauss $\Pi(x)$:

$$\boxed{\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt = \int_0^1 [-\ln(t)]^{x-1} dt, x > 0}$$

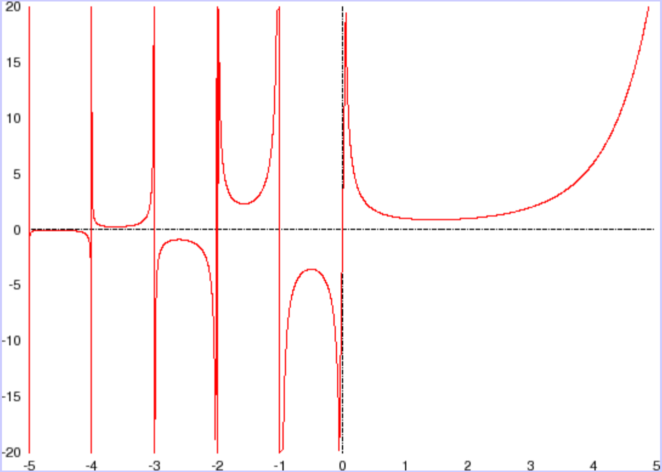
satisfies the functional equation:

$$\boxed{f(x+1) = x\, f(x),\ f(1) = 1,\ x > 0}$$

$$\Downarrow$$

$$\boxed{\Gamma(1) = 1,\ \Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \ldots = n!}$$

The Gamma-function $\Gamma(x)$ [3]



The Gamma-function $\Gamma(x)$ [4]

The function $\Gamma(x)$ is not the unique solution of the functional equation. Other solutions are, e.g.:

$\cos(2m\pi x)\Gamma(x), \; m \in \mathbb{N}$

$H(x) = \frac{1}{\Gamma(1-x)} \frac{d}{dx} \ln\left(\frac{\Gamma(\frac{1}{2}-\frac{1}{2}x)}{\Gamma(1-\frac{1}{2}x)}\right)$ Hadamard (1894)

$L(x) = \ldots$ Luschny (2006)

etcetera...

The Bohr-Mollerup theorem (1922): the Gamma function $\Gamma(x)$ is the unique solution of the functional equation, if we also demand that $f(x)$ is logarithmically convex.

Mittag-Leffler function [1]

The Mittag-Leffler function:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \; Re(\alpha) > 0,$$

A generalization with two parameters:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \; Re(\alpha) > 0, \; Re(\beta) > 0.$$

Note that: $E_{1,1}(z) = e^z$.

Mittag-Leffler function [2]

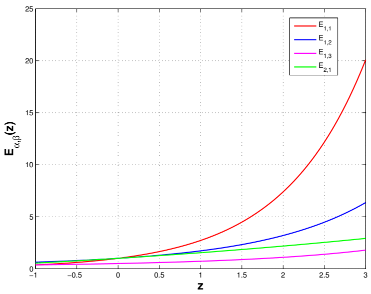
The solution of the *fractional* differential equation

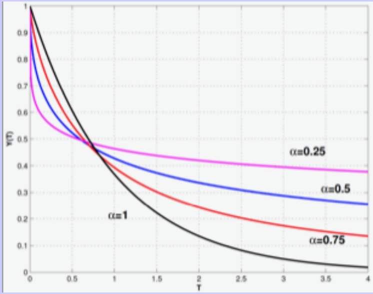
$$\begin{cases} D_t^{\alpha} u(t) = -\lambda u(t), \quad 0 < \alpha \leq 1 \\ u(0) = u_0 \end{cases}$$

reads

$$u(t) = u_0 \; E_{\alpha}(-\lambda t^{\alpha})$$

Mittag-Leffler function [3]





A mysterious "contradiction" (?)

1) $y(x) = x^k \Rightarrow \frac{d^n y}{dx^n} = \frac{k!}{(k-n)!} x^{k-n} = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} x^{k-n} \quad , k \geq n$

$\rightsquigarrow \frac{d^\alpha y}{dx^\alpha} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \quad , k \geq \alpha \in \mathbb{R}^{\geq 0}$

2) $y(x) = e^x \Rightarrow \frac{d^n y}{dx^n} = e^x \rightsquigarrow \frac{d^\alpha y}{dx^\alpha} = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} =$

$\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}, \ \alpha \in \mathbb{R}^{\geq 0} \quad [\star];$ BUT , on the other hand:

$y(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \frac{d^n y}{dx^n} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{k!}{(k-n)!} x^{k-n} = \sum_{k=0}^{\infty} \frac{x^{k-n}}{(k-n)!}$

$\sum_{k=0}^{\infty} \frac{x^{k-n}}{\Gamma(k-n+1)}, \ \rightsquigarrow \frac{d^\alpha y}{dx^\alpha} = \sum_{k=0}^{\infty} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)} \neq [\star]$!?!

Cauchy-formula for repeated integration

$$\mathcal{J}^0 f(x) = f(x)$$

$$\mathcal{J}^1 f(x) = \int_{-\infty}^{x} f(s)\ ds$$

$$\mathcal{J}^2 f(x) = \int_{-\infty}^{x} \mathcal{J}^1 f(s)\ ds$$

....

$$\mathcal{J}^n f(x) = \int_{-\infty}^{x} \mathcal{J}^{n-1} f(s)\ ds$$

for $f \in \tilde{S}(\mathbb{R})$, i.e. $P(x)\frac{d^k f}{dx^k} \to 0$ if $x \to -\infty$

Fractional integral of order $\alpha \geq 0$

$$I^n f(x) := \frac{1}{(n-1)!} \int_{-\infty}^{x} (x-s)^{n-1} f(s)\ ds \quad n! = \Gamma(n+1) \quad \text{for } n \in \mathbb{N}$$

It can be shown that: $\mathcal{J}^n f = I^n f, \quad n \in \mathbb{N}$.

Define for $\alpha \in \mathbb{R}^{\geq 0}$: $$\mathcal{J}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(s)}{(x-s)^{1-\alpha}}\ ds$$

Property: $$\begin{cases} \mathcal{J}^\alpha \mathcal{J}^\beta = \mathcal{J}^\beta \mathcal{J}^\alpha = \mathcal{J}^{\alpha+\beta} \quad \forall \alpha, \beta \geq 0 \\ \mathcal{J}^0 = \mathcal{I} \end{cases}$$

("the semi-group property of fractional differ-integral operators")

Fractional derivative of order $\alpha < m$

$$\alpha \in \mathbb{R}^{\geq 0} : \boxed{\mathcal{J}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(s)}{(x-s)^{1-\alpha}} \, ds}$$

Define the "fractional"-derivative:

$$\boxed{\mathcal{D}^{\alpha} f(x) := \mathcal{J}^{m-\alpha}\left(\frac{d^m}{dx^m} f(x)\right), \quad m > \alpha, \quad f \in \tilde{S}(\mathbb{R})}$$

Fractional derivatives: Caputo & Riemann-Liouville

The "Caputo"-derivative:

$$\boxed{\mathcal{D}_C^\alpha f(x) := \mathcal{J}_0^{m-\alpha}\left(\frac{d^m}{dx^m} f(x)\right), \quad x > 0}$$

and the "Riemann-Liouville-derivative":

$$\boxed{\mathcal{D}_{RL}^\alpha f(x) := \frac{d^m}{dx^m}\left(\mathcal{J}_0^{m-\alpha}(f(x))\right), \quad x > 0}$$

Here: $\mathcal{J}_0^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\, ds, \quad x > 0$

Note: $\mathcal{D}_C^\alpha(constant) = 0$ & $\mathcal{D}_{RL}^\alpha(constant) \sim x^{-\alpha} \neq 0$

"Consistency" of Caputo-derivative

For $f \in C^{m+1}([0,L]),\ \forall L > 0$:

$\mathcal{D}_C^{\alpha} f(x) =$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(s)}{(x-s)^{1-(m-\alpha)}} ds$$
$$= \frac{1}{\Gamma(m-\alpha)} \{ -\frac{(x-s)^{m-\alpha}}{m-\alpha} f^{(m)}(s)|_{s=0}^{s=x} + \int_0^x \frac{(x-s)^{m-\alpha}}{m-\alpha} f^{(m+1)}(s) ds \}$$
$$= \frac{1}{\Gamma(m-\alpha+1)} \{ 0 + x^{m-\alpha} f^{(m)}(0) + \int_0^x (x-s)^{m-\alpha} f^{(m+1)}(s) ds \}$$

(take limit: $\alpha \in \mathbb{R} \to m \in \mathbb{N}$)

$$= \frac{1}{\Gamma(1)} \{ f^{(m)}(0) + \int_0^x f^{(m+1)}(s) ds \}$$
$$= f^{(m)}(0) + f^{(m)}(x) - f^{(m)}(0) = \frac{d^m f}{dx^m}(x)$$

"Caputo" vs "Riemann-Liouville" [1]

(Luchko & Gorenflo, 1999, th.2.3, p.213)

Let $f \in L^1([0,\infty)) \cap C^m([0,\infty))$ and $m-1 < \alpha \leq m$ for some $m \in \mathbb{N}$. Then:

$$\mathcal{D}^{\alpha}_{RL} f(x) = \mathcal{D}^{\alpha}_{C} f(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{\Gamma(1+k-\alpha)} x^{k-\alpha} \quad (x > 0)$$

Notation: $f^{(k)}(0^+) = \lim_{x \downarrow 0} f^{(k)}(x)$

Corollary: if $f^{(k)}(0^+) = 0, \; k = 0, 1, \ldots, m-1,$ then $\mathcal{D}^{\alpha}_{RL} = \mathcal{D}^{\alpha}_{C}$

Grünwald-Letnikov-definition [1]

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f''(x) = \lim_{h_1 \to 0} \frac{\lim_{h_2 \to 0} \frac{f(x+h_1+h_2) - f(x+h_1)}{h_2} - \lim_{h_2 \to 0} \frac{f(x+h_2) - f(x)}{h_2}}{h_1}$$

Take $h = h_1 = h_2 \Rightarrow f''(x) = \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$

By induction:

$$f^{(n)}(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh), \;\; n \in \mathbb{N}$$

Grünwald-Letnikov-definition [2]

$$f^{(n)}(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \left(\begin{array}{c} n \\ m \end{array} \right) f(x - mh), \;\; n \in \mathbb{N}$$

Note: $\left(\begin{array}{c} n \\ m \end{array} \right) = \frac{n!}{m!(n-m)!}$, replace "!"-terms by "Γ"-values

Define: $$\mathcal{D}^{\alpha}_{GL} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\lceil \alpha \rceil} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha - m + 1)} f(x - mh)$$

Podlubny, 1999:

$$f \in C^{m+1}_{0-}(\mathbf{R}^{\geq 0}) := \{ f \in C^{m+1}([0,\infty)) \;\&\; f(x) = 0 \text{ for } x \leq 0 \}$$
$$\Rightarrow \qquad \mathcal{D}^{\alpha}_{GL} f(x) = \mathcal{D}^{\alpha}_{RL} f(x) = \mathcal{D}^{\alpha}_{C} f(x) = \mathcal{D}^{\alpha} f(x)$$

The fractional Laplacian

For $0 \leq \alpha \leq 2$, in one dimension:

$$-(-\Delta)^{\frac{\alpha}{2}} u(x) := \alpha \frac{2^{\alpha-1}\Gamma(\frac{1}{2}+\frac{\alpha}{2})}{\sqrt{\pi}\Gamma(1-\frac{\alpha}{2})} \int_{-\infty}^{\infty} \frac{u(x)-u(x+y)}{|y|^{1+\alpha}}\, dy$$

Theorem ($\alpha \neq 1$):

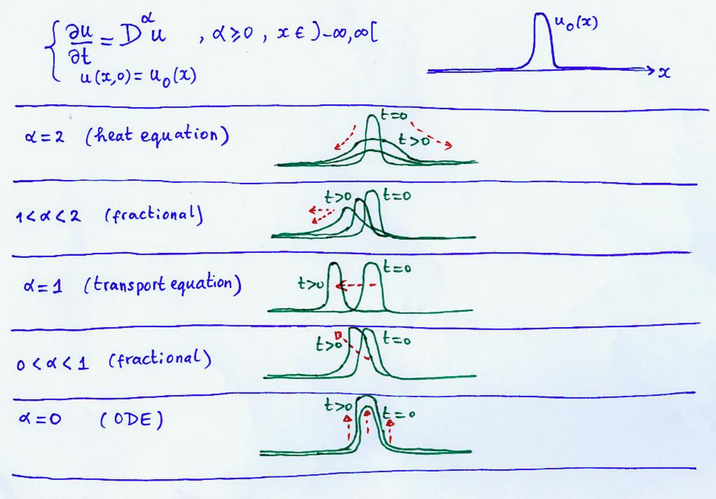
$$-(-\Delta)^{\frac{\alpha}{2}} u(x) = -\frac{1}{2\cos(\frac{\pi\alpha}{2})}[\mathcal{D}^{\alpha}_{Left} u(x) + \mathcal{D}^{\alpha}_{Right} u(x)]$$

$\alpha = 2$: $\quad -(-\Delta)^{\frac{\alpha}{2}} = \frac{\partial^2}{\partial x^2}$

$\alpha = 1$: $\quad -(-\Delta)^{\frac{\alpha}{2}} \neq \pm\frac{\partial}{\partial x}$

$\alpha = 0$: $\quad -(-\Delta)^{\frac{\alpha}{2}} = -\mathcal{I}$

The *left* space-fractional heat equation: α-variation



A *uniform* discretization for $\mathcal{D}_L^\alpha$

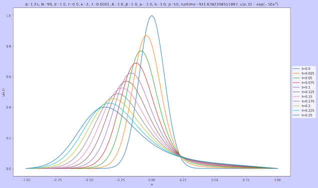
For $1 < \alpha < 2$:

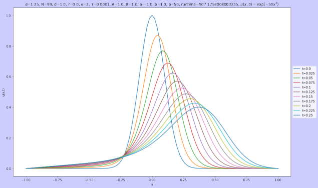
$$
\begin{aligned}
&\mathcal{D}_L^\alpha u|_{x_i} \\
&\quad = \frac{1}{\Gamma(2-\alpha)} \int_0^{x_i} \frac{u''(s)}{(x_i-s)^{\alpha-1}} ds \\
&\quad \approx \frac{1}{\Gamma(3-\alpha)} \sum_{j=1}^{i-1} \left\{ x_{j+1}^{2-\alpha} - x_j^{2-\alpha} \right\} \frac{u_{i-j+1}-2u_{i-j}+u_{i-j-1}}{h^2} \\
&\quad = \frac{1}{\Gamma(3-\alpha)} \sum_{j=1}^{i-1} \left\{ \frac{j^{2-\alpha}-(j-1)^{2-\alpha}}{h^{2-\alpha}} \right\} \left\{ \frac{u_{i-j+1}-2u_{i-j}+u_{i-j-1}}{h^2} \right\} \\
&\quad = \frac{1}{\Gamma(3-\alpha)h^\alpha} \sum_{j=1}^{i-1} \left\{ j^{2-\alpha} - (j-1)^{2-\alpha} \right\} \left\{ u_{i-j+1} - 2u_{i-j} + u_{i-j-1} \right\}
\end{aligned}
$$

Left-fractional vs left-fractional heat equation

$$\frac{\partial}{\partial t} u(x,t) = \mathcal{D}^{1.5}_{Left} u(x,t)$$

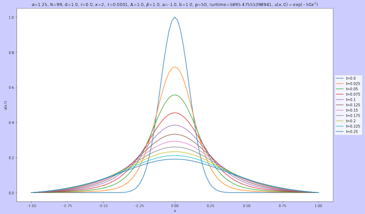
$$\frac{\partial}{\partial t} u(x,t) = \mathcal{D}^{1.5}_{Right} u(x,t)$$

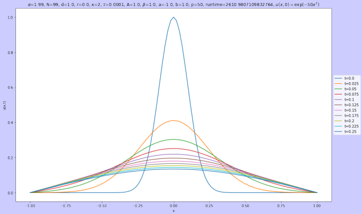




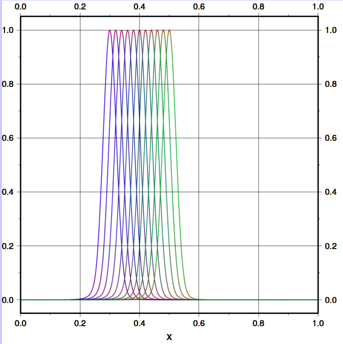
Space-fractional heat equation

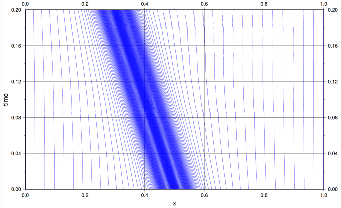
$$\frac{\partial}{\partial t} u(x,t) = -(-\Delta)^{\frac{\alpha}{2}} u(x,t) \qquad \alpha = 2 \text{ vs } 1.25$$





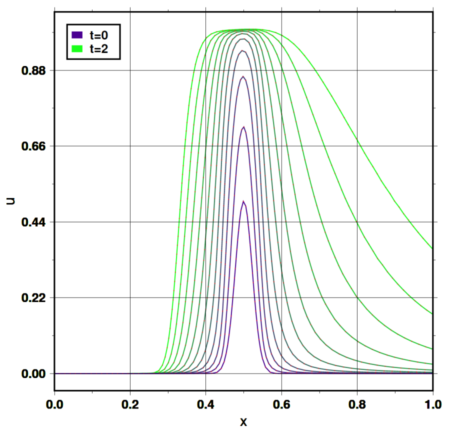
Left-fractional heat equation: $\lim_{\alpha \to 1}$





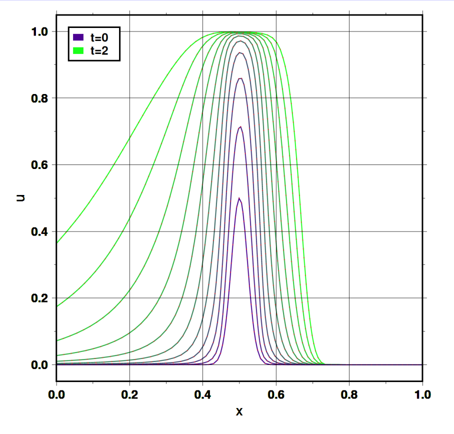
Left-fractional Fisher PDE

$$\frac{\partial}{\partial t} u(x,t) = \delta\ \mathcal{D}^{\alpha}_{Left} u(x,t) + \gamma\ u(x,t)(1-u(x,t)) \qquad \alpha = 1.5$$



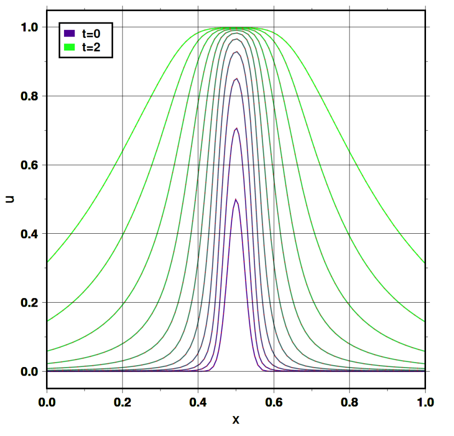
Right-fractional Fisher PDE

$$\frac{\partial}{\partial t} u(x,t) = \delta\, \mathcal{D}^{\alpha}_{Right} u(x,t) + \gamma\, u(x,t)(1-u(x,t)) \qquad \alpha = 1.5$$



Space-fractional Fisher PDE

$$\frac{\partial}{\partial t} u(x,t) = -\delta \, (-\Delta)^{\frac{\alpha}{2}} u(x,t) + \gamma \; u(x,t)(1-u(x,t)) \qquad \alpha = 1.5$$



The case $\alpha = 1$ [1]

$$(-\Delta)^{\frac{\alpha}{2}} = (-\Delta)^{\frac{1}{2}} = \mathcal{H}(\frac{\partial}{\partial x}),$$ where the *Hilbert transform*[1] $\mathcal{H}$

is given by:

$$[\mathcal{H}u](x) = u(x) \star \frac{1}{\pi x} = \frac{1}{\pi} \; p.v. \int_{-\infty}^{\infty} \frac{u(y)}{x-y} \, dy$$

Definition: $p.v. \int_{-a}^{a} f(x) \, dx = \lim_{\epsilon \to 0^+} [\int_{-a}^{-\epsilon} f(x) \, dx + \int_{\epsilon}^{a} f(x) \, dx]$

Example: $p.v. \int_{-\infty}^{\infty} \frac{1}{x} \, dx = \lim_{\epsilon \to 0} [\int_{-\infty}^{-\epsilon} \frac{1}{x} \, dx + \int_{\epsilon}^{\infty} \frac{1}{x} \, dx] = 0$

[1] used in signal processing

The case $\alpha = 1$ [2]

Properties of $\mathcal{H}$:

- $$\mathcal{H}^2 = -\mathcal{I}, \quad \mathcal{H}^4 = \mathcal{F}^4 = \mathcal{I}, \quad \mathcal{H}^{-1} = \mathcal{H}^3$$
- $$\mathcal{H} \circ \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \circ \mathcal{H}, \quad \mathcal{H}(f\, g) = f\, \mathcal{H}(g)$$
- $$\langle f, \mathcal{H} f \rangle_{L^2(\mathbb{R})} = 0, \quad ||f||_{L^2(\mathbb{R})} = ||\mathcal{F} f||_{L^2(\mathbb{R})} = ||\mathcal{H} f||_{L^2(\mathbb{R})}$$
- $$\mathcal{H}(\cos(x)) = \sin(x), \quad \mathcal{H}(\text{sinc})(x) = \frac{\pi t}{2} \text{sinc}^2(\frac{x}{2}), \ \ldots$$
- etcetera....

Finite-difference matrices [1]

Finite-difference matrices for $\boxed{\frac{\partial}{\partial x}}$ with periodic BCs

on a three-point stencil x_{i-1}, x_i, x_{i+1} (constant Δx)

$$D_{1\gamma} = \frac{1}{2\Delta x}\begin{pmatrix} 2\gamma & 1-\gamma & \cdots & \cdots & -(\gamma+1) \\ -(1+\gamma) & 2\gamma & 1-\gamma & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & 1-\gamma \\ 1-\gamma & & & -(1+\gamma) & 2\gamma \end{pmatrix} \in \mathbb{R}^{N\times N}$$

$\gamma \in \mathbb{R}$

$\gamma = 0$: $D_{1c} = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & \cdots & -1 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 1 & \cdots & -1 & 0 \end{pmatrix}$ central finite differences; error $= O((\Delta x)^2)$

$\lambda(D_{1c})$

$\gamma = -1$: $D_{1+} = \frac{1}{2\Delta x}\begin{pmatrix} -2 & 2 & & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & & \ddots & 2 \\ 2 & & 0 & -2 \end{pmatrix}$ forward finite differences; error $= O((\Delta x)^1)$ ("upwind")

$\gamma = +1$: $D_{1-} = \frac{1}{2\Delta x}\begin{pmatrix} 2 & 0 & & -2 \\ -2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & & -2 & 2 \end{pmatrix}$ backward finite differences; error $= O((\Delta x)^1)$ ("downwind")

$\gamma \in \mathbb{R}\setminus\{0\}$: error $= O((\Delta x)^1)$; Note: $\boxed{D_{1c} = \frac{1}{2}(D_{1+} + D_{1-})}$

Finite-difference matrices [2]

$$\frac{\partial^2}{\partial x^2} : \quad \boxed{D_{2\gamma_1\gamma_2} = D_{1\gamma_1} D_{1\gamma_2}}, \quad \gamma_1, \gamma_2 \in \mathbb{R}$$

examples:

$$D_2 = D_{1+} D_{1-} = \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & \ominus & \\ & 1 & -2 & 1 & \\ \ominus & & \ddots & \ddots & 1 \\ 1 & & & 1 & -2 \end{pmatrix} \quad \begin{cases} \text{error} = O((\Delta x)^2) \\ \lambda(D_2) \in \mathbb{R}^- \end{cases}$$

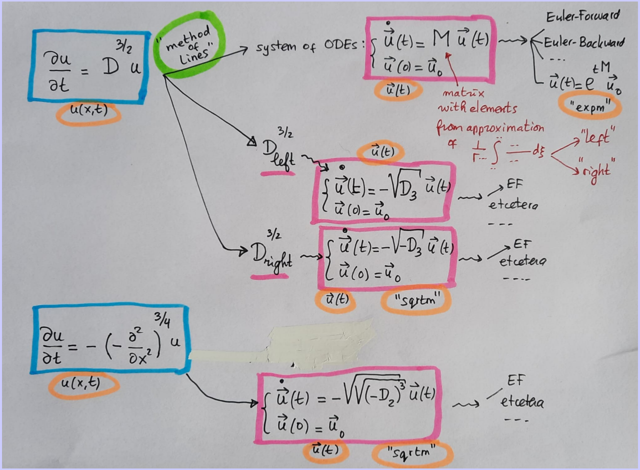
$$\frac{\partial^4}{\partial x^4} \rightarrow D_4 = (D_2)^2 = \frac{1}{(\Delta x)^4} \begin{pmatrix} 6 & -4 & 1 & & 1 & -4 \\ -4 & 6 & -4 & 1 & \ominus\ 0 & 1 \\ & \ddots & \ddots & \ddots & & \\ 1 & 0 & \ominus & \ddots & \ddots & -4 \\ -4 & 1 & & 1 & -4 & 6 \end{pmatrix} \quad \begin{cases} \lambda(D_4) \in \mathbb{R}^+ \\ \text{error} = O((\Delta x)^2) \end{cases}$$

and, for our purpose,

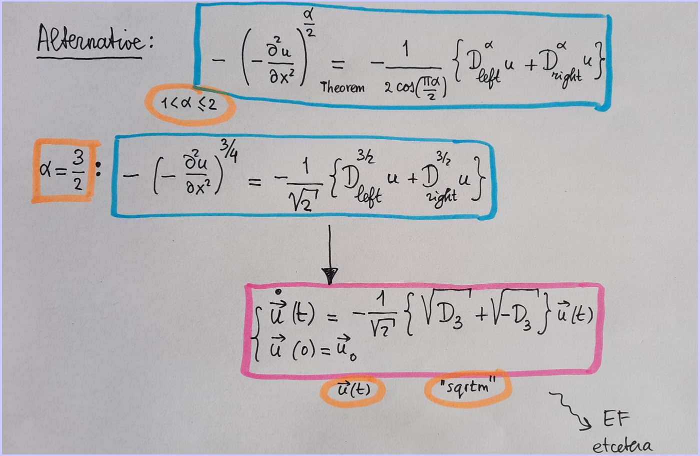
$$\frac{\partial^3}{\partial x^3} \rightarrow D_3 = \underbrace{D_{1+} D_{1-}}_{= D_2} D_{1c} = \frac{1}{(\Delta x)^3} \begin{pmatrix} 0 & -1 & 1/2 & & -1/2 & 1 \\ 1 & 0 & & & 0 & -1/2 \\ -1/2 & 1 & 0 & -1 & 1/2 & \ominus \\ & \ddots & \ddots & \ddots & \ddots & \\ 1/2 & 0 & \ominus & \ddots & \ddots & -1 \\ -1 & 1/2 & & -1/2 & 1 & 0 \end{pmatrix}$$

$$\begin{cases} \lambda(D_3) \in i\mathbb{R} \\ \text{error} = O((\Delta x)^2) \end{cases}$$

Fractional heat equation with square roots of matrices [1]

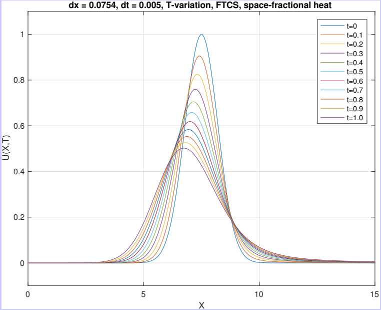


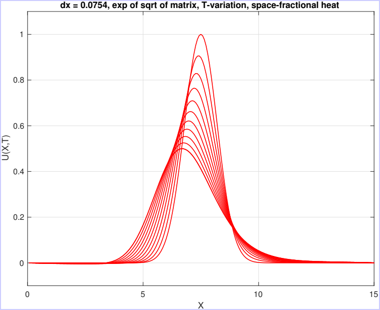
Fractional heat equation with square roots of matrices [2]



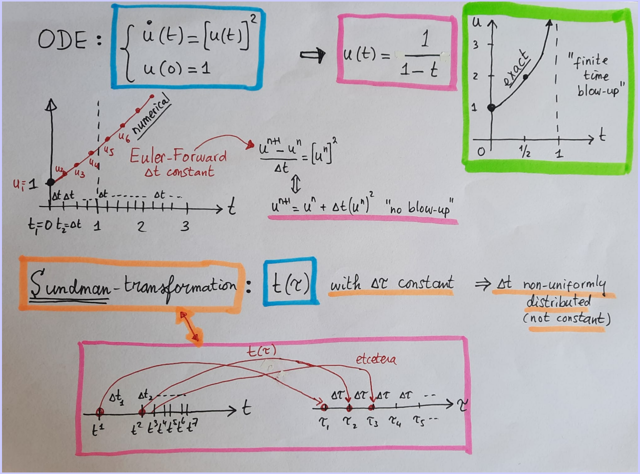
Fractional heat equation with square roots of matrices [3]

Left: approximation of fractional derivative $D^{3/2}$ and EF in time
Right: EXP in time and -SQRT(D3) in space





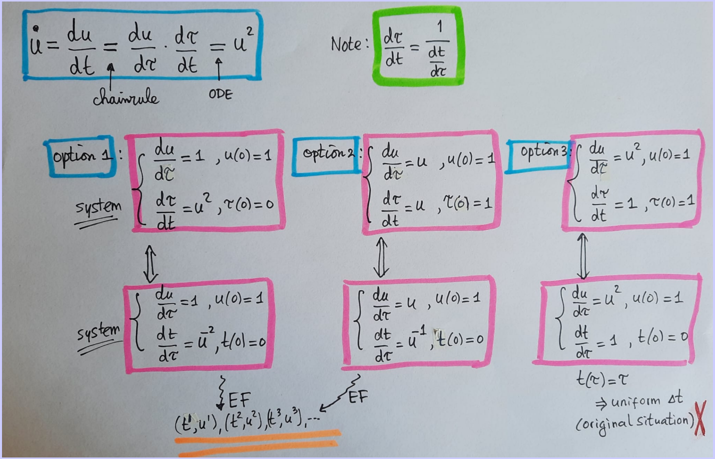
Blow-up and Sundman-transformation [1]



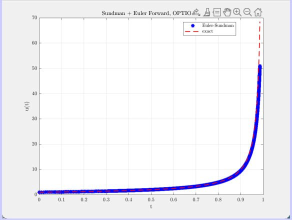
Blow-up and Sundman-transformation [2]

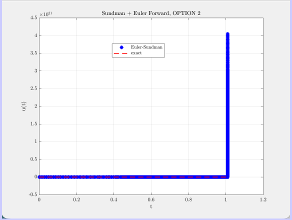
Karl Frithiof Sundman (1873 – 1949) was a Finnish mathematician. (Source: Wikipedia; May 9, 2026)

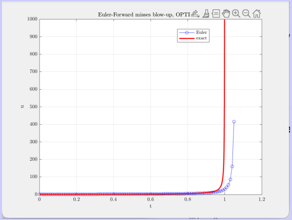
Blow-up and Sundman-transformation [3]



Blow-up and Sundman-transformation [4]







OPTION 1, OPTION 2, OPTION 3