1. Taylor series 1D, 2D, 3D, -----
2. Jacobian and Hessian (matrix)
3. Minimization of an energy surface
4. Stationary points : minimum, maximum, saddle, ---
5. Numerical approximation : Newton-Raphson 1D, 2D, --
6. Steepest descent method

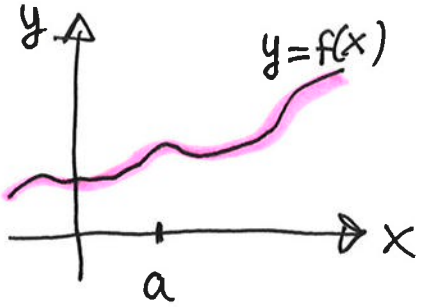
Power series (about 0):

$$\sum_{n=0}^{\infty} c_n x^n = \underbrace{c_0 x^0}_{n=0} + \underbrace{c_1 x^1}_{n=1} + \underbrace{c_2 x^2}_{n=2} + \cdots$$

$$= c_0 + c_1 x + c_2 x^2 + \cdots$$

This is a function of x : $f(x)$

about another point a : $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$



a = ... (?)

questions: , for which values of x does this series give a finite number?

, what are the c_n's ?

, how to calculate them, if $f(x)$ is known?

, why do you need such a series?
(solving DEs, approximating functions, etcetera)

, which $f(x)$, if you know the c_n's ?

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

given — given — to be calculated

1) substitute $x = a$:

$$\Rightarrow f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + \cdots$$

$$= c_0 + c_1 \cdot 0 + c_2 \cdot 0^2 + c_3 \cdot 0^3 + \cdots$$

$$= c_0$$

$$\Rightarrow c_0 = f(a)$$ ← given

2) calculate $f'(x)$:

$$f'(x) = (c_0)' + (c_1(x-a))' + (c_2(x-a)^2)' + \cdots$$

$$= 0 + c_1 + 2c_2(x-a) + \cdots$$

$$= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

substitute $x = a$:

$$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots$$

$$= c_1 + 2c_2 \cdot 0 + 3c_3 \cdot 0^2 + \cdots$$

$$= c_1$$

$$\Rightarrow c_1 = f'(a)$$ ← known

$\Rightarrow ?$ $c_2 = f''(a)$, $c_3 = f'''(a)$, $\cdots$??

3) calculate $f''(x)$:

$$f''(x) = 0 + 2c_2 + 6c_3(x-a) + \cdots$$

substitute $x = a$:

$$f''(a) = 2c_2 + 0 + 0 + \cdots$$

$$= 2c_2$$

$$\Rightarrow c_2 = \frac{f''(a)}{2}$$ ← known

continue in this manner: ----⇒ $f'''(a) = 6c_3$

$f^{(iv)}(a) = 24c_4$

$f^{(v)}(a) = 120c_2$

.....

$c_3 = \frac{1}{6} f'''(a)$

$\Rightarrow c_4 = \frac{1}{24} f^{(iv)}(a)$

$c_5 = \frac{1}{120} f^{(v)}(a)$

....

Remember: $n! = n(n-1)(n-2)(n-3)\cdots 3.2.1$

define $0! = 1$

$\Downarrow$

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$

$5! = 120$

$\vdots$

"factorial"

$\Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$

$n = 0, 1, 2, 3, \ldots\ldots$

We conclude: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

Taylor series about the point a

Example: $f(x) = e^x$ about 0 $(a = 0)$

calculate: $f'(x) = e^x, f''(x) = e^x, f'''(x) = e^x, \ldots$

and $f'(0) = 1, f''(0) = 1, f'''(0) = 1, \ldots$

$\Rightarrow c_n = \frac{1}{n!}$

$$\Rightarrow \quad f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

approximations:

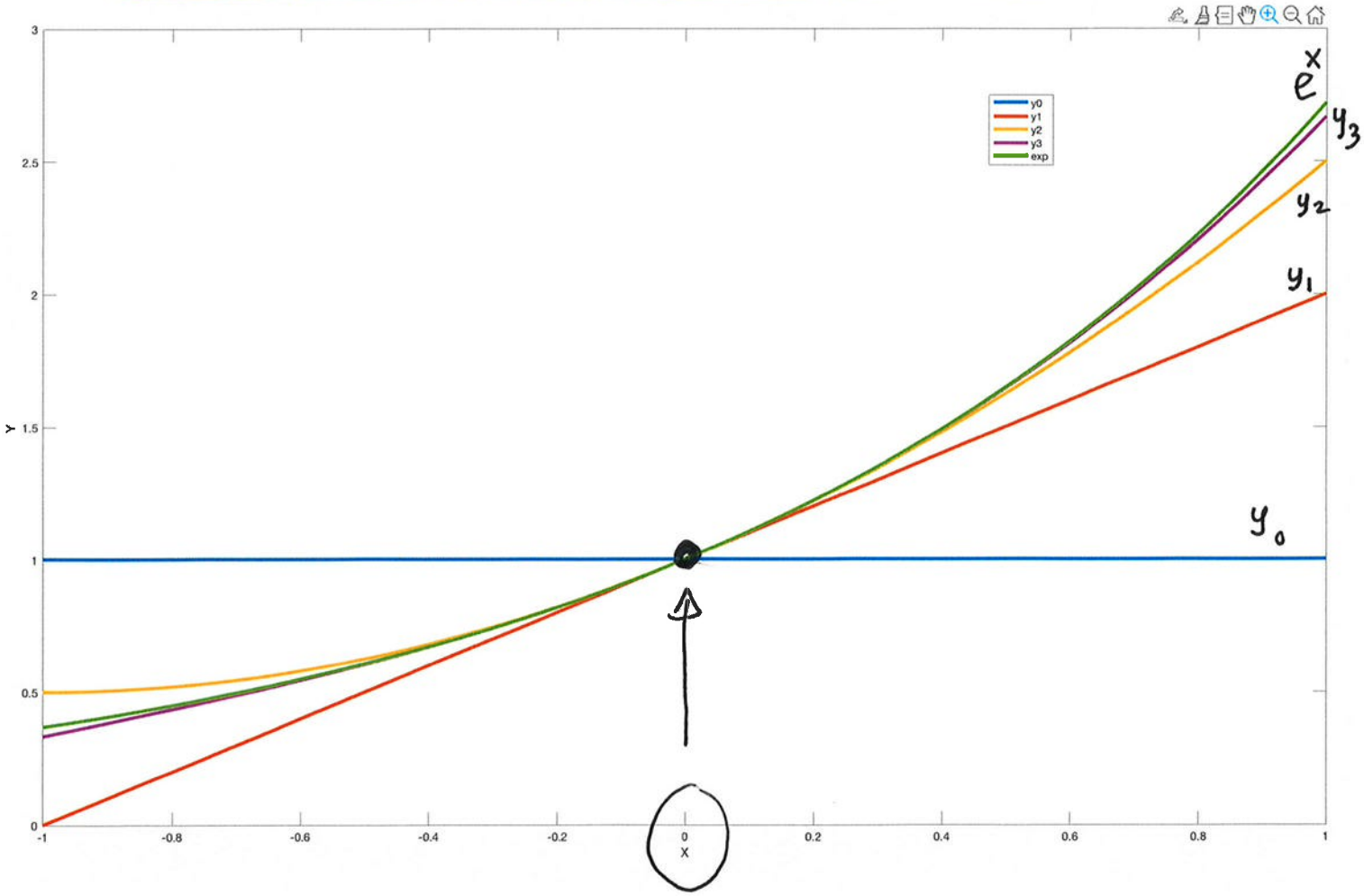
$e^x \approx 1 = y_0$

$e^x \approx 1 + x = y_1$

$e^x \approx 1 + x + \frac{x^2}{2} = y_2$

$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = y_3$

.....



Theorem (without a proof):

Calculate $R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$ "the radius of convergence"

Then: the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges (a finite number as a result)

for $-R < x-a < R$ (or $|x-a| < R$)

or $a-R < x < a+R$

if $R=0$, then convergence only for $x=a$

if $R=\infty$, then convergence for all values of x

example 1 $f(x) = e^x$, $c_n = \frac{1}{n!}$ (Taylor series about 0)

$$\left| \frac{c_n}{c_{n+1}} \right| = \left| \frac{1/n!}{1/(n+1)!} \right| = \frac{(n+1)!}{n!} = \frac{(n+1)n(n-1)\cdots 3\cdot 2\cdot 1}{n(n-1)\cdots 3\cdot 2\cdot 1} = n+1$$

$\lim_{n\to\infty} (n+1) = \infty \Rightarrow R = \infty$

this series converges for all x-values

example 2 $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ ($\neq e^x$, which f ???)

$$R = \lim_{n\to\infty} \left| \frac{1/n}{1/(n+1)} \right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$$

$\Rightarrow$ this series / power series, not a Taylor series!
converges for $-1 < x < 1$ (or $|x| < 1$)

example 2 (continued) What happens for $x=1$?

" " " $x=-1$?

$x=1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ (it diverges: $\sum_{n=1}^{\infty} = \infty$)

$x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \cdots > -1$

also $= -\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \cdots < -\frac{1}{2}$

$\Rightarrow -1 < \sum_{n=1}^{\infty} \frac{(-1)^n}{n} < -\frac{1}{2} \Rightarrow$ convergent

"Applications" (many!)

1) $\lim_{x\to 0} \frac{\sin(x)}{x} = ?$ "$\frac{0}{0}$"

Taylor series (about 0): $\frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \cdots$

$\lim_{x\to 0} = 1 \; (-0+0-0\cdots)$

$= 1$

2) $\lim_{x\to 0} \frac{x\sin(x)}{\cos(x)-1} = ?$ "$\frac{0}{0}$"

$= \frac{x\,(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)}{(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots) - 1} = \frac{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} \cdots}{-\frac{1}{2}x^2 + \frac{1}{24}x^4 \cdots}$

Taylor series about 0 (sin, cos)

divide by x^2: $= \frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots}{-\frac{1}{2} + \frac{1}{24}x^2 \cdots}$

Take now limit $x\to 0$: $\Rightarrow \frac{1-0+0\cdots}{-\frac{1}{2}+0\cdots} = \frac{1}{-1/2} = -2$

If you approximate a function by a finite version of the Taylor series, how big is the error?

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad (\infty\text{-many terms})$$

$$\approx \sum_{n=0}^{N} c_n (x-a)^n \qquad (\text{finite number of terms})$$

Taylor polynomials of degree N

$$f(x) = \underbrace{\sum_{n=0}^{N} c_n (x-a)^n}_{\text{approximation}} + \underbrace{\frac{f^{(N+1)}(\xi)}{(N+1)!}(x-a)^{N+1}}_{\text{the error}}$$

$f(x)$: exact

(ξ is an unknown point between x and a)

---

special case: $N=0$

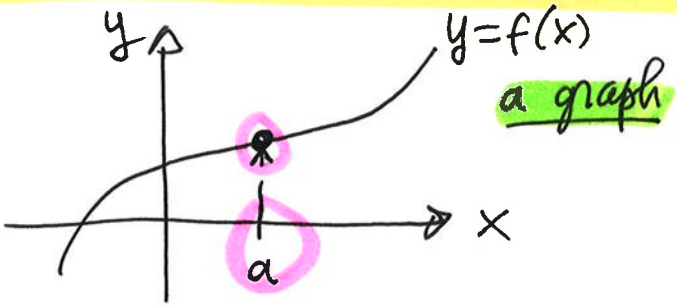
$$f(x) = \underbrace{f(a)}_{=c_0} + f'(\xi)(x-a)$$

$$\Leftrightarrow \quad f'(\xi) = \frac{f(x)-f(a)}{x-a}$$

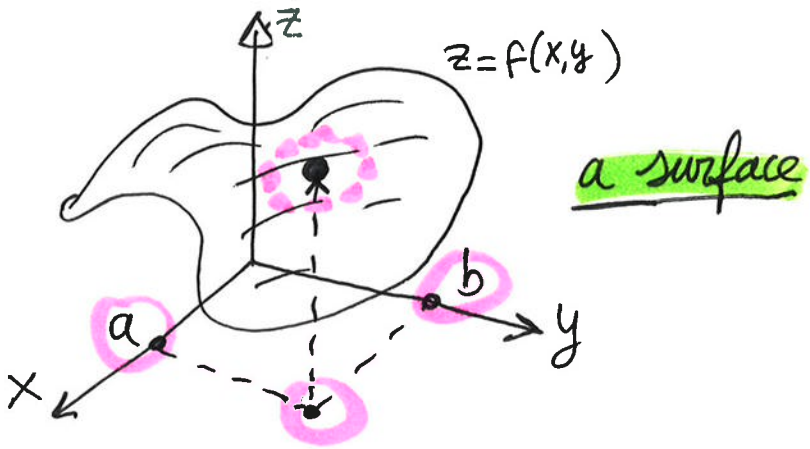
the intermediate value theorem.

Taylor in two dimensions; 2D

in 1D : $y = f(x)$



in 2D : $z = f(x,y)$



Taylor approximates $f(x,y)$ about (a,b)

in 3D : $w = f(x,y,z)$

---

etcetera

about the point (a,b) :

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)$$

$$+ \frac{1}{2!}(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + \frac{1}{2!}(y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)$$

$$+ \frac{1}{2!}(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + \frac{1}{2!}(y-b)(x-a)\frac{\partial^2 f}{\partial y \partial x}(a,b)$$

$$+ \frac{1}{3!}\left\{(x-a)^3\frac{\partial^3 f}{\partial x^3}(a,b) + \text{---------- even more terms ----}\right\}$$

$$+ \frac{1}{4!}\{--\} + \text{------}$$

in 3D : about the point (a,b,c)

$$f(x,y,z) = f(a,b,c) + (x-a)\frac{\partial f}{\partial x}(a,b,c) + \cdots\cdots\cdots$$

check it!

in "any" D :

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{R}^d$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{pmatrix} \in \mathbb{R}^d \qquad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \in \mathbb{R}^d$$

inner product

transpose

$$f(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \bullet (\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T \bullet \mathcal{H}(\vec{f}(\vec{a}))(\vec{x}-\vec{a}) + \text{H.O.T.}$$

"higher order terms"

matrix of second derivatives "the Hessian"

in two dimensions :

$$\mathcal{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

a symmetric matrix, since $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ for "smooth" functions f

counter example: $f(x,y)=\begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$

f is continuous everywhere, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as well!

However: $\frac{\partial^2 f}{\partial x \partial y}(0,0)=1 \neq -1=\frac{\partial^2 f}{\partial y \partial x}(0,0)$ ("not smooth")

Jacobian matrix for a vector function $\vec{f}(x,y)=\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$ (not a surface)

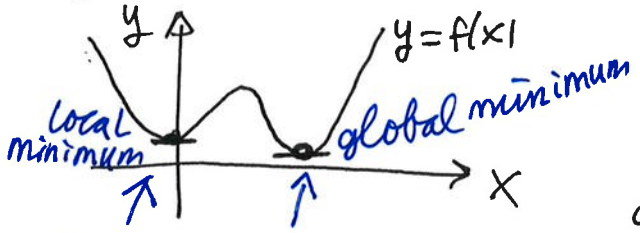
$$J=\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

"first derivative matrix"

Note: $H = J(\nabla f)$, where $\nabla f=\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$

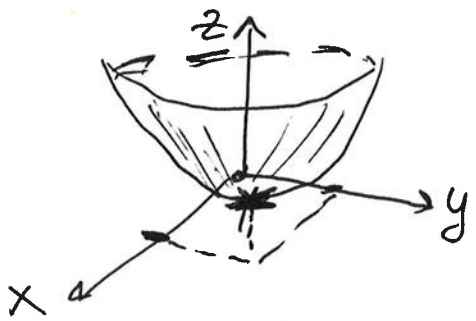
minimization of an energy function f

1D:



$f'=0$ at those points

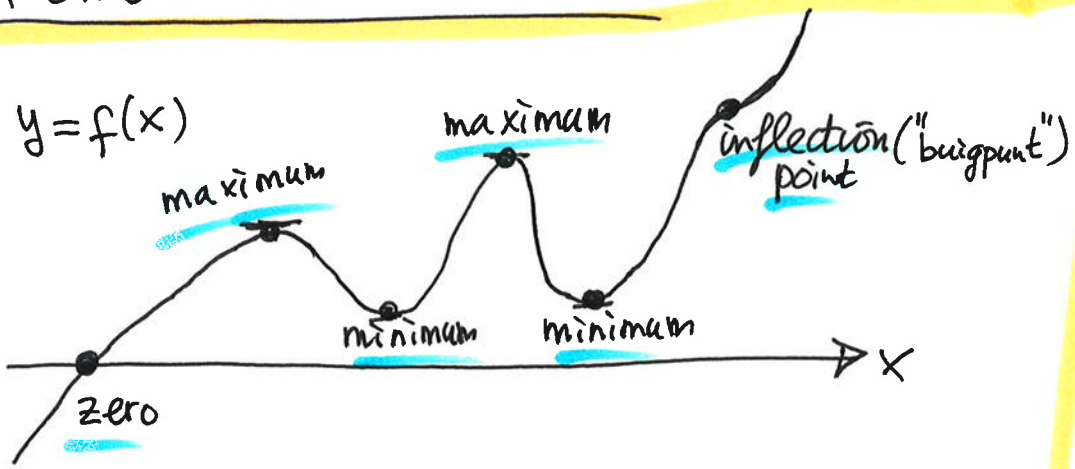
2D:



$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ must be zero

Let us work this out in more details:

in 1D:



for a minimum: $f'=0$ and $f''>0$

for a maximum: $f'=0$ and $f''<0$

for an inflection point: $f''=0$ (not necessarily $f'=0$)

for a zero: $f=0$

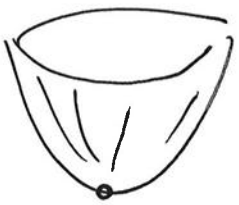
in 2D: $z=f(x,y)$ (a surface in (x,y,z)-space)

"energy"

for a minimum: $\nabla f = \text{grad}(f) = \begin{pmatrix}0\\0\end{pmatrix}$

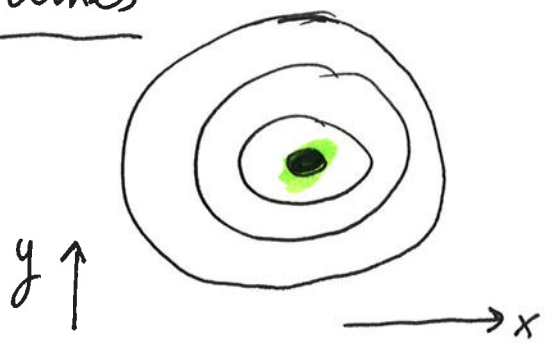
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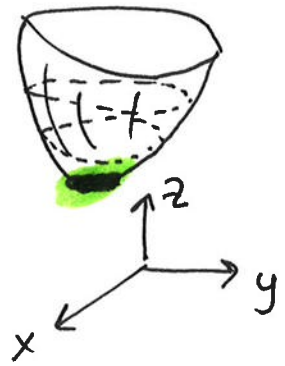
and $\begin{cases} \dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 = \det(\mathcal{H}) > 0 \\ \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \Delta f > 0 \end{cases}$



1) find (x,y) that satisfy $\nabla f = \begin{pmatrix}0\\0\end{pmatrix}$
2) calculate $\det(\mathcal{H})$ and Δf
3) conclude minimum (or not)

contourlines





for a maximum: $\begin{cases} \det(\mathcal{H}) > 0 \\ \Delta f < 0 \end{cases}$

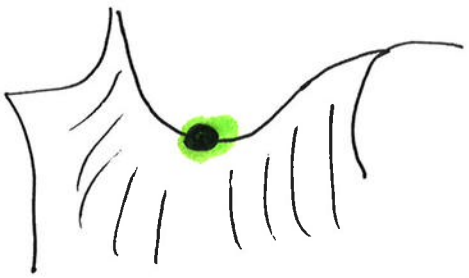
II



[the (x,y)-values must satisfy: $\nabla f = \vec{0}$]

what if $\nabla f = \text{grad}\, f = \vec{0}$, but $\det(\mathcal{H}) < 0$

III $\Rightarrow$ "a saddle point":



in a contourplot:



what if $\text{grad}(f) = \vec{0}$ and we do not have I, II or III?

$\Rightarrow$ unclear, we would need extra analysis.

Example: $z = f(x,y) = x^3 - 3xy^2 + y^4$

1) calculate ∇f and find (x,y) such that $\nabla f = \vec{0}$

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - 3y^2 \overset{\text{set}}{=} 0 \\ \text{AND} \\ \frac{\partial f}{\partial y} = -6xy + 4y^3 \overset{\text{set}}{=} 0 \end{cases} \Leftrightarrow \begin{cases} x^2 = y^2 \\ \text{AND} \\ y(-6x + 4y^2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y \ \underline{\text{or}} \ x = -y \\ \text{AND} \\ y = 0 \ \underline{\text{or}} \ y^2 = \frac{3}{2}x \end{cases} \Leftrightarrow \begin{cases} x = y \ \text{AND} \ y = 0 \Rightarrow (0,0) \\ \underline{\text{or}} \\ x = y \ \text{AND} \ y^2 = \frac{3}{2}x \Rightarrow \left(\frac{3}{2}, \frac{3}{2}\right) \\ \underline{\text{or}} \\ x = -y \ \text{AND} \ y = 0 \Rightarrow (0,0) \\ \underline{\text{or}} \\ x = -y \ \text{AND} \ y^2 = \frac{3}{2}x \Rightarrow \left(\frac{3}{2}, -\frac{3}{2}\right) \end{cases}$$

equal

$\Rightarrow$ <u>three</u> points satisfy $\nabla f = \vec{0}$

2) $$\mathcal{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & -6y \\ -6y & -6x + 12y^2 \end{pmatrix}$$

$$\Rightarrow \det(\mathcal{H}) = 6x(-6x + 12y^2) - (-6y)\cdot(-6y)$$
$$= -36x^2 + 72xy^2 - 36y^2$$
$$= 36(-x^2 + 2xy^2 - y^2)$$

and $$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x + (-6x + 12y^2) = 12y^2$$

3) create a <u>table</u>:

point	$\det(\mathcal{H})$	Δf	character
$(0,0)$	0	0	<u>UNCLEAR</u> ---
$\left(\frac{3}{2}, \frac{3}{2}\right)$	>0	>0	Minimum
$\left(\frac{3}{2}, -\frac{3}{2}\right)$	>0	>0	Minimum

What if finding (x,y) such that ∇f can not be done analytically?

$\rightsquigarrow$ numerical method needed

first in 1D and first how to find a zero of $f(x)$?

example: $f(x) = x^2 - 2 = 0$, $x > 0$

(we know: $x = \sqrt{2} = 1.41421356624\ldots$)

re-write: $2x^2 = x^2 + 2$

$\Leftrightarrow 2x = x + \frac{2}{x}$

$\Leftrightarrow x = \frac{1}{2}\left(x + \frac{2}{x}\right)$

satisfies!!

create an iteration method: $\begin{cases} x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right), i = 0,1,2,\ldots \\ x_0 = 2 \end{cases}$

$\lim_{i\to\infty} x_i = ?$

Note: $\lim_{i\to\infty} x_i = \lim_{i\to\infty} x_{i+1}$ (!)

$\Rightarrow$ indeed, if it converges, the limit will be $\sqrt{2}$.

"check": $x_0 = 2$

$x_1 = \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{1}{2}\cdot 3 = \frac{3}{2} = 1.5$

$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{1}{2}\frac{17}{6} = \frac{17}{12} = 1.41667\ldots$

$x_3 = \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \ldots = 1.4142\ldots$

etcetera

Note: $x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right)$ $\quad (f(x) = x^2 - 2)$

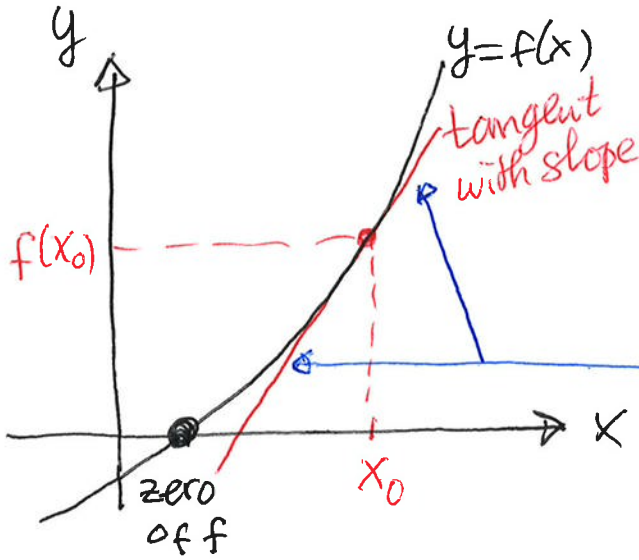
$$\Leftrightarrow x_{i+1} = \frac{1}{2}x_i + \frac{1}{x_i}$$

$$= x_i - \frac{1}{2}x_i + \frac{1}{x_i}$$

$$= x_i - \frac{x_i^2 - 2}{2x_i}$$

$$= x_i - \frac{f(x_i)}{f'(x_i)}$$

method of Newton-Raphson "NR"!



in $(x_0, f(x_0))$:

$$y = f'(x_0)x + f(x_0) - f'(x_0)x_0$$

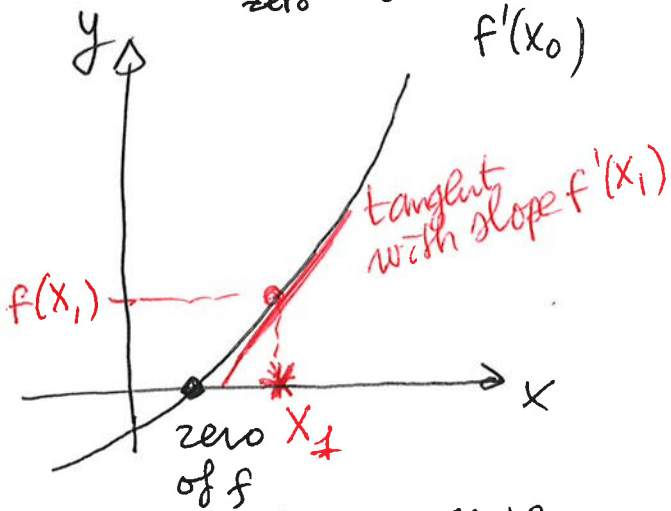
(linear function in x)

the zero of this straight line: set $y = 0$

$$x_{zero} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

call this x_{zero}: x_1

and repeat:



calculate zero of this new straight line; call this x_2

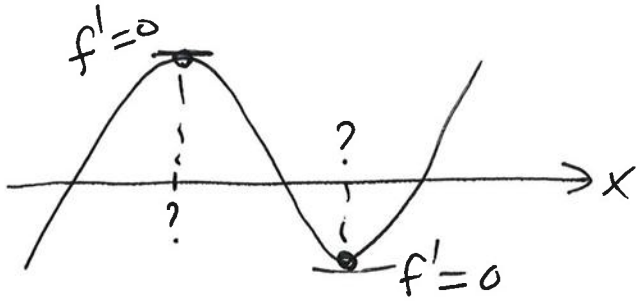
$$\Rightarrow x_{zero} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

etcetaa

for a minimum (or maximum) of f ("the energy")

find x such that $f'(x) = 0$

(zero's of f')



Replace in Newton-Raphson: f by f' (and also f' by f'')

$$\Rightarrow \begin{cases} x_{i+1} = x_i - \dfrac{f'(x_i)}{f''(x_i)} & i = 0, 1, 2, \cdots \\ x_0 = \text{starting value}. \end{cases}$$

in 2D (or higher dimensions)

$$z = f(x, y)$$

$$\Rightarrow \begin{cases} \vec{X}_{i+1} = \vec{X}_i - \mathcal{H}^{-1}\Big|_{\vec{X}_i} \cdot \text{grad}(f)\Big|_{\vec{X}_i}, & i = 0, 1, 2, \cdots \\ \vec{X}_0 = \text{starting point} \end{cases}$$

$\vec{X}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ $\quad$ $\text{grad}(f)\Big|_{\vec{X}_i}$ = gradient of f at $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$

$\mathcal{H}^{-1}\Big|_{\vec{X}_i}$ = inverse of Hessian matrix at $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$

Newton-Raphson in 2D for finding $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $\nabla f = \vec{0}$

Steepest descent method

main difference with NR (there is no Hessian matrix)

$$\begin{cases} \vec{X}_{i+1} = \vec{X}_i - \alpha_i \nabla f\big|_{\vec{X}_i} \\ \vec{X}_0 = \text{starting value} \end{cases} \quad , i = 0, 1, 2, \cdots$$

$(\alpha_i > 0)$ "step size" → constant or → updated each iteration step

∇f is vector: the gradient of "energy" f

in 2D: $\vec{X}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ and $\vec{X}_{i+1} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}$

Example $z = f(x,y) = x^2 + 2y^2$ "paraboloid" (minimum in (0,0))

1) compute gradient vector

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$$

2) choose an arbitrary initial guess $\vec{X}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and a constant step size $\alpha = 0.25$.

a) evaluate the gradient at $\vec{X}_0$: $\nabla f(2,1) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

b) calculate the updated position $\vec{X}_1$:

$$\vec{X}_1 = \vec{X}_0 - \alpha \nabla f(\vec{X}_0)$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 0.25 \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

c) evaluate f-values: $i=0$ $f(2,1) = 2^2 + 2\cdot 1^2 = 6$

$i=1$ $f(1,0) = 1^2 + 2\cdot 0^2 = 1$ ⇒ successful descent!

3) repeat the iteration process starting from $\vec{x}_1 = (1,0)^T$ with the same step size $\alpha = 0.25$.

a) evaluate the new gradient: $\nabla f(1,0) = \begin{pmatrix} 2*1 \\ 4*0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

b) calculate the updated position $\vec{x}_2$:

$$\vec{x}_2 = \vec{x}_1 - \alpha \cdot \nabla f(\vec{x}_1)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0.25 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2} \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

c) evaluate f-values: $i = 1 \quad f(1,0) = 1$

$i = 2 \quad f(\frac{1}{2}, 0) = \frac{1}{4}$ $\Rightarrow$ again successful descent!

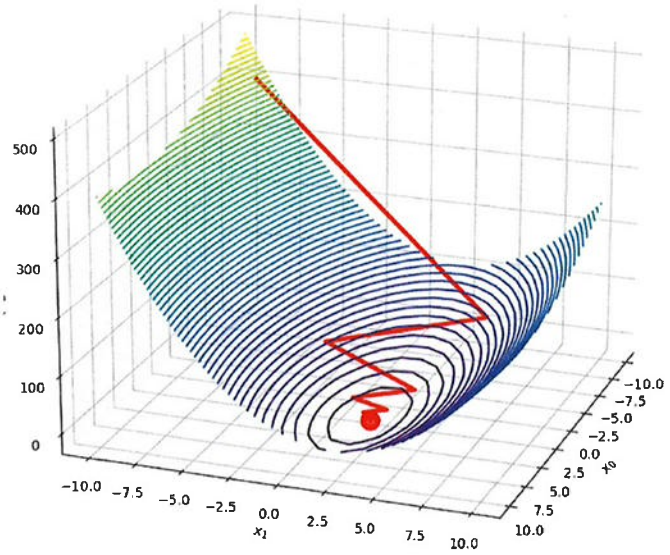
etcetera $\Rightarrow$ convergence to $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (0,0)^T$ the (known) minimum

Important remark:

The gradient vector $\nabla f(\vec{x}_i)$ points in the direction of the steepest ascent.

$\Rightarrow$ the direction of steepest descent is given by $-\nabla f(\vec{x}_i)$

steepest descent on a surface towards the minimum



the same but now in a contourplot

