Introduction Scientific Computing

Assignment 3a (Bisection Method)

The flow diagram can be found in the lecture notes.

1. How fast does this method converge? How do you measure this?
2. Write a MATLAB function file `bisection.m` that performs the method with input x_0 and x_1, and the number of iterations.
3. Apply this to the equation
$$f(x) = x^3 + 3x - 4 = 0.$$
4. Do the same for
$$f(x) = x^3 - 3x + 2 = 0.$$
(discuss the differences with the previous case)

Assignment 3b (Newton–Raphson)

1. Write a MATLAB function file `NR.m` that implements the Newton–Raphson iteration for solving $f(x) = 0$ (input: x_0 and maximum number of iterations). Alternatively: use a while-loop (which criterion(s) could you use?).
2. Apply this to:
$$\begin{aligned}
&f(x) = \sin(\pi x), \quad x_0 \in (0, 1/4), \quad \text{(what happens, if } x_0 = 1/2\text{?)}\\
&f(x) = x^3 + 3x - 4 = 0, \quad x_0 = ...\\
&f(x) = x^3 - 3x + 2 = 0, \quad x_0 = ...\\
&f(x) = \tan(x - 1) = 0, \quad x_0 = ...\\
&f(x) = x + e^{-Bx^2}\cos(x), \quad B = 1, 5, 10, 25, 50, \quad x_0 = ...
\end{aligned}$$
3. Make plots of the functions f and of the iterates x_i as a function of i.

4. Consider the extended method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} - \frac{1}{2}\frac{f(x_i)^2 f''(x_i)}{(f'(x_i))^3}.$$

Try this method and compare with Newton–Raphson. Is the convergence "faster" or "slower"?

Assignment 3c (Denman–Beavers)

Define the Denman–Beavers iteration:

$$\begin{cases} x_{k+1} = \frac{1}{2}\left(x_k + \frac{1}{y_k}\right), & x_0 = a > 0, \\ y_{k+1} = \frac{1}{2}\left(y_k + \frac{1}{x_k}\right), & y_0 = 1, \end{cases} \qquad k = 0, 1, 2, \ldots$$

1. Write a MATLAB function DB.m with input a and maximum number of iterations K. This method can be used as an alternative to Newton–Raphson to solve
$$x^2 - a = 0.$$
2. Does it converge and to which value(s)? Distinguish between x_k and y_k $(k \to \infty)$.
3. Plot the iterates as a function of k.

Assignment 3d (Gerlach Method)

The Gerlach scheme for approximating $\sqrt{a}$ $(a > 0)$ is:

$$x_{i+1} = x_i - \frac{(x_i^2 - a)(3x_i^2 + a)(3x_i^6 + 27ax_i^4 + 33a^2x_i^2 + a^3)}{2x_i(x_i^4 + 10ax_i^2 + 5a^2)(5x_i^4 + 10ax_i^2 + a^2)}, \quad i = 0, 1, 2, \ldots$$

with $x_0 = a/2$.

1. Write a MATLAB function GE.m with input a and maximum number of iterations I.
2. Plot the iterates x_i as a function of i (e.g. for $a = 2$ and $a = 5$).
3. Does it converge? (slow or fast?)

Assignment 3e (2D Newton)

Consider:

$$f(x,y) = xy - 1, \quad g(x,y) = x^2 + y^2 - 4.$$

1. Plot the curves $f(x,y) = 0$ and $g(x,y) = 0$. How many solutions (x,y) for $f = g = 0$ exist?
2. Describe the Newton–Raphson method for:

   $$\begin{cases} f(x,y) = 0, \\ g(x,y) = 0, \end{cases}$$

   and determine the Jacobian matrix.
3. Perform iterations starting from $(x_0, y_0) = (2, 3/5)$ and compute

   $$(x_1, y_1), (x_2, y_2), \ldots, (x_{10}, y_{10}).$$

4. Plot these points in the (x,y)-plane.

Assignment 3f (Eigenvalues)

Consider the eigenvalue problem:

$$A\mathbf{x} = \lambda\mathbf{x}, \quad A \in \mathbb{R}^{n\times n}.$$

Normalize:

$$\|\mathbf{x}\| = 1 \quad \Rightarrow \quad \mathbf{x}\cdot\mathbf{x} = 1.$$

We solve:

$$\begin{cases} A\mathbf{x} - \lambda\mathbf{x} = 0, \\ \mathbf{x}\cdot\mathbf{x} - 1 = 0. \end{cases}$$

This gives a nonlinear system. Apply Newton–Raphson:

$$\mathbf{z}_{k+1} = \mathbf{z}_k - J^{-1}(\mathbf{z}_k)F(\mathbf{z}_k),$$

where

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix}.$$

Case $n = 2$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 - \lambda x_1 = 0, \\ a_{21}x_1 + a_{22}x_2 - \lambda x_2 = 0, \\ x_1^2 + x_2^2 - 1 = 0. \end{cases}$$

Try this for a few 2×2-matrices <u>and</u> compare your numerical solution with the function `eig` in MATLAB.