Introduction Scientific Computing

Assignment 2a ("Method of Jacobi")

We consider the linear system

$$A\mathbf{x} = \mathbf{b},$$

where A is a given matrix and $\mathbf{b}$ a given vector. The vector $\mathbf{x}$ is to be found numerically.
In the following exercises it is useful to compute (why?):

$$\max_{i=1,2,3} |\lambda_i(I - D^{-1}A)|.$$

Rewriting

Let D be the diagonal matrix consisting of the diagonal elements of A. Then:

$$D^{-1}A\mathbf{x} = D^{-1}\mathbf{b}.$$

Rewriting:

$$0 = D^{-1}\mathbf{b} - D^{-1}A\mathbf{x},$$

$$\mathbf{x} = D^{-1}\mathbf{b} - D^{-1}A\mathbf{x} + \mathbf{x},$$

$$\mathbf{x} = D^{-1}\mathbf{b} + (I - D^{-1}A)\mathbf{x}.$$

Iteration

Choose an initial guess $\mathbf{x}_0$. Define the iteration:

$$\mathbf{x}_{j+1} = D^{-1}\mathbf{b} + (I - D^{-1}A)\mathbf{x}_j, \quad j = 0, 1, 2, \dots$$

Theorem (no proof)

If A is strictly diagonally dominant, then the iteration converges to $\mathbf{x}$. (what is ment by diagonally dominant?).

Apply the Jacobi-method to the linear system $A\mathbf{x} = \mathbf{b}$ for the following cases:

1)

$$A = \begin{pmatrix} 3 & 0 & 4 \\ 7 & 4 & 2 \\ -1 & 1 & 2 \end{pmatrix}$$

2)

$$A = \begin{pmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{pmatrix}$$

3)

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & -6 \end{pmatrix}$$

4)

$$A = \begin{pmatrix} 7 & 6 & 9 \\ 4 & 5 & -4 \\ -7 & -3 & 8 \end{pmatrix}$$

- Choose an initial guess $\mathbf{x}_0$ and a vector $\mathbf{b}$.
- Compare with the "exact" solution $A^{-1}\mathbf{b}$.
- Does the Jacobi-method converge? Also compute $\max_{i=1,2,3} |\lambda_i(I - D^{-1}A)|$.
- Calculate the residual $R := ||A\mathbf{x_j} - \mathbf{b}||_\mathbf{2}$ and plot the residual as function of the index in the iterative process.
- Choose a relevant final index of the iterations in the Jacobi-method.