① Schedule

Day 3 Thursday May 1 (611 AB)
(today)
$\begin{cases} f(x)=0 \\ f(x,y)=0,\ g(x,y)=0 \end{cases}$

Monday May 5 No class "bevrijdingsdag"

Day 4 Thursday May 8 (611 AB)
$\begin{cases} \text{FD-matrices } D_1, D_2, \ldots \\ \text{heat equation} \\ \text{nonlinear PDE} \leftrightarrow \text{IMEX-method} \end{cases}$

Day 5 Monday May 12 (BBG 001, KBG 208),
$\begin{cases} \text{square roots of matrices "}\sqrt{A}\text{"} \\ \text{matrix-Newton, Denman-Beavers} \end{cases}$

Day 6 Thursday May 15 (611 AB)
$\begin{cases} \text{fractional derivatives} \\ \text{matrix } D_3 \\ \text{expm, sqrtm} \\ \sqrt{D_3}, \sqrt{-D_2}, \ldots \\ \text{space-fractional heat equation} \end{cases}$

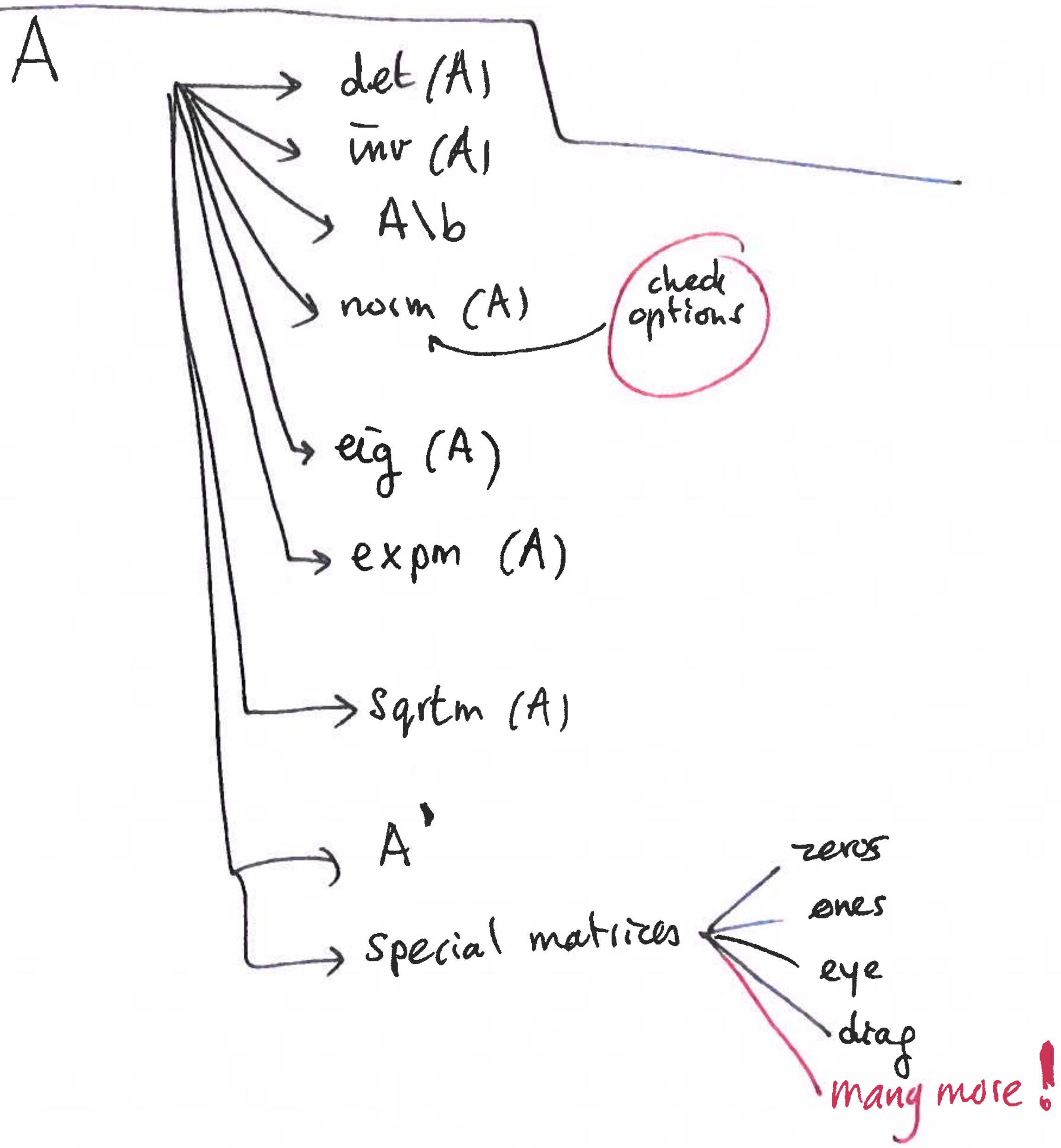
Monday May 19 13^{15}-15^{00} (BBG 001) Questions can be asked about Project/Report 1

Friday May 23 ≤ 23:59 Deadline Report 1

Monday May 26 Start Module 2

MatLab

← Matrix



Useful for lectures 4,5,6

Consider the linear system of ODEs: $\dot{\vec{y}} = A\vec{y}$ ← vector

time-derivative ↗ ↑ matrix (given)

with initial vector: $\vec{y}(0)=\vec{y}_0$ ← given

Two basic methods for ↗

Euler-Forward:

$$\frac{\vec{y}^{n+1}-\vec{y}^{n}}{\Delta t} = A\vec{y}^{n}, \quad n=0,1,2,\dots,N-1$$

choose stepsize → Δt ; choose ↑ N

$\vec{y}^{0}=\vec{y}_0$

$$\Rightarrow \begin{cases} \vec{y}^{0}=\vec{y}_0 \\ n=0,1,\dots,N-1 \\ \vec{y}^{n+1}=(I+\Delta t A)\vec{y}^{n} \end{cases}$$

"for"

Euler-Backward:

$$\frac{\vec{y}^{n+1}-\vec{y}^{n}}{\Delta t} = A\vec{y}^{(n+1)}, \quad n=0,1,2,\dots,N-1$$

$$\Rightarrow \vec{y}^{n+1}=(I-\Delta t A)^{-1}\vec{y}^{n}$$

$$\begin{cases} \vec{y}^{0}=\vec{y}_0 \\ \underbrace{(I-\Delta t A)}_{\overset{\text{define}}{=} B \text{ (matrix)}}\vec{y}^{n+1}=\vec{y}^{n} \end{cases} \rightsquigarrow B\vec{y}^{n+1}=\vec{y}^{n} \rightarrow \begin{cases} n=0,1,\dots,N-1 \\ \vec{y}^{n+1}=B\backslash\vec{y}^{n} \end{cases}$$

Matlab

"Exact" $\vec{y}(t)=e^{tA}\vec{y}_0$ ← expm in matlab

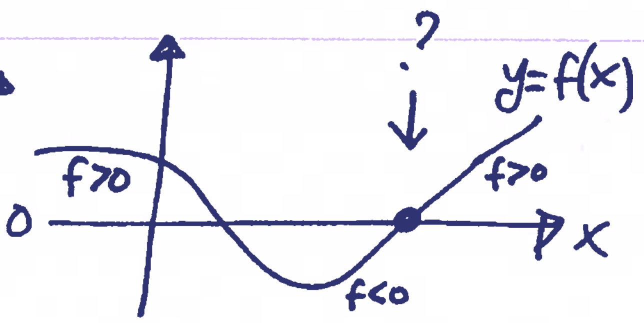
IMEX: $\dot{\vec{y}} = A\vec{y} + B\vec{y} \rightsquigarrow \frac{\vec{y}^{n+1}-\vec{y}^{n}}{\Delta t} = A\vec{y}^{n+1} + B\vec{y}^{n} \Rightarrow (I-\Delta t A)\vec{y}^{n+1}$

loop $n=0,1,2,\dots,N-1 \}$ $=(I+\Delta t B)\vec{y}^{n}$

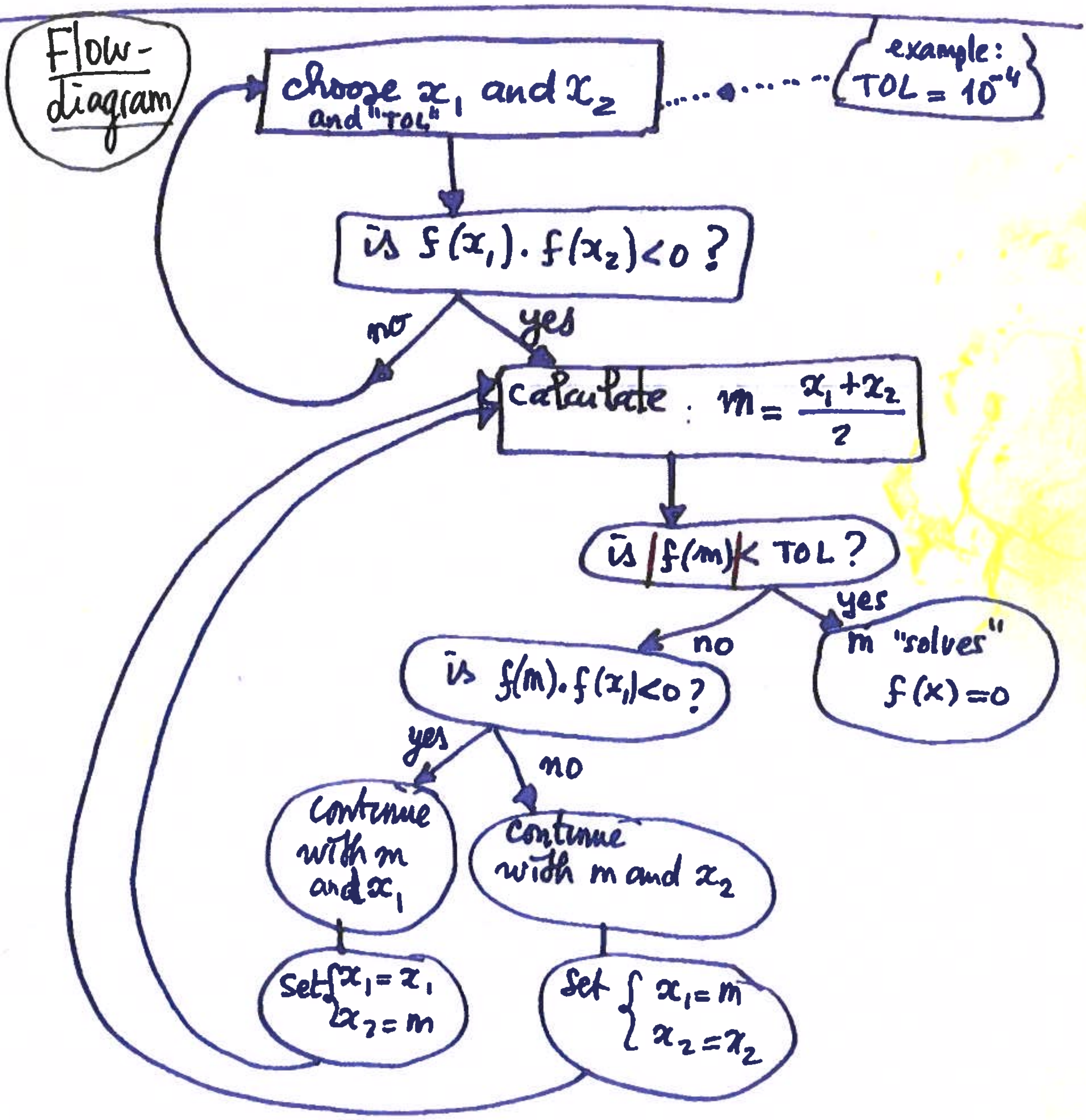
④ solving nonlinear equations: $f(x)=0$

Bisection method

$f(x)$ in f.m



{makes use of intermediate value theorem}



Consider the sequence $x_0, x_1, x_2, \ldots\ldots$

Does this sequence <u>converge</u> ?

<u>Example</u>

$$\begin{cases} x_0 = 2 \\ x_{i+1} = \underbrace{\frac{1}{2}\left(x_i + \frac{2}{x_i}\right)}_{\substack{\text{a nonlinear recursion} \\ \text{("successive substitution")}}} , \quad i = 0, 1, 2, 3, \ldots\ldots \end{cases}$$

Calculate:

$$x_1 = \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{1}{2} \cdot 3 = \frac{3}{2} = 1.5$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{1}{2} \cdot \frac{17}{6} = \frac{17}{12} \approx 1.41667$$

$$x_3 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \cdots \approx 1.4142\ldots$$

etcetera

Note: $\underbrace{\lim_{i\to\infty} x_{i+1}}_{\text{call this limit } x \text{ (if it exists)}} = \lim_{i\to\infty} x_i = \lim_{i\to\infty} x_{i-1} = \cdots$

$$\Rightarrow \lim_{i\to\infty} : \quad x = \frac{1}{2}\left(x + \frac{2}{x}\right) \Leftrightarrow 2x = x + \frac{2}{x}$$

$$\Leftrightarrow 2x^2 = x^2 + 2$$

$$\Leftrightarrow \boxed{x^2 - 2 = 0}$$

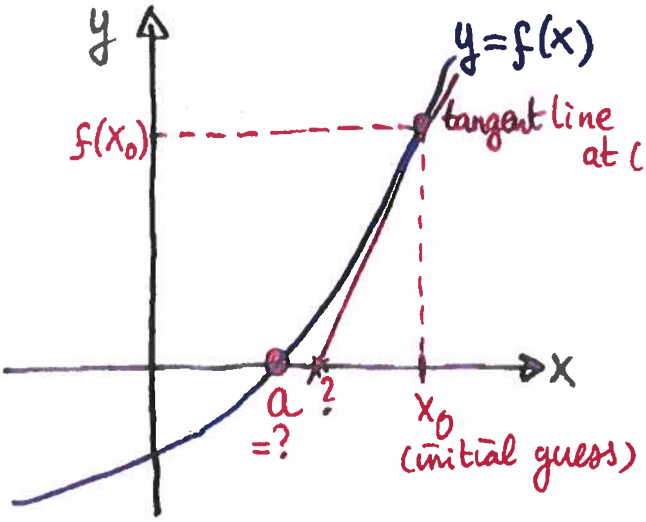
$$\Leftrightarrow \underline{\underline{x_{1,2} = \pm\sqrt{2}}}$$

Check:

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right)$$

$$\Leftrightarrow x_{i+1} = x_i - \frac{x_i^2-2}{2x_i} = x_i - \frac{f(x_i)}{f'(x_i)}, f(x)=x^2-2$$

A derivation using tangent lines



The tangent line at $(x_0, f(x_0))$: $y = Ax+B$ with $A = f'(x_0)$ $\Rightarrow y = f'(x_0)x+B$

this straight line goes through $(x_0, f(x_0))$: $f(x_0) = f'(x_0)x+B$

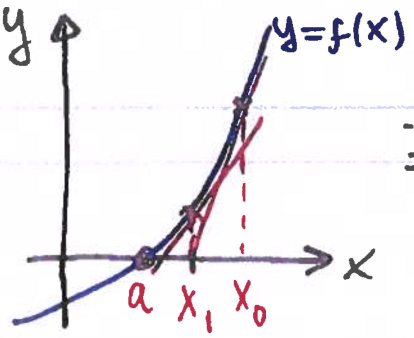
$$\Rightarrow B = f(x_0) - f'(x_0)x_0$$

$$\Rightarrow y = f'(x_0)x + f(x_0) - f'(x_0)x_0$$

The zero of this line: $0 = f'(x_0)x + f(x_0) - f'(x_0)x_0$

$$\Rightarrow x = \frac{f'(x_0)x_0 - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

call this zero x : x_1 and repeat the process



... $\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

etcetera $x_3 = --$

$x_4 = --$

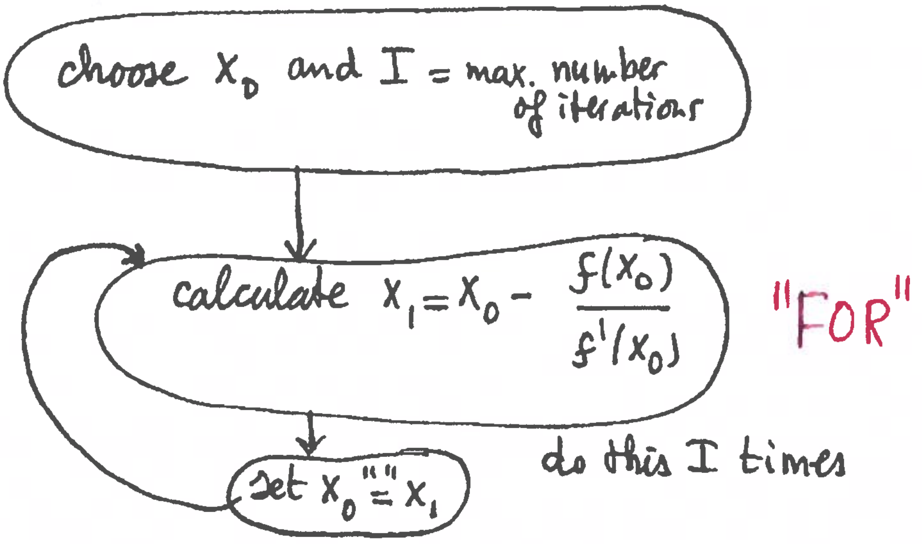
...

Newton-Raphson

- starting value (initial guess): $x_0 = ?$
- how/when to stop the process (iteration)?

Newton's method in Matlab

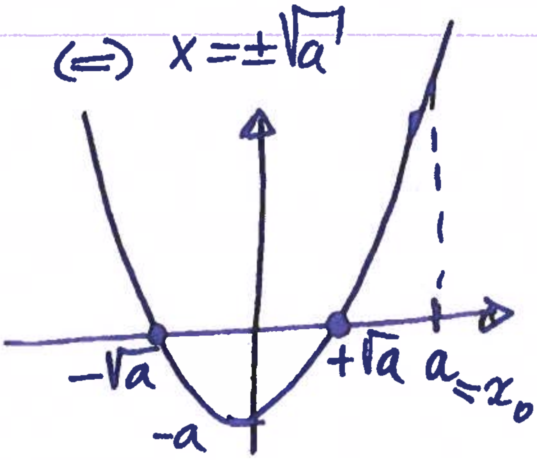
- define f(x) in separate file f.m (or using "@")
- similarly f'(x) in fp.m



instead of I, you could make use of a while loop, where you test, whether $|x_1 - x_0| < TOL$ defined at the beginning

$f(x)=x^2-a=0 \iff x=\pm\sqrt{a}$

$(a>0)$



Newton-Raphson:

$$(*)\quad x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}=x_k-\frac{x_k^2-a}{2x_k}=\frac{1}{2}\left(x_k+\frac{a}{x_k}\right)$$

$k=0,1,2,\dots$ ("$k\to\infty$")

initial guess: $x_0=a>\sqrt{a}$

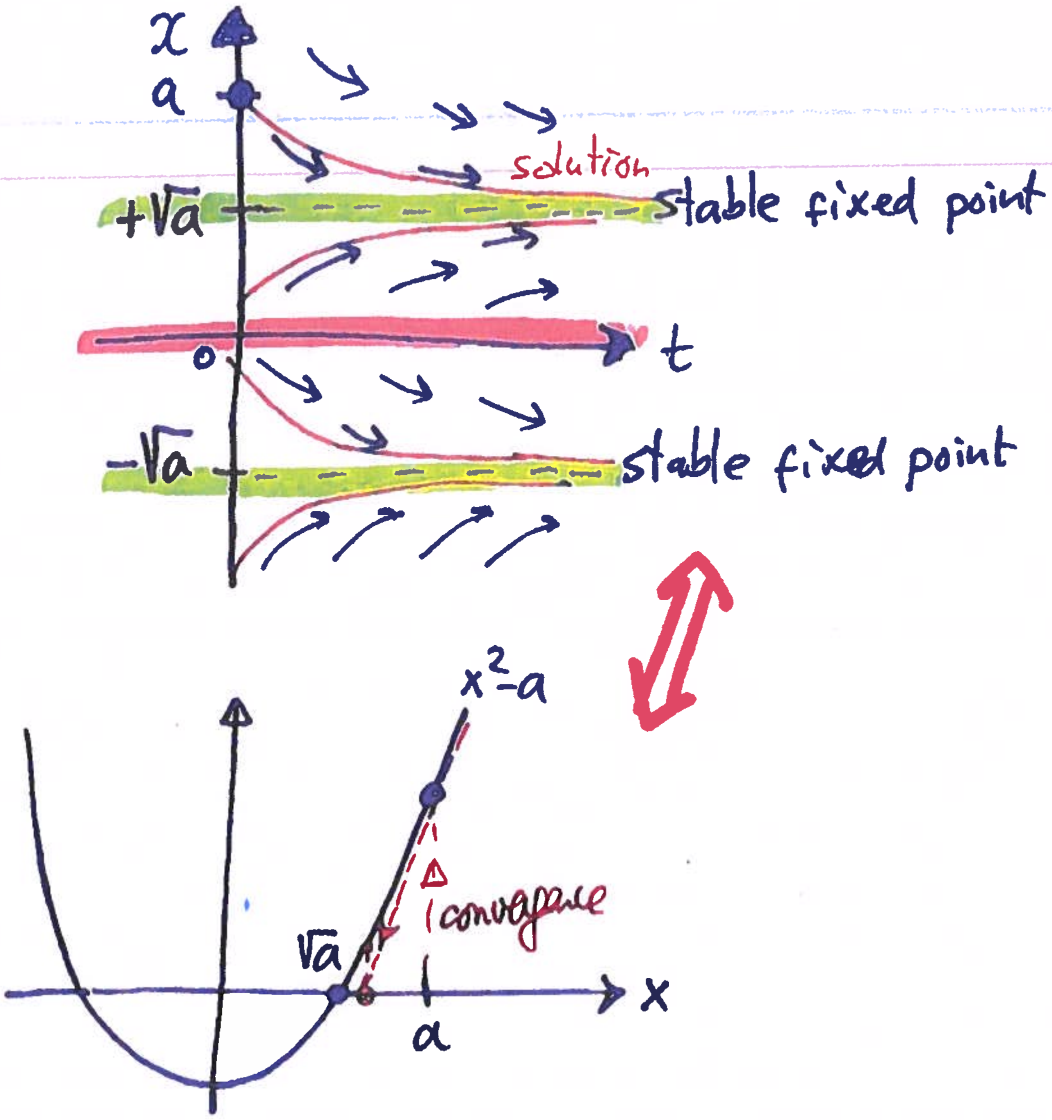
Continuous Newton: ODE $\begin{cases}\dot{x}=-\dfrac{x^2-a}{2x}\\ x(0)=a\end{cases}$ $\Longrightarrow x(t)=\dots$ take limit "$t\to\infty$"

index

apply Euler-Forward:

$$\begin{cases}x^{n+1}=x^n+\Delta t\left(-\dfrac{(x^n)^2-a}{2x^n}\right)\\ x^0=a\end{cases},\ n=0,1,2,\dots$$

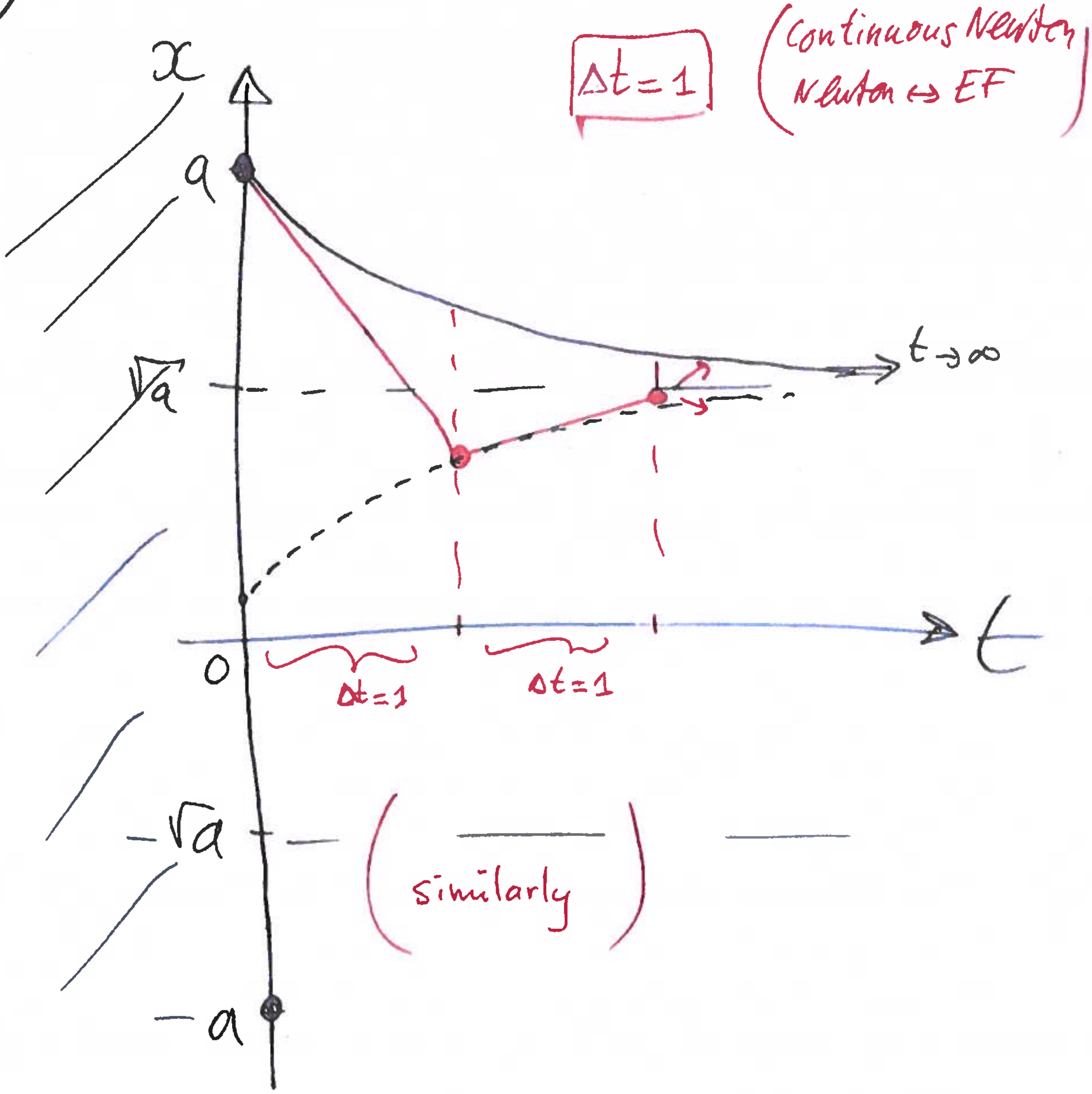
"take" $\Delta t=1$: $\begin{cases}x^{n+1}=x^n-\dfrac{(x^n)^2-a}{2x^n}\\ x^0=a\end{cases},\ n=0,1,2,\dots$ $=(*)$



$$x_0 > 0 \quad \Rightarrow \lim_{t \to \infty} x(t) = \sqrt{2}$$

$$x_0 = 0 \quad \lightning$$

$$x_0 < 0 \quad \Rightarrow \lim_{t \to \infty} x(t) = -\sqrt{2}$$



$\boxed{\Delta t = 1}$

(continuous Newton
Newton ↔ EF)

Alternative

for the equation $f(x) = x^2 - a = 0$:

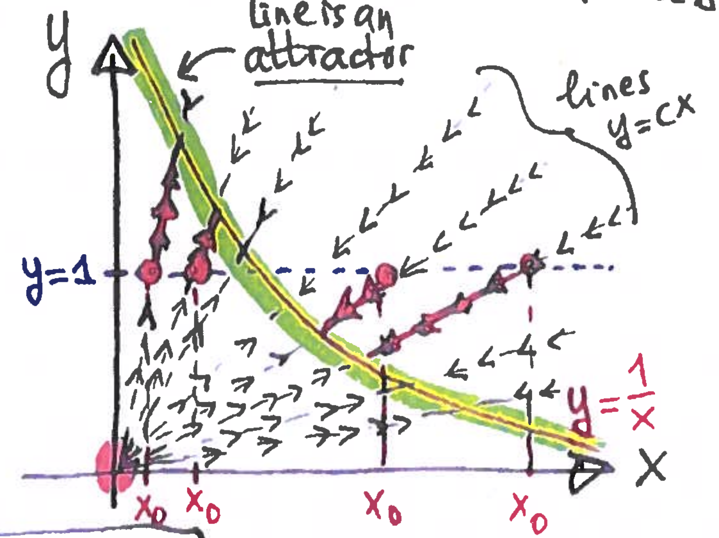
$$k=0,1,2,\dots \quad \begin{cases} x_{k+1} = \frac{1}{2}\left(x_k + \frac{1}{y_k}\right), & x_0 = a > 0 \\ y_{k+1} = \frac{1}{2}\left(y_k + \frac{1}{x_k}\right), & y_0 = 1 \end{cases}$$

"Denman-Beavers" method

Continuous Denman Beavers : $\begin{cases} \dot{x} = -\frac{1}{2}x + \frac{1}{2y}, & x(0) = a > 0 \\ \dot{y} = -\frac{1}{2}y + \frac{1}{2x}, & y(0) = 1 \end{cases}$

Euler-Forward with step-size $\Delta t = 1$!

$\boxed{\text{dynamical system}}$



$\boxed{\text{phase-plane}}$

$$\lim_{t\to\infty} (x(t), y(t)) = \left(\sqrt{a}, \frac{1}{\sqrt{a}}\right)$$

⑫

$$\begin{cases} \overset{a}{X} = \;-- \\ \overset{a}{y} = \;- \end{cases}$$

$$\Big\downarrow \text{EF}$$

$$\begin{cases} \dfrac{x^{n+1} - x^{n}}{\Delta t} = -\frac{1}{2}x^{n} + \dfrac{1}{2y^{n}} \\ \dfrac{y^{n+1} - y^{n}}{\Delta t} = -\frac{1}{2}y^{m} + \dfrac{1}{2x^{n}} \end{cases}$$

$$\text{"}\Delta t = 1\text{"} \xrightarrow{\quad} \begin{cases} x^{n+1} = \frac{1}{2}x^{n} + \dfrac{1}{2y^{n}} = \frac{1}{2}\left(x^{n} + \dfrac{1}{y^{n}}\right) \\ y^{n+1} = \frac{1}{2}y^{n} + \dfrac{1}{2x^{n}} = \frac{1}{2}\left(y^{n} + \dfrac{1}{x^{i}}\right) \end{cases}$$

DB

$(\text{for } x^2 - a = 0)$

$$\frac{dy}{dx} = \frac{-\frac{1}{2}y + \frac{1}{2x}}{-\frac{1}{2}x + \frac{1}{2y}} \overset{(* 2xy)}{\overset{\downarrow}{=}} \frac{-xy^2+y}{-x^2y+x} = \frac{y(1-xy)}{x(1-xy)}$$

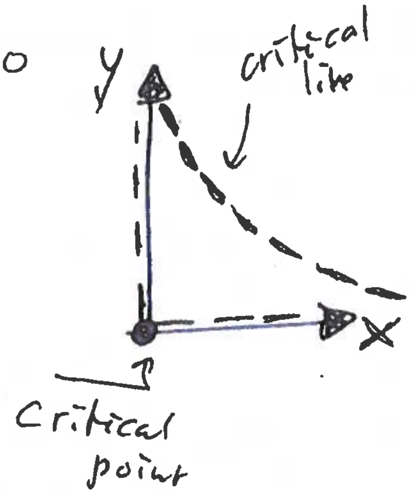
$$\overset{y \neq \frac{1}{x}}{=} \frac{y}{x}$$

$y = \frac{1}{x}$: $\frac{"0"}{0}$

critical (stationary) "point" (line,) curve

$x=0$: $\frac{dy}{dx} = \infty$

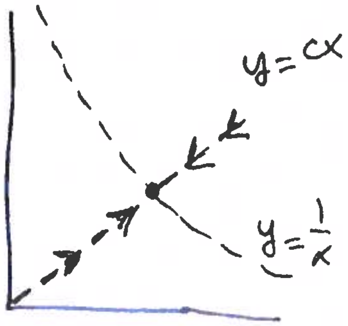
$y=0$: $\frac{dy}{dx} = 0$



on line: $y = cx$

$(y \neq \frac{1}{x})$ $\Rightarrow$ $\frac{dy}{dx} = c$ (constant

>0



$y = \frac{1}{x}$ "attractor"

(can be checked via further analysis)

can be proved (not here): $\lim_{t\to\infty} x(t) = \sqrt{a}$

$(\Rightarrow \lim_{t\to\infty} y(t) = \frac{1}{\sqrt{a}})$

Newton-Raphson: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

$f(x)=0$

$\underset{\text{can be shown}}{\Longrightarrow}$ (if x_0 is chosen appropriately) error $= x_i - a$ decreases quadratically (second order)

$$x_{i+1} - a \approx c \cdot (x_i - a)^2$$

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define $x_i - \dfrac{f(x_i)}{f'(x_i)} \overset{d}{=} \varphi(x_i)$

and $x_{i+1} = \Phi(x_i) \overset{d}{=} \dfrac{x_i \varphi(\varphi(x_i)) - (\varphi(x_i))^2}{\varphi(\varphi(x_i)) - 2\varphi(x_i) + x_i}$

then higher-order convergence is expected/obtained
(check this with Matlab for $f(x) = x^2 - 2$)
which order do you find?

A "very high" order (see exercises) can be obtained with the Gerlach-scheme:

$$\begin{cases} x_0 = \dfrac{a}{2} \\ x_{i+1} = x_i - \dfrac{(x_i^2 - a)(3x_i^2 + a)(3x_i^6 + 27a x_i^4 + 33a^2 x_i^2 + a^3)}{2x_i(x_i^4 + 10a x_i^2 + 5a^2)(5x_i^4 + 10a x_i^2 + a^2)} \end{cases}$$

$i = 0, 1, 2, \ldots$

which order?

Two dimensions

* The **gradient** of a **scalar** function $f(x,y): \underset{\text{"vector"}}{\mathbb{R}^2} \to \underset{\text{number}}{\mathbb{R}}$

$$\operatorname{grad}(f) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \frac{\partial f}{\partial x} \cdot \underset{\begin{pmatrix}1\\0\end{pmatrix}}{\underset{\|}{\vec{\imath}}} + \frac{\partial f}{\partial y} \cdot \underset{\begin{pmatrix}0\\1\end{pmatrix}}{\underset{\|}{\vec{\jmath}}}$$

$$= \underset{\text{"habla" or "del"}}{\nabla} f \qquad \boxed{\text{scalar} \to \text{vector}}$$

example: $f(x,y) = 6xy - y^2$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(6xy - y^2) = 6y \\ \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(6xy - y^2) = 6x - 2y \end{cases}$$

$$\Rightarrow \operatorname{grad}(f) = \begin{pmatrix} 6y \\ 6x-2y \end{pmatrix}$$

* the Jacobian (Jacobi matrix) of a vectorfunction $\vec{f}(x,y)=\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

, a 2x2 matrix

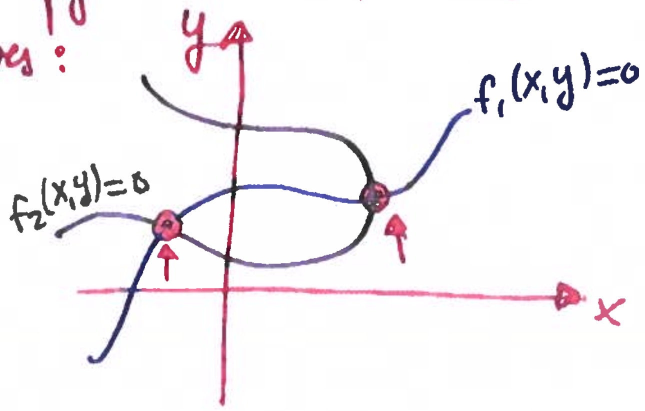
Given are two functions $f_1(x,y)$ and $f_2(x,y)$

Find $(x,y) \in \mathbb{R}^2$ such that $\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases}$ ⊛

$f_1(x,y)=0$ is a curve in the x-y plane

$f_2(x,y)=0$ is another curve in the x-y plane

the (x,y) that satisfy ⊛ are the points of intersection of the two curves:



$$\begin{cases} f_1(x,y) \approx f_1(a,b) + \frac{\partial f_1}{\partial x}(a,b)\,(x-a) + \frac{\partial f_1}{\partial y}(a,b)\,(y-b) \\ f_2(x,y) \approx f_2(a,b) + \frac{\partial f_2}{\partial x}(a,b)\,(x-a) + \frac{\partial f_2}{\partial y}(a,b)\,(y-b) \end{cases}$$

2D-Taylor expansion for f_1 and f_2 near the point $(a,b) \in \mathbb{R}^2$

"set" $\approx \to =$ and $= 0$ $\Rightarrow$ solve the two linear equations in the two unknowns x and y

$$\Rightarrow \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}}_{\text{Jacobian } J}\Bigg|_{(a,b)} \begin{pmatrix} x-a \\ y-b \end{pmatrix} = -\begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} - \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}^{-1}}_{= J^{-1}}\Bigg|_{(a,b)} \begin{pmatrix} f_1(a,b) \\ f_2(a,b) \end{pmatrix}$$

repeat this process as in 1D:

$$\begin{cases} \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} - J^{-1}\Big|_{(x_i,y_i)} \begin{pmatrix} f_1(x_i,y_i) \\ f_2(x_i,y_i) \end{pmatrix} \qquad i = 0,1,2,3,\dots \\ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ starting values} \end{cases}$$

Newton-fractals

Apply Newton-Raphson to $f(z)=0$, $z \in \mathbb{C}$

↑

$$\Rightarrow \begin{cases} z_{i+1} = z_i - \dfrac{f(z_i)}{f'(z_i)}, & i = 0,1,2,\dots \\ z_0 = \text{--} \in \mathbb{C} & \end{cases}$$

Example: $f(z) = z^3 - 1$, $f'(z) = 3z^2$

there are three complex solutions z:

$$\begin{cases} z_1 = 1 \\ z_2 = \frac{1}{2}\left(-1+\sqrt{-3}\right) \\ z_3 = \frac{1}{2}\left(-1-\sqrt{-3}\right) \end{cases}$$

Run NR "until convergence" and check to which of the three roots (solutions) the method converges.

Assign the color "1", if it converges to z_1,
the color "2", " " " " z_2,
and color "3", " " " " z_3.

$\Rightarrow$ (-- ?! --) (check the Matlab file fractal.m on the webpage, and play with different functions f to create different patterns/fractals)