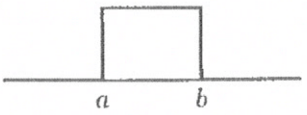
Section 15.6

Find the Fourier transform:

15. $f(x)=\begin{cases}1, & \text{if } a<x<b\\ 0, & \text{otherwise}\end{cases}$ Figure

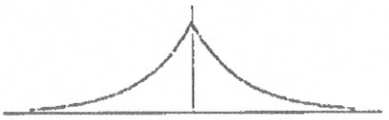


We have $g(y)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} f(x)e^{-ixy}\,dx$

$$=\frac{1}{\sqrt{2\pi}}\int_a^b e^{-ixy}\,dx=\frac{1}{\sqrt{2\pi}}\left[-\frac{1}{iy}e^{-ixy}\right]_{x=a}^{x=b}=\frac{e^{-iay}-e^{-iby}}{i\sqrt{2\pi}\,y}$$

16. $f(x)=e^{-a|x|}\quad (a>0)$

We have $e^{-a|x|}=\begin{cases}e^{-ax} & x>0\\ e^{ax} & x<0\end{cases}$ Figure



Therefore $g(y)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} f(x)e^{-ixy}\,dx$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-ax}e^{-ixy}\,dx+\int_{-\infty}^{0} e^{ax}e^{-ixy}\,dx\right]$$

Replacement of x by $-x$ in the second integral gives

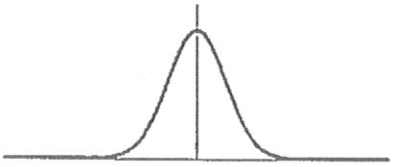
$$g(y)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\left[e^{-ax}e^{-ixy}+e^{-ax}e^{+ixy}\right]dx=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\left[e^{-(a+iy)x}+e^{-(a-iy)x}\right]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[-\frac{e^{-(a+iy)x}}{a+iy}-\frac{e^{-(a-iy)x}}{a-iy}\right]_0^{\infty}=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+iy}+\frac{1}{a-iy}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+y^2}\right)$$

17. Show that the Fourier transform of e^{-ax^2} $(a > 0)$ is

$$\frac{1}{\sqrt{2a}} e^{-y^2/4a}$$ (see Figure 15.14b)



[Hint : change variable to $t = \sqrt{a}\,x + \dfrac{iy}{2\sqrt{a}}$ and use $\displaystyle\int_{-\infty}^{\infty} e^{-\alpha t^2}\, dt = \sqrt{\frac{\pi}{\alpha}}$]

We have $$g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ax^2}\, e^{-ixy}\, dx$$

Let $$t = \sqrt{a}\,x + \frac{iy}{2\sqrt{a}} \quad \rightarrow \quad dt = \sqrt{a}\,dx\,.$$

Then $$t^2 = ax^2 + ixy - \frac{y^2}{4a}$$

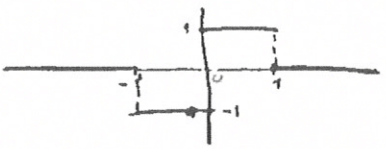
$$e^{-t^2} = e^{y^2/4a} \times e^{-ax^2} e^{-ixy}$$

and $$g(y) = \frac{e^{-y^2/4a}}{\sqrt{2\pi a}} \int_{-\infty}^{+\infty} e^{-t^2}\, dt = \frac{e^{-y^2/4a}}{\sqrt{2\pi a}} \times \sqrt{\pi}$$

$$= \frac{e^{-y^2/4a}}{\sqrt{2a}}$$

FT1

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

$$= \frac{1}{\pi}\left(\int_{-1}^{0} -\cos\omega t\, dt + \int_{0}^{1}\cos\omega t\, dt\right)$$

$$= \frac{1}{\omega\pi}\left(\left[-\sin\omega t\right]_{-1}^{0} + \left[\sin(\omega t)\right]_{0}^{1}\right)$$

$$= \frac{1}{\omega\pi}\left(-\sin(-\omega) + \sin(\omega)\right) = 0$$

$f(x)$ is odd

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

$$= \frac{1}{\omega\pi}\left(\left[\cos\omega t\right]_{-1}^{0} + \left[-\cos(\omega t)\right]_{0}^{1}\right)$$

$$= \frac{1}{\omega\pi}\left(1 - \cos\omega - \cos\omega + 1\right)$$

$$= \frac{2}{\omega\pi}\left(1 - \cos(\omega)\right)$$

FT2

. We have

$$f(x) = \begin{cases} 1 - \cos x & \text{if } -\pi/2 < x < \pi/2, \\ 0 & \text{otherwise.} \end{cases}$$

The solution follows by combining Examples 1 and 2 and using the linearity of the Fourier integral, or we can compute directly and basically repeat the computation in these examples. Since f is even, $B(w) = 0$ for all w, and

$$\begin{aligned}
A(w) &= \frac{2}{\pi}\int_0^{\pi/2} (1-\cos t)\cos wt\, dt \\
&= \frac{2\sin(\pi w/2)}{\pi w} - \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{2}\left[\cos(w+1)t + \cos(w-1)t\right] dt \\
&= \frac{2\sin(\pi w/2)}{\pi w} - \frac{1}{\pi}\left[\frac{\sin(w+1)t}{w+1} + \frac{\sin(w-1)t}{w-1}\right]\Bigg|_0^{\pi/2} \qquad (w \neq 1) \\
&= \frac{2\sin(\pi w/2)}{\pi w} - \frac{1}{\pi}\left[\frac{\sin(w+1)\frac{\pi}{2}}{w+1} + \frac{\sin(w-1)\frac{\pi}{2}}{w-1}\right] \qquad (w \neq \pm 1) \\
&= \frac{2\sin(\pi w/2)}{\pi w} - \frac{1}{\pi}\left[\frac{\cos\frac{w\pi}{2}}{w+1} - \frac{\cos\frac{w\pi}{2}}{w-1}\right] \\
&= \frac{2\sin(\pi w/2)}{\pi w} - \frac{2}{\pi}\frac{1}{1-w^2}\cos\frac{w\pi}{2},
\end{aligned}$$

If $w = \pm 1$,

$$\begin{aligned}
A(w) &= \frac{2\sin(\pm\pi/2)}{\pm\pi} - \frac{2}{\pi}\int_0^{\pi/2} \cos^2 t\, dt \\
&= \frac{2}{\pi} - \frac{2}{\pi}\int_0^{\pi/2} \frac{1+\cos 2t}{2} dt = \frac{2}{\pi} - \frac{1}{2}.
\end{aligned}$$

(Another way to get this answer is to use the fact that $A(w)$ is continuous (whenever f is integrable) and take the limit of $A(w)$ as w tends to ± 1. You will get $\frac{1}{2}$. Applying the Fourier integral representation, we obtain:

$$\frac{2}{\pi}\int_0^{\infty} \left(\frac{\sin(\pi w/2)}{w} - \frac{\cos\frac{\pi w}{2}}{1-w^2}\right)\cos wx\, dw = f(x) = \begin{cases} 1-\cos x & \text{if } |x| \leq \frac{\pi}{2}. \\ 0 & \text{if } |x| > \frac{\pi}{2}. \end{cases}$$

So $f(x) = \int_0^\infty \frac{2}{\omega\pi}(1-\cos(\omega))\sin\omega x \, d\omega$

FT2 $f(x)$ is even, $B(\omega) = 0$

$$A(\omega) = \frac{2}{\pi}\int_0^{\pi/2}(1-\cos t)\cos\omega t \, dt$$

$$= \frac{2\sin\left(\frac{\pi\omega}{2}\right)}{\pi\omega} - \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{2}\left[\cos(\omega+1)t + \cos(\omega-1)t\right]dt$$

$$= \frac{2\sin\left(\frac{\pi\omega}{2}\right)}{\pi\omega} - \frac{2}{\pi}\frac{1}{1-\omega^2}\cos\left(\frac{\omega\pi}{2}\right) \qquad \omega \neq \pm 1$$

If $\omega = \pm 1$ $\quad A(\omega) = \frac{2\sin(\pm\pi/2)}{\pm\pi} - \frac{2}{\pi}\int_0^{\pi/2}\cos^2 t \, dt$

$$= \frac{2}{\pi} - \frac{2}{\pi}\int_0^{\pi/2}\frac{1+\cos 2t}{2}dt = \frac{2}{\pi} - \frac{1}{2}$$

So $f(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{\sin\frac{\pi\omega}{2}}{\omega} - \frac{\cos\frac{\pi\omega}{2}}{1-\omega^2}\right]\cos\omega x \, d\omega$

FT3 even $\rightarrow B(\omega) = 0$

$$A(\omega) = \frac{2}{\pi}\int_0^1 (1-t)\cos(\omega t)\, dt$$

$$= \frac{2}{\pi}\left[\frac{\sin(\omega t)}{\omega}\right]_0^1 - \frac{2}{\pi}\int_0^1 t\cos(\omega t)\, dt$$

$$= \frac{2}{\pi}\left[\frac{\sin(\omega)}{\omega} - \frac{\omega\sin(\omega) + \cos(\omega) - 1}{\omega^2}\right]$$

So $f(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{1-\cos(\omega)}{\omega^2}\right]\cos\omega x \, d\omega$

FT4

In computing $\widehat{f}$, the integral depends on the values of f on the interval $(-1, 1)$. Since on this interval f is odd, it follows that $f(x)\cos wx$ is odd and $f(x)\sin wx$ is even on the interval $(-1, 1)$. Thus

$$\begin{aligned}
\widehat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\overbrace{\int_{-1}^{1} f(x)\cos wx\,dx}^{=0} - \frac{i}{\sqrt{2\pi}}\int_{-1}^{1} f(x)\sin wx\,dx \\
&= -\frac{2i}{\sqrt{2\pi}}\int_{0}^{1}\sin wx\,dx \\
&= \frac{2i}{\sqrt{2\pi}}\frac{\cos wx}{w}\Big|_0^1 \\
&= i\sqrt{\frac{2}{\pi}}\frac{\cos w - 1}{w}.
\end{aligned}$$

FT6

We evaluate the Fourier transform with the help of Euler's identity: $e^{-iwx} = \cos wx - i\sin wx$. We also use the fact that the integral of on odd function over a symmetric interval is 0, while the integral of an even function over a symmetric interval $[-a, a]$ is twice the integral over $[0, a]$.

$$\begin{aligned}
\widehat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\sin x(\cos wx - i\sin wx)\,dx \\
&= \frac{1}{\sqrt{2\pi}}\overbrace{\int_{-\pi}^{\pi}\sin x\cos wx\,dx}^{=0} - \frac{i}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\sin x\sin wx\,dx \\
&= -\frac{2i}{\sqrt{2\pi}}\int_{0}^{\pi}\frac{1}{2}\left[(-\cos(w+1)x + \cos(w-1)x\right]\,dx.
\end{aligned}$$

If $w = 1$,

$$\widehat{f}(1) = -\frac{i}{\sqrt{2\pi}}\int_0^{\pi}\left[(-\cos 2x + 1\right]\,dx = -\frac{i\sqrt{\pi}}{\sqrt{2}}.$$

If $w = 1$, we find similarly that

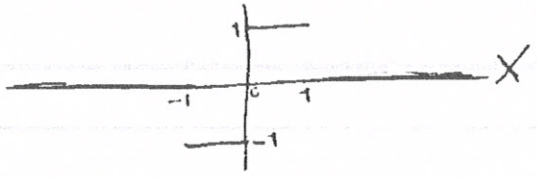
$$\widehat{f}(-1) = \frac{i\sqrt{\pi}}{\sqrt{2}}.$$

(You may also note that $\widehat{f}$ is odd in this case.) For $w \neq \pm 1$, we have

$$\begin{aligned}
\widehat{f}(w) &= \frac{i}{\sqrt{2\pi}}\left(\frac{1}{w+1}\sin(w+1)x - \frac{1}{w-1}\sin(w-1)x\right)\Big|_0^{\pi} \\
&= \frac{i}{\sqrt{2\pi}}\left(\frac{1}{w+1}\sin(w+1)\pi - \frac{1}{w-1}\sin(w-1)\pi\right) \\
&= \frac{i}{\sqrt{2\pi}}\left(\frac{1}{w+1}(-1)\sin w\pi - \frac{1}{w-1}(-1)\sin w\pi\right) \\
&= \frac{i}{\sqrt{2\pi}}\sin w\pi\left(-\frac{1}{w+1} + \frac{1}{w-1}\right) \\
&= i\sqrt{\frac{2}{\pi}}\frac{\sin w\pi}{w^2-1}.
\end{aligned}$$

Check that as $w \to \pm 1$, $\widehat{f}(w) \to \widehat{f}(\pm 1)$.

FT4 a)



$f(x)$ is odd → $f \cos \omega x$ is odd and $f \sin \omega x$ is even

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x) \cos \omega x \, dx - \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} f(x) \sin \omega x \, dx$$

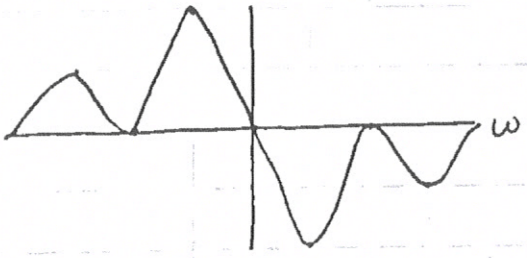
$$= \frac{-2i}{\sqrt{2\pi}} \int_{0}^{1} \sin \omega x \, dx$$

$$= \frac{-2i}{\sqrt{2\pi}} \left[-\frac{\cos \omega x}{\omega} \right]_0^1$$

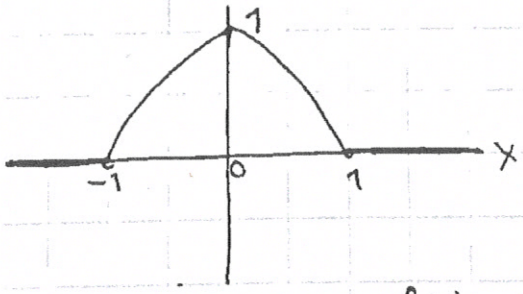
$$= \frac{-2i}{\sqrt{2\pi}} \left[-\frac{\cos \omega}{\omega} + \frac{1}{\omega} \right]$$

$$= i\sqrt{\frac{2}{\pi}} \frac{\cos(\omega) - 1}{\omega}$$

b)



FT5 a)



$f(x)$ is even → $f(x) \sin \omega x$ is odd → 0

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x) \cos \omega x \, dx$$

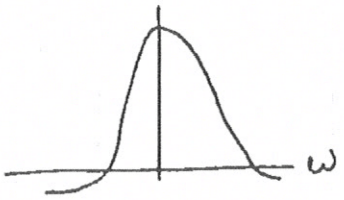
$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^1 (1-x^2) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega x}{\omega} \right]_0^1 - \sqrt{\frac{2}{\pi}} \int_0^1 x^2 \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega}{\omega} - \frac{\omega^2 \sin \omega - 2\sin \omega + 2\omega \cos \omega}{\omega^3} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2\sin \omega - 2\omega \cos \omega}{\omega^3} \right)$$

b)



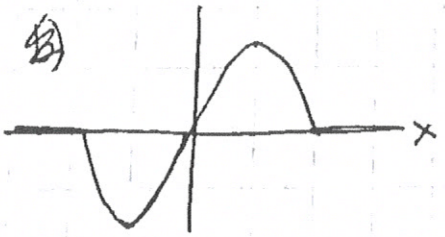
FT6

a) f is odd → f cos is odd → 0

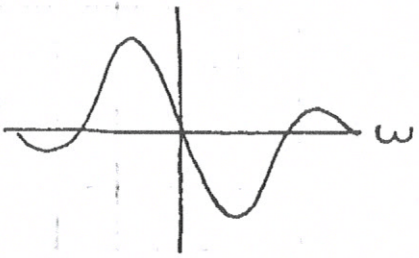
$$\hat{f}(\omega) = i\sqrt{\frac{2}{\pi}} \int_0^{\pi} \sin x \sin \omega x \, dx$$

$$= i\sqrt{\frac{2}{\pi}} \frac{\sin(\pi\omega)}{\omega^2 - 1} \quad \text{for } \omega \neq \pm 1$$

a)



b)



for $\omega = 1$ $\quad \hat{f}(1) = -\frac{i\sqrt{\pi}}{\sqrt{2}}$

for $\omega = -1$ $\quad \hat{f}(-1) = \frac{i\sqrt{\pi}}{2}$