Answer Key: OEFEN-DEELTOETS A (SK-B2WDMM)

⇒ Draft solutions – subject to typos and verification!

Question W1

(a) $\nabla T(x,y,z) = \begin{bmatrix} 4x - y\sin(xy) + 2z \\ 3y^2 - x\sin(xy) \\ 6z + 2x \end{bmatrix}$

(b) $\nabla \cdot \vec{v} = yze^{xz} + x^2e^y + (2z + z^2)e^{y+z}$

(c) $\frac{\partial \rho}{\partial v} = 4u^2 - 4u^2v + 4u + 8v - 12v^2 + 2v\sin(u - v^2)$

Question W2

(a)
- Expansion about $(0,0)$: $P_2(x,y) = 2024 - 3y$
- Expansion about $(0,-1)$: $P_2(x,y) = 2026 - x^2 - 3(y+1)^2$

(b) $\nabla E(0,-1) = [0,0]^T$ and $\nabla E(0,1) = [0,0]^T$

(c)
- Hessian: $H(x,y) = \begin{bmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6y + 6x^2y \end{bmatrix}$
- Laplacian: $\nabla^2 E = 2y^3 + 6y + 6x^2y$
- Minimum point: $(0,1)$
- Minimum value: $E(0,1) = 2022$

(d)
- Steepest Descent: $\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha\nabla E(\mathbf{x}_k)$
- Newton-Raphson: $\mathbf{x}_{k+1} = \mathbf{x}_k - [H(\mathbf{x}_k)]^{-1}\nabla E(\mathbf{x}_k)$

Question W3

(a) $\frac{\partial H}{\partial x} = x, \quad \frac{\partial H}{\partial y} = 4\pi^2 y$

(b)
- System: $\begin{cases} y' = x \\ x' = -4\pi^2 y \end{cases}$
- Matrix A: $\begin{bmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{bmatrix}$
- Hamiltonian relations: $y' = \frac{\partial H}{\partial x}$ and $x' = -\frac{\partial H}{\partial y}$

(c) $\frac{dH}{dt} = 0 \implies H(t) = \text{constant} = \frac{9}{2} = 4.5$

(d)
- Euler-Forward: $y_{n+1} = y_n + \Delta t\, x_n, \quad x_{n+1} = x_n - 4\pi^2\Delta t\, y_n$
- Störmer-Verlet: $x_{n+1} = x_n - 4\pi^2\Delta t\, y_n, \quad y_{n+1} = y_n + \Delta t\, x_{n+1}$
- Advantage: Störmer-Verlet is a symplectic integrator that conserves energy and long-term stability without artificial blow-up.

Question W4

(a) $y(t) = e^{2t}\big(C_1\cos(3t) + C_2\sin(3t)\big)$

(b) $y(t) = e^{2t}\big(2\cos(3t) + 2\sin(3t)\big)$

(c) System matrix $A = \begin{bmatrix} 0 & 1 \\ -13 & 4 \end{bmatrix}$ with initial state $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

Question W5

W5 i) C (Calculated value: $\nabla^2 f(1,0,2) = 8$)

W5 ii) B) (Calculated value: $D_{\mathbf{u}}C(2,1) = -\frac{2\sqrt{5}}{9} \approx -0.4969$)