Solutions: Practice/Oefen Exam Deeltoets B

Solution 1

(a) The gradient $\mathbf{G} = \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$ is:

$$\mathbf{G} = \left(3x^2z,\ -4yz^2,\ x^3 - 4y^2z\right)$$

(b) The divergence $\nabla \cdot \mathbf{G}$ is:

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial x}(3x^2z) + \frac{\partial}{\partial y}(-4yz^2) + \frac{\partial}{\partial z}(x^3 - 4y^2z) = 6xz - 4z^2 - 4y^2$$

Solution 2

(a) Setting first partial derivatives to zero:

$$f_x = 4x - 8 = 0 \implies x_0 = 2, \quad f_y = 2y + 4 = 0 \implies y_0 = -2$$

The critical point is $(2, -2)$.

(b) The second partial derivatives are $f_{xx} = 4$, $f_{yy} = 2$, and $f_{xy} = f_{yx} = 0$. The Hessian matrix is:

$$H = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

Since both eigenvalues ($\lambda_1 = 4, \lambda_2 = 2$) are strictly positive, the Hessian is positive definite, meaning $(2, -2)$ is a **local minimum**.

Solution 3

(a) Expressing as a system $\dot{x} = v$ and $\dot{v} = -2\gamma v - \omega_0^2 x$, the Explicit Euler update scheme with step size Δt is:

$$x_{n+1} = x_n + \Delta t\, v_n, \quad v_{n+1} = v_n - \Delta t\left(2\gamma v_n + \omega_0^2 x_n\right)$$

(b) For $\gamma = 0$, Explicit Euler amplifies the mechanical energy $E_n = \frac{1}{2}v_n^2 + \frac{1}{2}\omega_0^2 x_n^2$ by a factor of $(1 + \omega_0^2 \Delta t^2) > 1$ at every step, causing the numerical trajectory to spiral outwards and explode over long runtimes.

Solution 4

The characteristic equation is $r^2 + 4r + 4 = 0 \implies (r + 2)^2 = 0$, yielding a repeated root $r = -2$. The general solution and its derivative are:

$$y(t) = (C_1 + C_2 t)e^{-2t}, \quad y'(t) = C_2 e^{-2t} - 2(C_1 + C_2 t)e^{-2t}$$

Applying initial conditions:

$$y(0) = C_1 = 3, \quad y'(0) = C_2 - 2C_1 = -2 \implies C_2 - 6 = -2 \implies C_2 = 4$$

The unique solution is $y(t) = (3 + 4t)e^{-2t}$.

Solution 5

Evaluating the integral for $t \geq 0$:

$$\hat{f}(\omega) = \int_0^\infty e^{-at} e^{-i\omega t} dt = \int_0^\infty e^{-(a+i\omega)t} dt = \left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)} \right]_0^\infty = 0 - \frac{1}{-(a+i\omega)} = \frac{1}{a+i\omega}$$

Multiplying numerator and denominator by the complex conjugate $(a - i\omega)$ gives:

$$\hat{f}(\omega) = \frac{a - i\omega}{a^2 + \omega^2}$$

Solution 6

(a) Equating the exponents $\frac{4}{d^*} = \frac{1}{2} \implies d^* = 8$.

(b) For $d = 300$, Simpson's rule error scales as $\mathcal{O}(N^{-4/300}) \approx \mathcal{O}(N^{-0.013})$, meaning the error barely decreases with more grid points. Monte Carlo retains its dimension-independent rate of $\mathcal{O}(N^{-0.5})$, making it vastly superior.

Solution 7:

The linear system $Xk \approx \mathbf{y}$ is:

$$X = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 0.8 \\ 1.7 \\ 3.5 \end{pmatrix}$$

Computing $X^T X$ and $X^T \mathbf{y}$:

$$X^T X = 1^2 + 2^2 + 4^2 = 1 + 4 + 16 = 21$$

$$X^T \mathbf{y} = 1(0.8) + 2(1.7) + 4(3.5) = 0.8 + 3.4 + 14.0 = 18.2$$

Applying the OLS formula:

$$\hat{k} = (X^T X)^{-1} X^T \mathbf{y} = \frac{18.2}{21} = \frac{26}{30} \approx 0.867 \text{ s}^{-1}$$