Scalar functions ("values")

: Temperature $T(x,y,z,t)$

density $\rho(x,y,z,t)$

energy $E(x,y,z,t)$

length l (a scalar)

.....

Vector functions : ("size/value & direction")

velocity $\vec{v}(x,y,z,t)$

acceleration $\vec{a}(x,y,z,t)$

position $\vec{r}(x,y,z,t)$

force $\vec{F}(x,y,z,t)$

pressure $\vec{P}(x,y,z,t)$

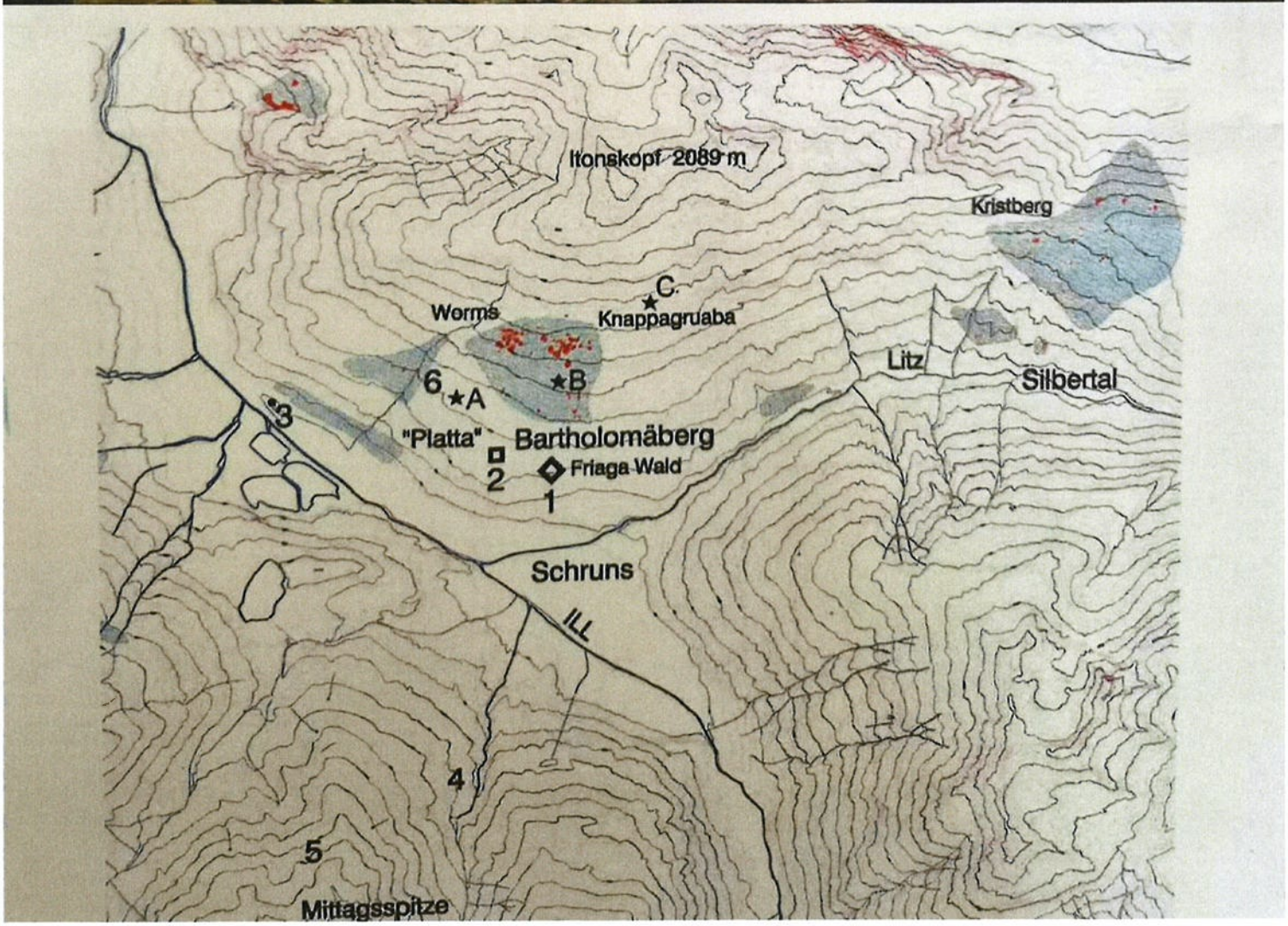
magnetic induction $\vec{B}(x,y,z,t)$

.....

contourlines



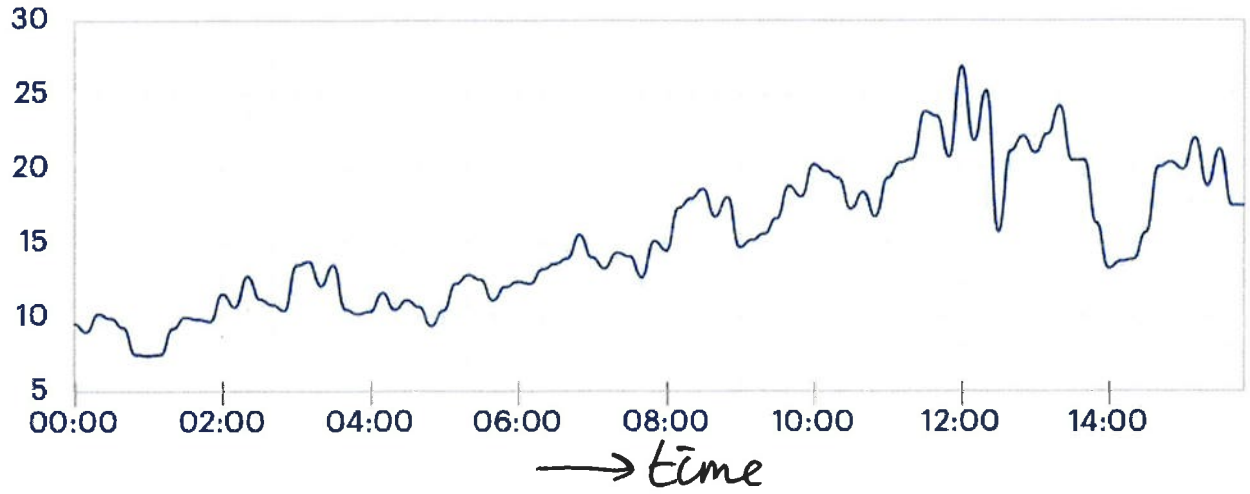



Itonskopf 2089 m
Kristberg
C.
Worms
Knappagruaba
Litz
Silbertal
6
A
B
3
"Platta"
Bartholomäberg
2
Friaga Wald
1
Schruns
ILL
4
5
Mittagsspitze

Windsnelheid in km/u

Meetstation De Bilt

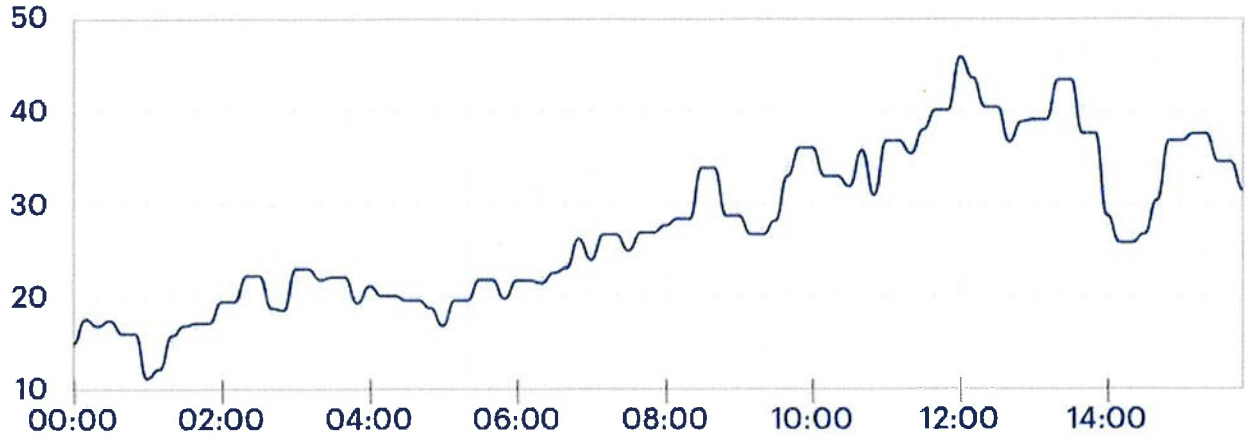
Actueel (15:50): **17,5** Min: **7,4** Max: **26,9**



Windstoten in km/u

Meetstation De Bilt

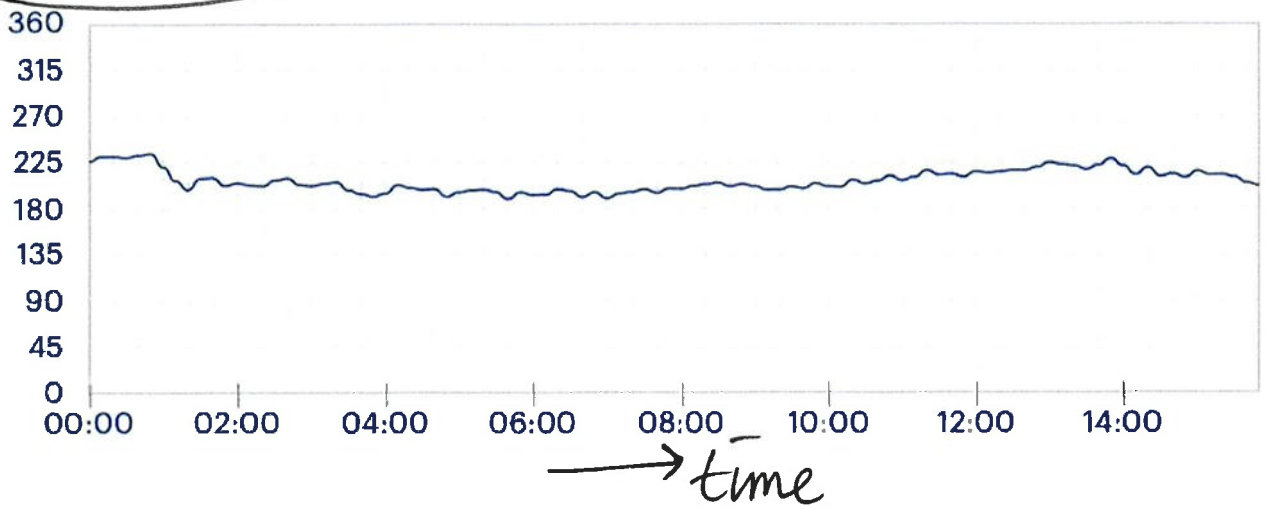
Actueel (15:50): **31,5** Min: **11,2** Max: **45,9**

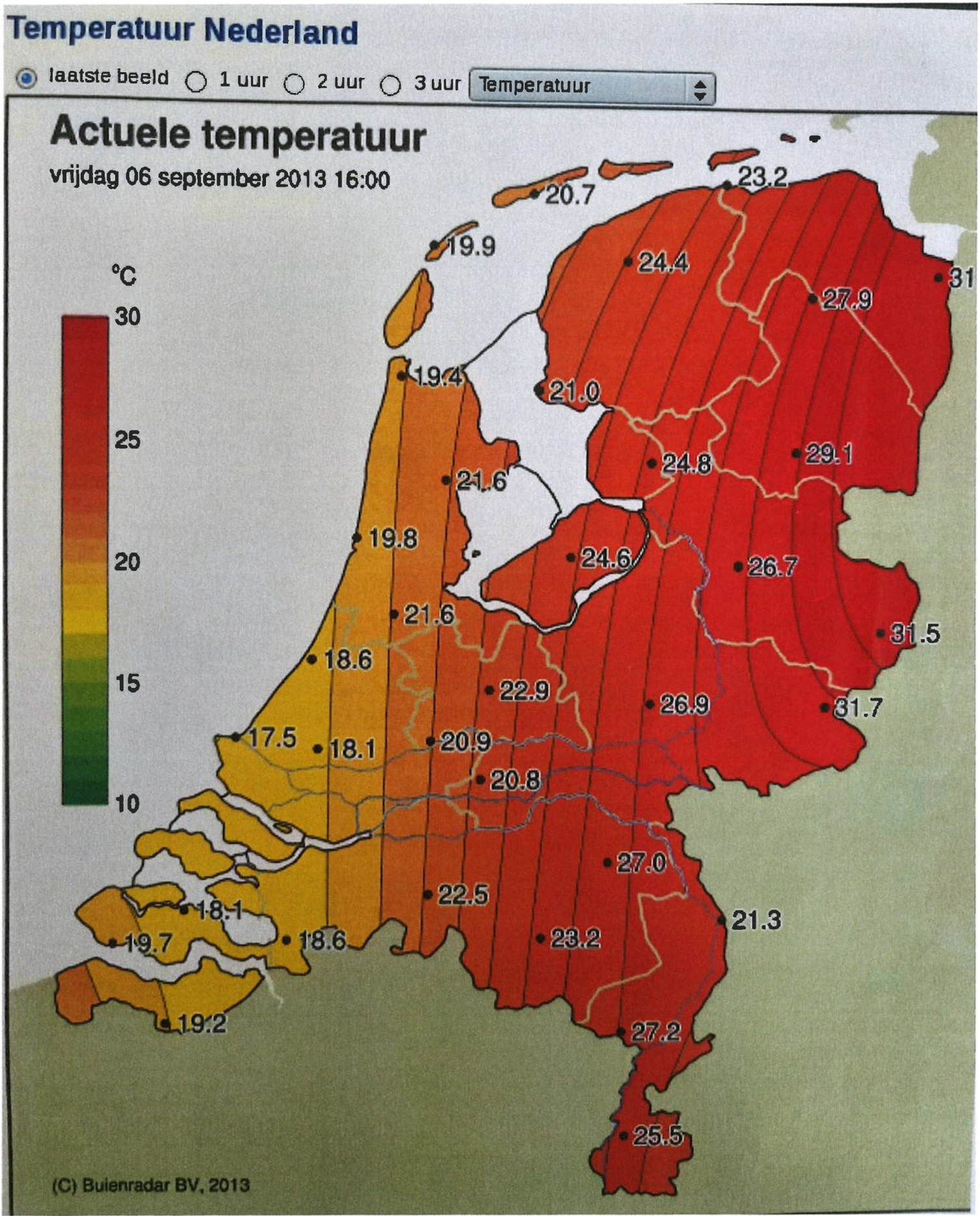


Windrichting in graden

Meetstation De Bilt

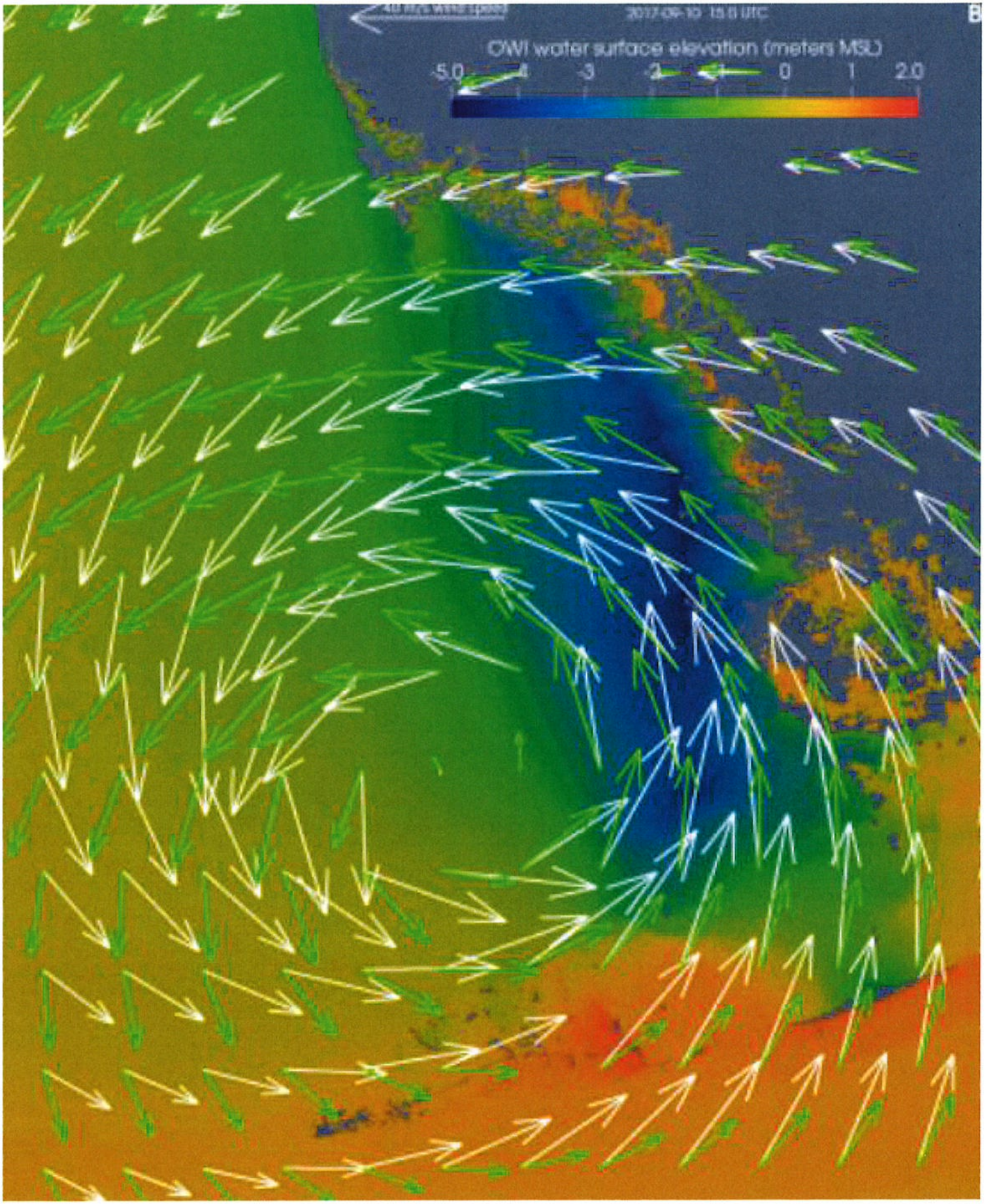
Actueel (15:50): **202** Min: **189** Max: **234**





temperatures + contour lines

hurricane + velocity field + temperature



Vectors

notation $\vec{a}$ (or $\vec{a}$, or $\underline{a}$, or $\mathbf{a}$, ---)



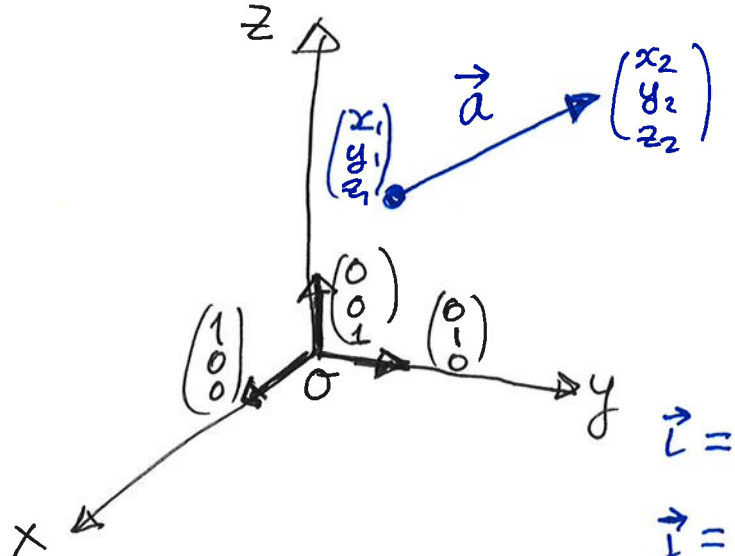
* it has a starting point
  and an end point
* it has a direction "➚"
* it has a length : $|\vec{a}|$ (or $\|\vec{a}\|$)

Cartesian coordinate system

starting point $P : \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, endpoint $Q : \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

vector $\vec{a}$

(with respect to the origin $O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$)



$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\vec{a} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

"transposed" =

$$= \left(x_2 - x_1, y_2 - y_1, z_2 - z_1\right)^T$$

components of the vector

Unit vectors: $\vec{i}, \vec{j}, \vec{k}$

$$\vec{a} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \overset{\text{name}}{\underset{\text{re-write or define}}{=}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

the zero-"vector", the origin $\vec{0} = (0,0,0)^T = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

has length $|\vec{0}| = 0$ ← number

↑ "vector"

The length of the vector $\vec{a}$:

$$|\vec{a}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\left(= \sqrt{a_1^2 + a_2^2 + a_3^2} \right)$$

two vectors are identical

$\Leftrightarrow \vec{a} = \vec{b} \Leftrightarrow$ all components are equal

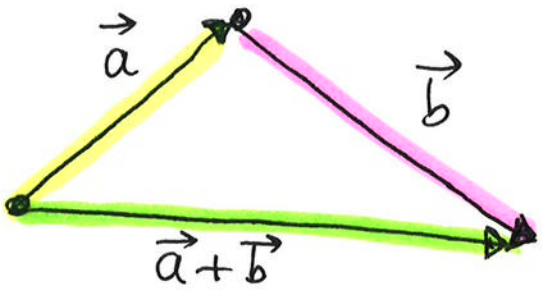
$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}''$ $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}''$ $\Leftrightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$

adding two vectors $\vec{a}$ and $\vec{b}$: ↓ T

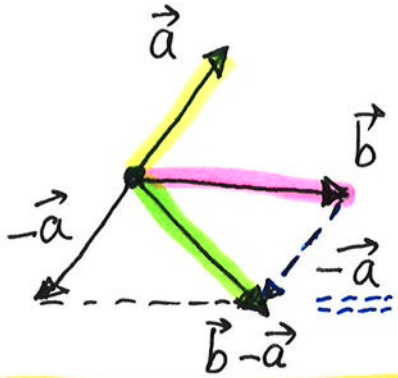
$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)^T$$

$$= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}$$

in a figure:



subtracting vectors: $\vec{b}-\vec{a}=\vec{b}+(-\vec{a})$



"multiplying" vectors:

① scalar product (inner product)

$$\vec{a}\bullet\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos(\gamma)=a_1b_1+a_2b_2+a_3b_3$$

number!!!

$=$

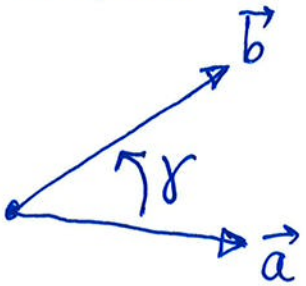
$(\vec{a},\vec{b})$

$=$

$\langle\vec{a},\vec{b}\rangle$

$|\vec{a}|$: number; $|\vec{b}|$: times; $\cos(\gamma)$: number; $a_1b_1+a_2b_2+a_3b_3$: times, number

γ is the angle between the two vectors $\vec{a}$ and $\vec{b}$

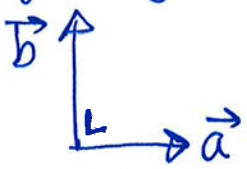


Some relations with the inner product:

1) $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a}$ and $\vec{b}$ are perpendicular

"orthogonality" $\Leftrightarrow \gamma = \frac{\pi}{2}$ $\Leftrightarrow \vec{a} \cdot \vec{b} = 0$

$(90^\circ) \rightarrow \cos(\gamma) = 0$



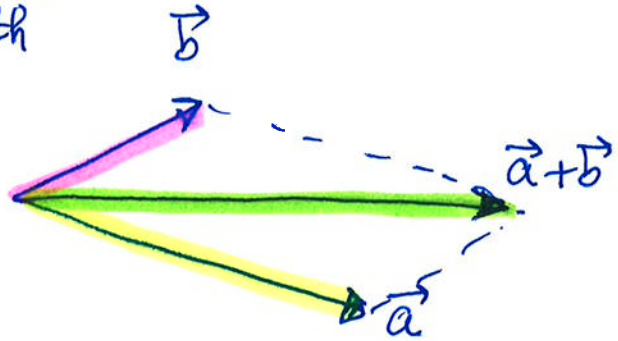
2) $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$

3) $|\vec{a} \cdot \vec{b}| \leq |\vec{a}| \cdot |\vec{b}|$ (Cauchy-Schwarz inequality)

absolute value — length

4) $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$ (triangle inequality)

length

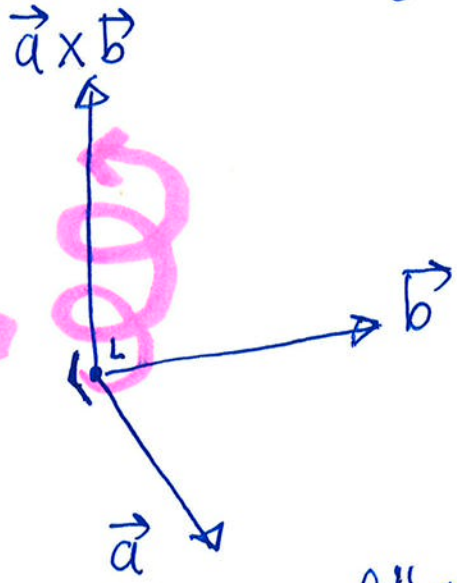


② vector product (outer product)

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \quad (*)$$

$\vec{a} \times \vec{b}$: vector

$\begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$: vector



$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$!

"the corkscrew rule" ("the right-hand rule")

$$(*) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

determinant

number

$\Rightarrow$ vector !

Some relations with the outer product:

1) $\vec{i} \times \vec{j} = \vec{k}$ $\quad \vec{j} \times \vec{i} = -\vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$ $\quad \vec{k} \times \vec{j} = -\vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$ $\quad \vec{i} \times \vec{k} = -\vec{j}$

check in a 3D figure!

2) $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$

$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

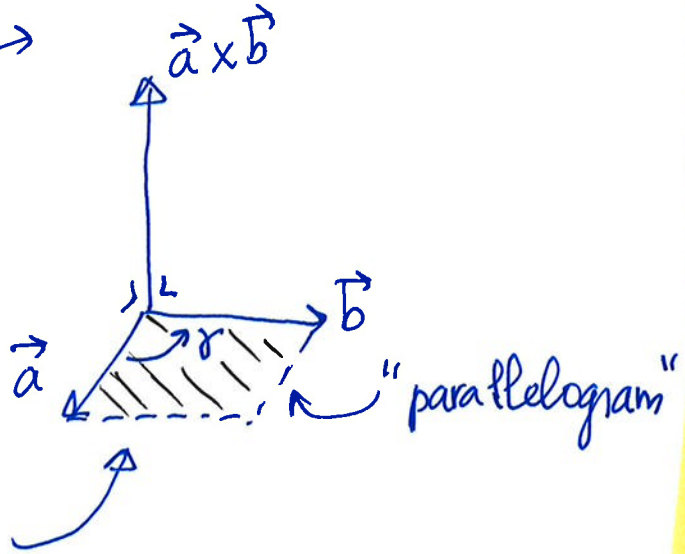
(not equal)

3) $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\gamma)$

$\vec{a} \times \vec{b}$: vector

number (length)

numbers



= area

3) component wise product

$$\vec{a} .* \vec{b} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{pmatrix}$$

Let us call the vector functions $\vec{v}$ and the scalar functions f

$$\vec{v} = \vec{v}(x,y,z) = \begin{pmatrix} v_1(x,y,z) \\ v_2(x,y,z) \\ v_3(x,y,z) \end{pmatrix} \leftrightarrow \text{"vector field"}$$

$$f = f(x,y,z) \leftrightarrow \text{"scalar field"}$$

in one dimension (space x or time t):

derivative $\frac{df}{dx}, f', \ldots$ $\frac{df}{dt}, \dot{f}, \ldots$

"ordinary derivative" (if $f(x)$) (if $f(t)$)

in dimensions > 1 several types of "first derivatives" exist!

1 The gradient of $f(x,y,z)$ ← scalar function

$$\text{grad}(f) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

a vector

"partial derivative"

keep y and z constant

keep x and z constant

keep x and y constant

other notation

$$\text{grad}(f) = \nabla f$$

∇ "nabla" or "del"

$$\nabla \dots = \frac{\partial \dots}{\partial x}\cdot\vec{i} + \frac{\partial \dots}{\partial y}\cdot\vec{j} + \frac{\partial \dots}{\partial z}\vec{k}$$

"operator"

scalar $f \longrightarrow$ vector ∇f

Example: $f(x,y,z) = 6xy - z^2$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(6xy - z^2) = 6y\underbrace{\frac{\partial}{\partial x}(x)}_{=1} - \underbrace{\frac{\partial}{\partial x}(z^2)}_{=0} = 6y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(6xy - z^2) = 6x\underbrace{\frac{\partial}{\partial y}(y)}_{=1} - \underbrace{\frac{\partial}{\partial y}(z^2)}_{=0} = 6x$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(6xy - z^2) = \underbrace{\frac{\partial}{\partial z}(6xy)}_{=0} - \underbrace{\frac{\partial}{\partial z}(z^2)}_{=-2z} = -2z$$

$$\Rightarrow \text{grad}(f) = \nabla f = \begin{pmatrix} 6y \\ 6x \\ -2z \end{pmatrix} = 6y\vec{i} + 6x\vec{j} - 2z\vec{k}$$

2 The divergence of $\vec{v}(x,y,z)$

($\vec{v}$: vector function)

$$\vec{v} = \begin{pmatrix} v_1(x,y,z) \\ v_2(x,y,z) \\ v_3(x,y,z) \end{pmatrix}$$

$$\Rightarrow \quad \operatorname{div}(\vec{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$\operatorname{div}(\vec{v})$: vector (function); $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$: scalar!! (function) — !

$$= \nabla \bullet \vec{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \bullet \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{\partial}{\partial x} v_1 + \frac{\partial}{\partial y} v_2 + \frac{\partial}{\partial z} v_3$$

read as an "operator"

$\bullet$: inner product

Example: $\vec{v}(x,y,z) = \begin{pmatrix} 3xz \\ 2xy \\ -yz^2 \end{pmatrix} \Rightarrow \operatorname{div}(\vec{v}) = \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(-yz^2)$

$$= 3z + 2x - 2yz$$

$$\operatorname{div}(\operatorname{grad}(f)) = \operatorname{div} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$\operatorname{grad}(f)$: vector; f: scalar; $\operatorname{div}(\operatorname{grad}(f))$: scalar

other notation: Δf "laplace of f"

or: $\nabla^2 f$ "nabla squared" of f

div appears in several "conservation laws":

example 1: mass conservation in a flow

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) = 0$$

ρ: density, $\vec{v}$: velocity field

if ρ = constant, then $\frac{\partial \rho}{\partial t} = 0$ and $\text{div}(\rho \vec{v}) = \rho \cdot \text{div}(\vec{v})$

the flow is called incompressible ("niet samendrukbaar")

$$\Rightarrow \text{div}(\vec{v}) = 0$$

example 2: "no magnetic monopoles exist"

$$\text{div}(\vec{B}) = 0$$

$\vec{B}$: magnetic induction

this is an important conservation law in magneto-hydro-dynamics

In general: divergence = "outflow" − "inflow"

(check exercise R1)

3. The rotation (or curl) of a vectorfunction

given $\vec{v} = \begin{pmatrix} v_1(x,y,z) \\ v_2(x,y,z) \\ v_3(x,y,z) \end{pmatrix}$

then $\operatorname{rot}(\vec{v}) \overset{\text{other name}}{=} \operatorname{curl}(\vec{v}) = \nabla \times \vec{v}$

"nabla operator" — outa product

vectorfunction → vectorfunction !

$$= \begin{pmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{pmatrix} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{pmatrix}$$

unit vectors

first partial derivatives

components of $\vec{v}$

formal expression

"vector" $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

a flow is called "irrotational", if $\operatorname{rot}(\vec{v}) = \vec{0}$

in classical mechanics: $\vec{v} = \vec{\omega} \times \vec{r}$

$\vec{v}$: velocity (vector) — $\vec{\omega}$: angle velocity (vector) — $\vec{r}$: position vector

"velocity field in a rotating object"

$\Rightarrow \operatorname{rot}(\vec{v}) = \ldots = 2 \cdot \vec{\omega}$ (check!!)

↑ a "scaling constant"

Example: $\vec{v} = \begin{pmatrix} yz \\ 3xz \\ z \end{pmatrix}$

$$\Rightarrow rot(\vec{v}) = \begin{pmatrix} 0 - 3x \\ y - 0 \\ 3z - z \end{pmatrix} = \begin{pmatrix} -3x \\ y \\ 2z \end{pmatrix}$$

$$= -3x\vec{i} + y\vec{j} + 2z\vec{k}$$

properties:

1) $div(rot(\vec{v})) = 0$ ← number

vector ← vector ← scalar

2) $rot(grad(f)) = \vec{0}$ ← vector

scalar ← vector ← vector

3) $\nabla(fg) = f\nabla g + g\nabla f$

$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

$\Delta(fg) = g\Delta f + 2\nabla f \cdot \nabla g + f\Delta g$

$div(\vec{u} \times \vec{v}) = \vec{v} \cdot rot(\vec{u}) - \vec{u} \cdot rot(\vec{v})$

$rot(f\vec{v}) = (grad(f)) \times \vec{v} + f\, rot(\vec{v})$

ETCETERA

4 The directional derivative

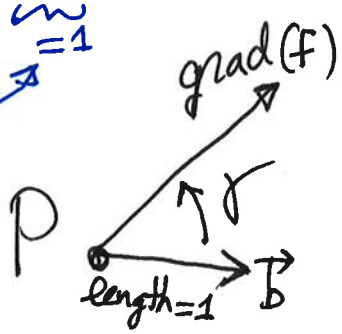
$$D_{\vec{b}} f\big|_P = \vec{b} \cdot \operatorname{grad}(f)$$

- D: derivative
- $\vec{b}$: in the direction of the vector $\vec{b}$
- f: function f
- $|_P$: at the point P
- $\cdot$: inner product
- $\operatorname{grad}(f)$: at point P

$\vec{b}$ of length $= 1$ (!)

$$= |\vec{b}| \cdot |\operatorname{grad}(f)| \cdot \cos(\gamma)$$

($|\vec{b}| = 1$)



Example: $f = x^2 + 3y^2 + xz$

$P = (1,2,3)$

$\vec{b} = \frac{1}{\sqrt{3}}(1,-1,1)$ $\quad (\Leftrightarrow |\vec{b}| = 1)$

$$\operatorname{grad}(f) = \begin{pmatrix} 2x+z \\ 6y \\ x \end{pmatrix} \text{ in the point } P: \begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix}$$

$$\Rightarrow D_{\vec{b}} f\big|_P = \operatorname{grad}(f)\big|_P \cdot \vec{b}$$

$$= \begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \frac{5}{\sqrt{3}} - \frac{12}{\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{6}{\sqrt{3}}$$

a number !! negative

Remark

The direction of the vector grad(f) at a point P is the direction of the maximum increase of the function f in this point P.

Note: if $\vec{b}$ and grad(f) have the same direction, then $\gamma = 0$ and $\cos(\gamma) = 1$ yields a maximum value of the directional derivative $D_{\vec{b}} f|_P$.

And so, $D_{\vec{b}} f|_P$ is maximal in the direction of grad(f) $\underset{\text{in this case}}{=}$ direction of $\vec{b}$

* Maximum decrease in the direction of $-$ grad(f)

The gradient vector is perpendicular to the level sets $f(x,y) = c$ (constant). Each c gives another curve!

"proof": choose a curve in the level set: $\vec{r}(t) = (x(t), y(t))^T$

$\Rightarrow f(x(t), y(t)) = c$.

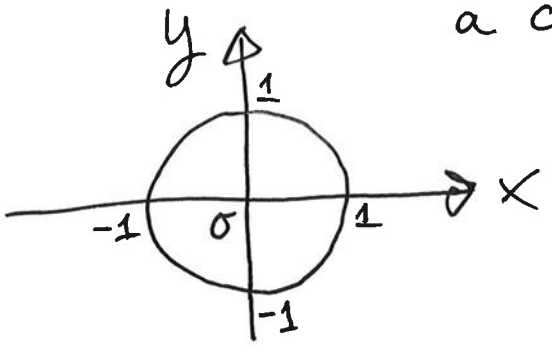
$\xrightarrow[\text{chain rule}]{}$ $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{d}{dt}(c) = 0$

(see later --) $\Leftrightarrow$ $\text{grad}(f) \bullet \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = 0$ ← tangent vector

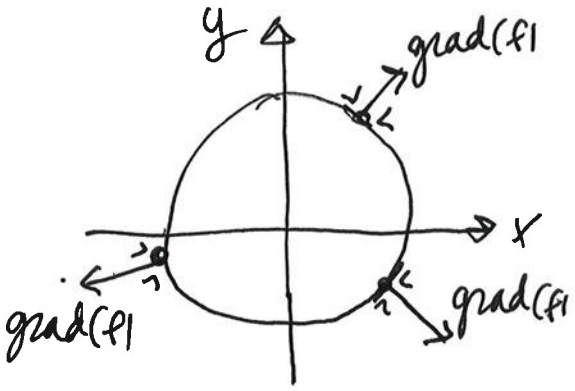
tangent vector $\perp$ normal vector ☒

Example $f(x,y) = x^2 + y^2 = c$ (take $c=1$)

a circle in the (x,y)-plane.



grad(f) is perpendicular to the circle which is given by $f(x,y) = x^2 + y^2 - 1 = 0$

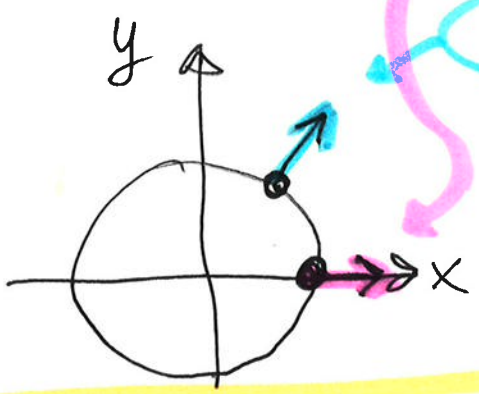


for example $(1,0)$ is on the circle.

$$\text{grad}(f) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

in $(1,0)$: $\text{grad}(f) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

in $(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$: $\text{grad}(f) = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$



This property also holds in 3D: $f(x,y,z) = c$ ("surface")

Chain rule

"we know": $\frac{d}{dt} f(\underbrace{g(t)}_{=u}) = \frac{df}{du} \cdot \frac{dg}{dt}$

$= f'(g(t)) \cdot g'(t)$

example: $f(u) = e^u$

$u = g(t) = 4t^3$

we want $\frac{d}{dt}(e^{4t^3})$:

1) $g'(t) = 12t^2$

2) $\frac{df}{du} = \frac{d}{du} e^u = e^u$

$\Rightarrow \frac{d}{dt} f(g(t)) = e^{4t^3} \cdot 12t^2$

now in "more" dimensions $\Rightarrow$ we need $\frac{\partial}{\partial x}$ notation.

suppose $w = w(x,y,z)$ and $x = x(u,v), y = y(u,v), z = z(u,v)$

given

Then:

$$\begin{cases} \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \end{cases}$$

Special cases

"no z":
$$\begin{cases} \dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} \\ \dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} \end{cases}$$
(the third terms "disappear")

"with z" but only $x=x(t)$, $y=y(t)$, $z=z(t)$:

$$\frac{\partial w}{\partial t} = \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

may use now, because there is only t-dependence at the end

previous case but with "no z":
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

"no z, no y":
$$\frac{dw}{dt} = \frac{dw}{dx}\cdot\frac{dx}{dt}$$
(this is the "old" well-known chain rule)

Note: use ∂ for partial derivatives! (no δ)

δ means something else! ("variational derivative" ...)