Extra Formula Sheet: Fourier Analysis and Trigonometric Integrals

1 Fourier Integral and Fourier Transforms

When a function $f(x)$ is defined on the infinite interval $(-\infty, \infty)$, is piecewise smooth, and is absolutely integrable, it can be mathematically expressed as a non-periodic continuous superposition of harmonic waves using amplitude coefficients.

1.1 The Fourier Integral Representation via $A(\omega)$ and $B(\omega)$

The real-valued Fourier integral representation of $f(x)$ is defined as:

$$f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\, d\omega \tag{1}$$

Where the spectral amplitude functions $A(\omega)$ and $B(\omega)$ act as continuous analogues to the classical Fourier series coefficients and are calculated via:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\, dx \tag{2}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\, dx \tag{3}$$

1.2 Symmetric Fourier Transform Pair

Alternatively, using complex exponential notation, the continuous symmetric Fourier transform pair converts a function between the spatial domain x and the angular frequency domain ω:

- **Forward Fourier Transform ($\mathcal{F}$):**

$$F(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx \tag{4}$$

- **Inverse Fourier Transform ($\mathcal{F}^{-1}$):**

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\, d\omega \tag{5}$$

2 Essential Trigonometric Identities

These core goniometric relations are vital for simplifying products of sines and cosines before integration.

2.1 Product-to-Sum and Fundamental Identities

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \tag{6}$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] \tag{7}$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \tag{8}$$

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \quad \text{(Fundamental Pythagorean Identity)} \tag{9}$$

2.2 Double-Angle Formulas

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{10}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{11}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{12}$$

3 Trigonometric Integrals and Orthogonality

When evaluating Fourier coefficients over a periodic boundary (interval $[-\pi, \pi]$ or $[0, L]$), the orthogonality of trigonometric functions determines whether an integral vanishes (equals zero) or yields a non-zero value. Let $n, m \in \mathbb{Z}^+$ be positive integers.

3.1 Integrals Over a Symmetric Period $[-\pi, \pi]$

- **Sine-Sine Product (Orthogonality Boundary):**

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases} \tag{13}$$

- **Cosine-Cosine Product (Orthogonality Boundary):**

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases} \tag{14}$$

3.2 Generalized Boundary Scales Over Interval $[0, L]$

For arbitrary domain scales, let the spatial frequency adapt as $\frac{n\pi x}{L}$:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases} \tag{15}$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases} \tag{16}$$

3.3 Examples of Single Sine and Cosine Integrals

To build quick intuition, here are foundational examples of single trigonometric functions over symmetric and non-symmetric periods:

- **Symmetric Period (Vanishing due to Odd Symmetry):** Any single sine wave integrated over a full symmetric period yields zero.

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = \left[-\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} = 0 \tag{17}$$

- **Symmetric Period (Vanishing due to Periodic Waves):** A single cosine wave over a full symmetric period also yields zero.

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = \left[\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} = 0 \tag{18}$$

- **Non-Symmetric Half-Period (Non-Zero Example):** Integrating a sine over a positive half-period leaves a net positive area.

$$\int_{0}^{\pi} \sin(x)\,dx = [-\cos(x)]_{0}^{\pi} = -\cos(\pi) - (-\cos(0)) = 1 + 1 = 2 \tag{19}$$

3.4 Single Integrals Over the Interval $[-1, 1]$ (Non-Zero Example)

When integrating trigonometric functions over the specific symmetric interval $[-1, 1]$, the angular frequencies are no longer locked to π. This changes the evaluation at the boundaries:

- **The Sine Integral (Vanishing due to Odd Parity):** Even though the boundaries do not contain π, the odd symmetry of the sine function guarantees the integral vanishes.

$$\int_{-1}^{1} \sin(x)\,dx = \Big[-\cos(x)\Big]_{-1}^{1} = -\cos(1) - (-\cos(-1)) = 0 \tag{20}$$

  (Since $\cos(-1) = \cos(1)$ *due to even parity).*

- **The Cosine Integral (Non-Zero Result due to Even Parity):** Because the cosine is an even function, the areas on both sides of the y-axis reinforce each other, resulting in a non-zero value:

$$\int_{-1}^{1} \cos(x)\,dx = \Big[\sin(x)\Big]_{-1}^{1} = \sin(1) - \sin(-1) = \sin(1) - (-\sin(1)) = 2\sin(1) \tag{21}$$

  Numerical valuation: Since $\sin(1\ rad) \approx 0.84147$*, the definite integral yields* $\approx 1.68294 \neq 0$.