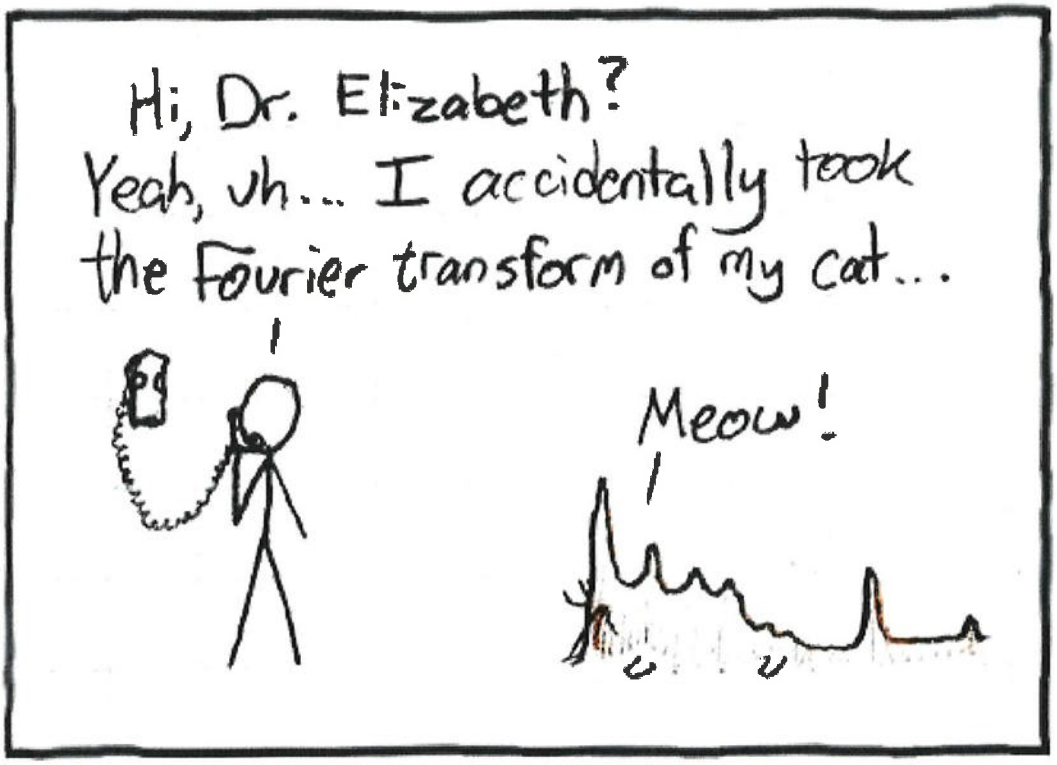
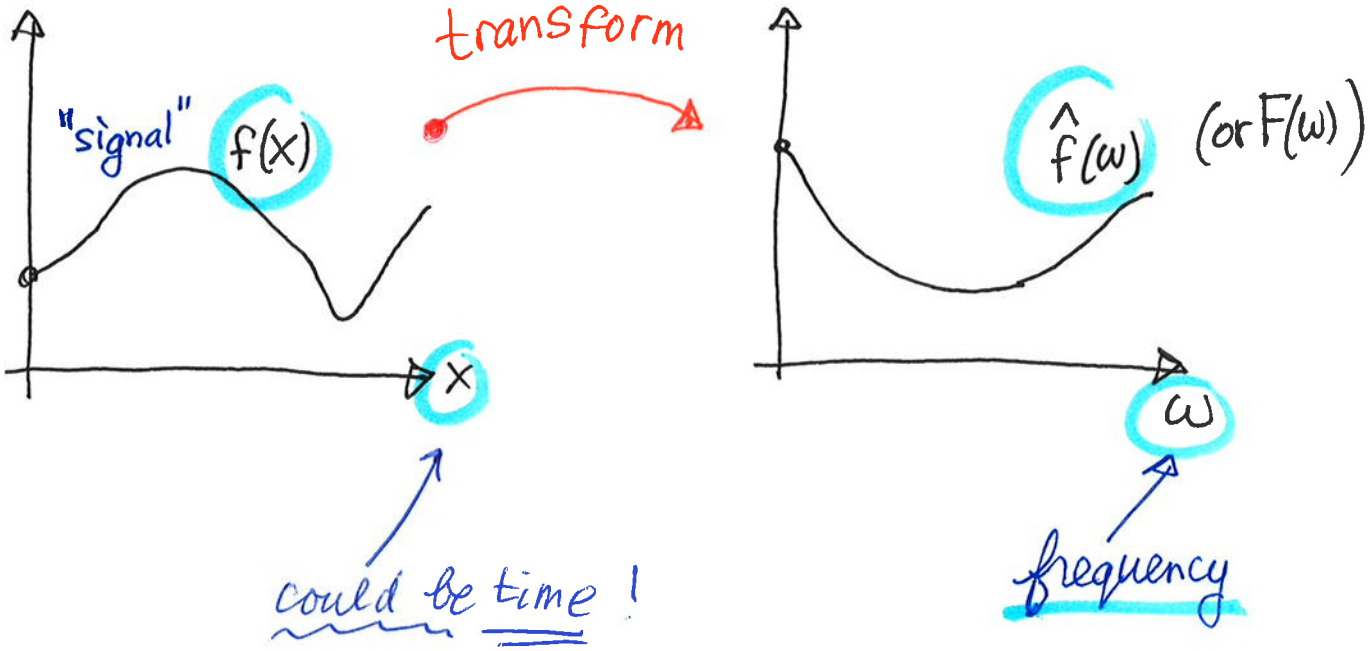
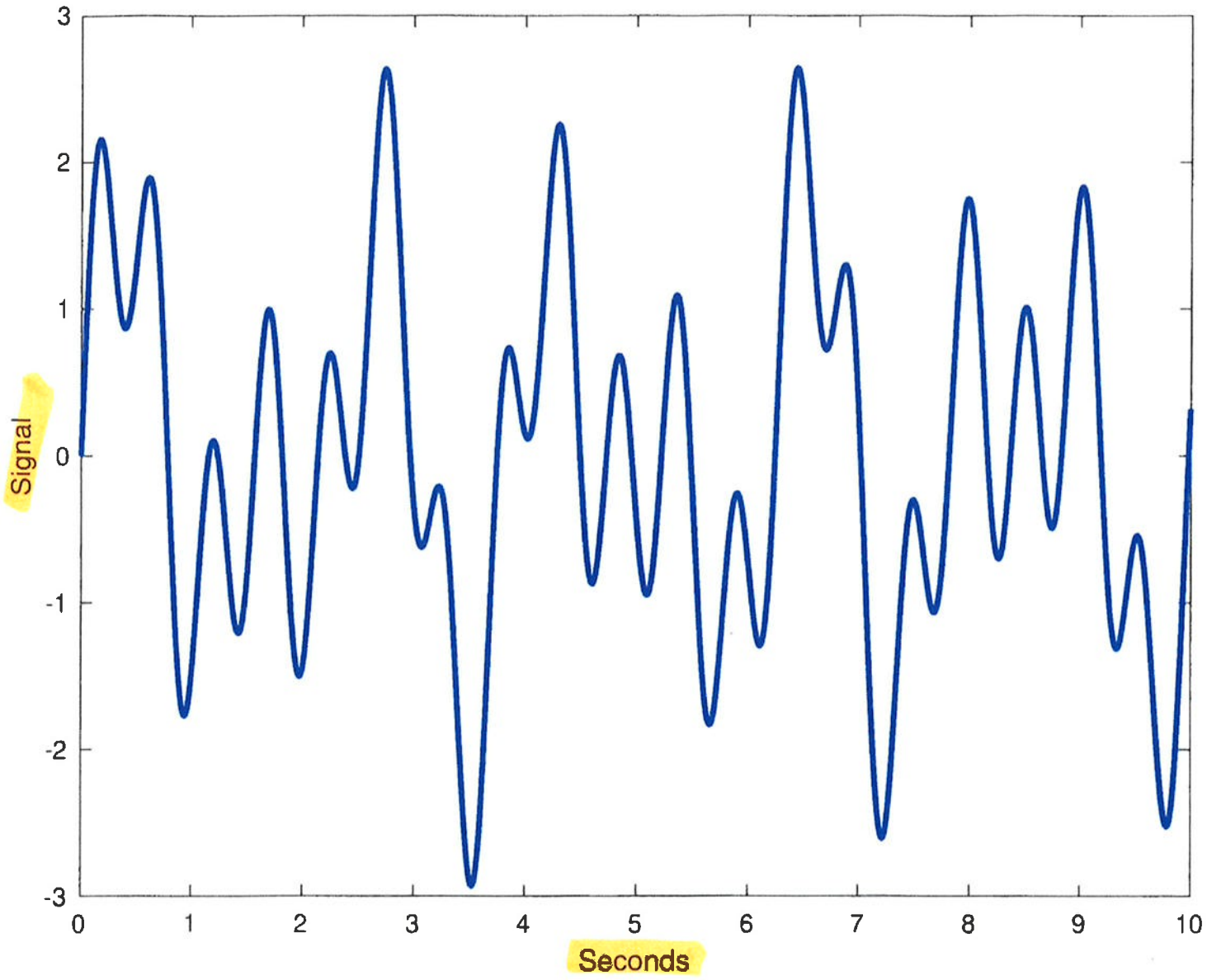
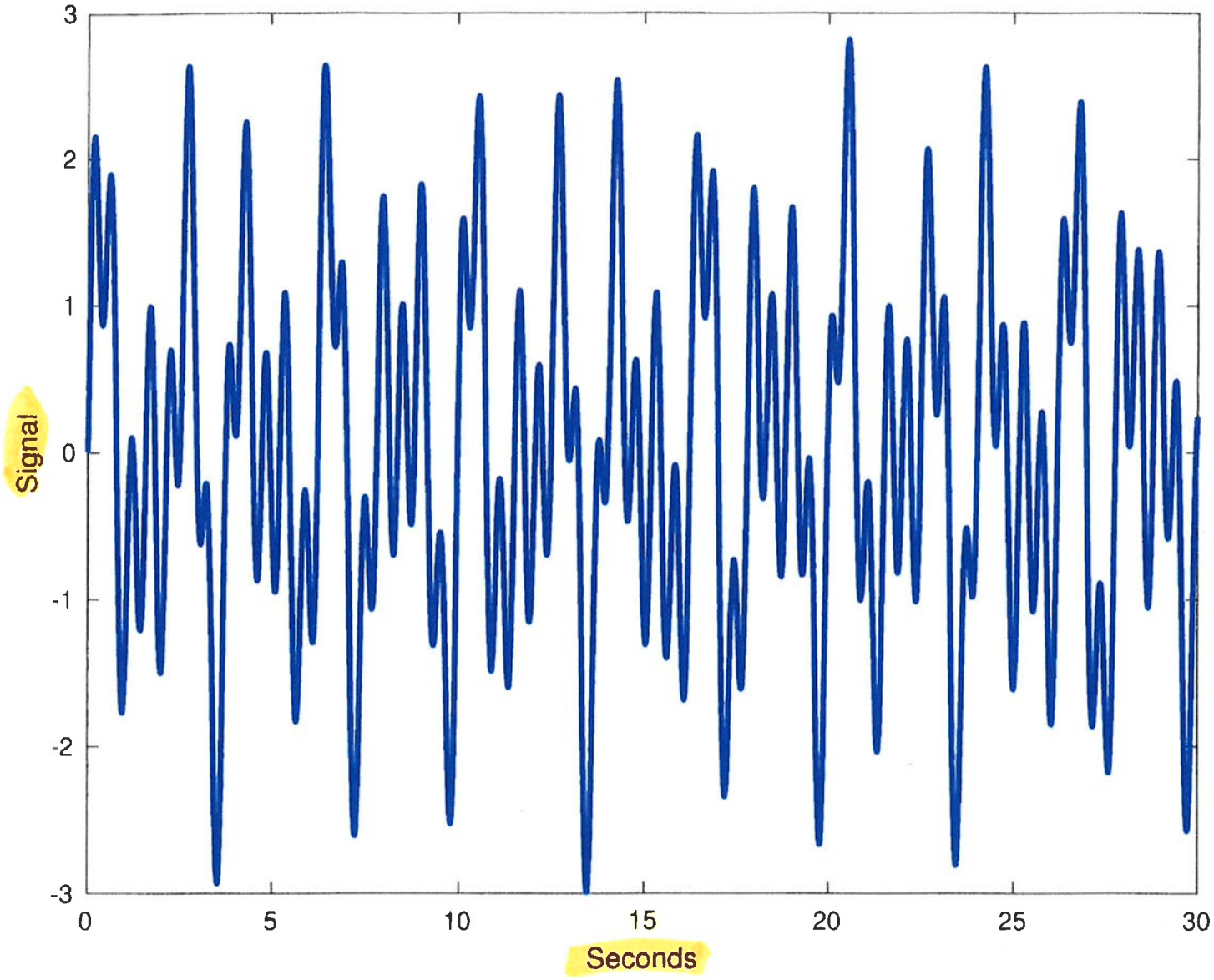

Hi, Dr. Elizabeth?
Yeah, uh... I accidentally took the Fourier transform of my cat...
Meow!


transform
"signal"
f(x)
x
could be time!
$\hat{f}(\omega)$
(or $F(\omega)$)
ω
frequency



periodic signal ?



periodic signal ?

(how can we analyze this ???)

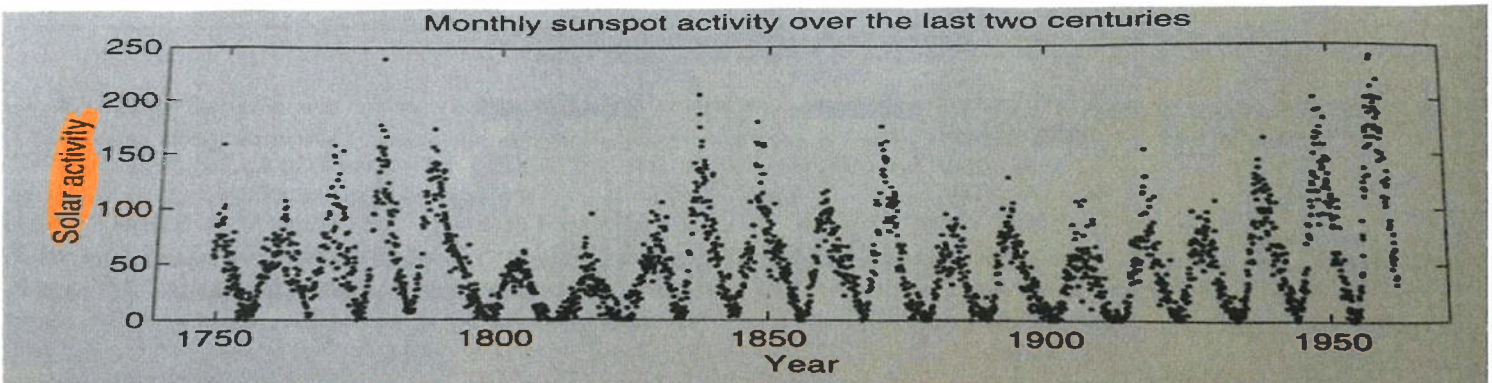


Figure 17.8 *Sunspot activity over the last two centuries. The points are monthly measurements.*

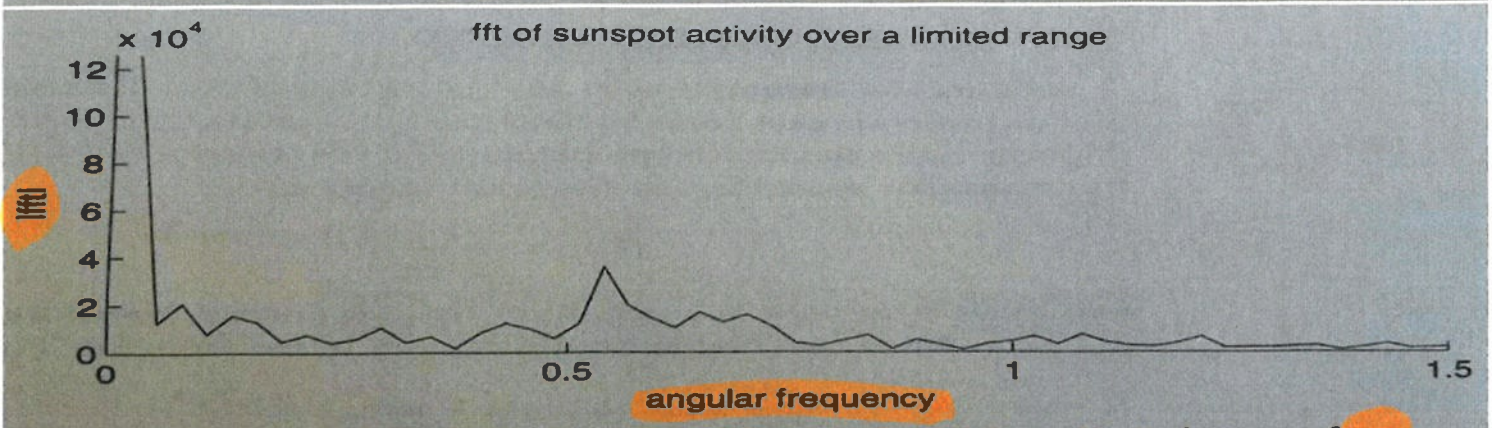


Figure 17.9 *The FFT of the sunspot activity is plotted for a limited range of* ω_n.

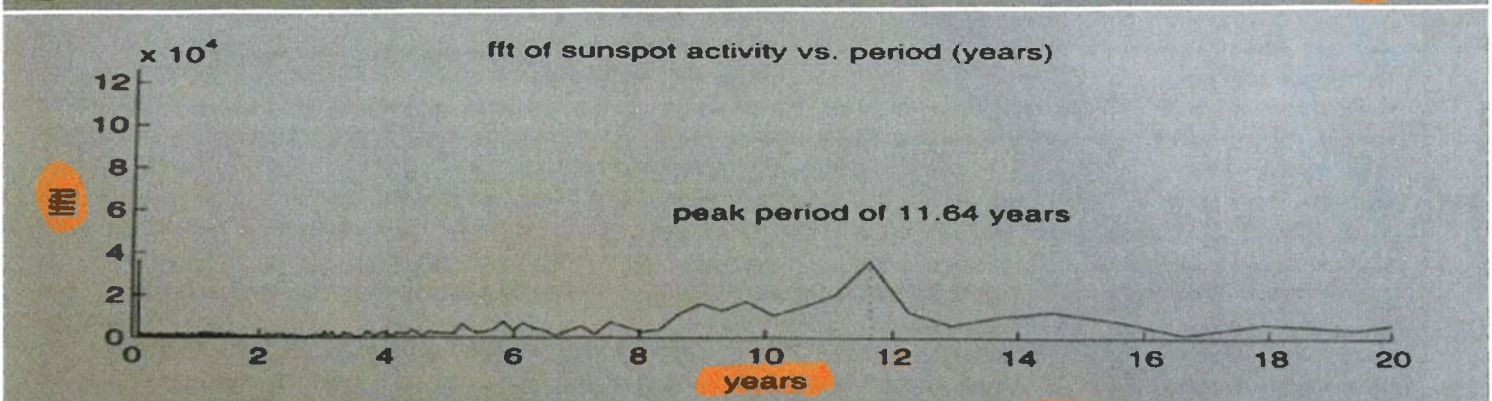
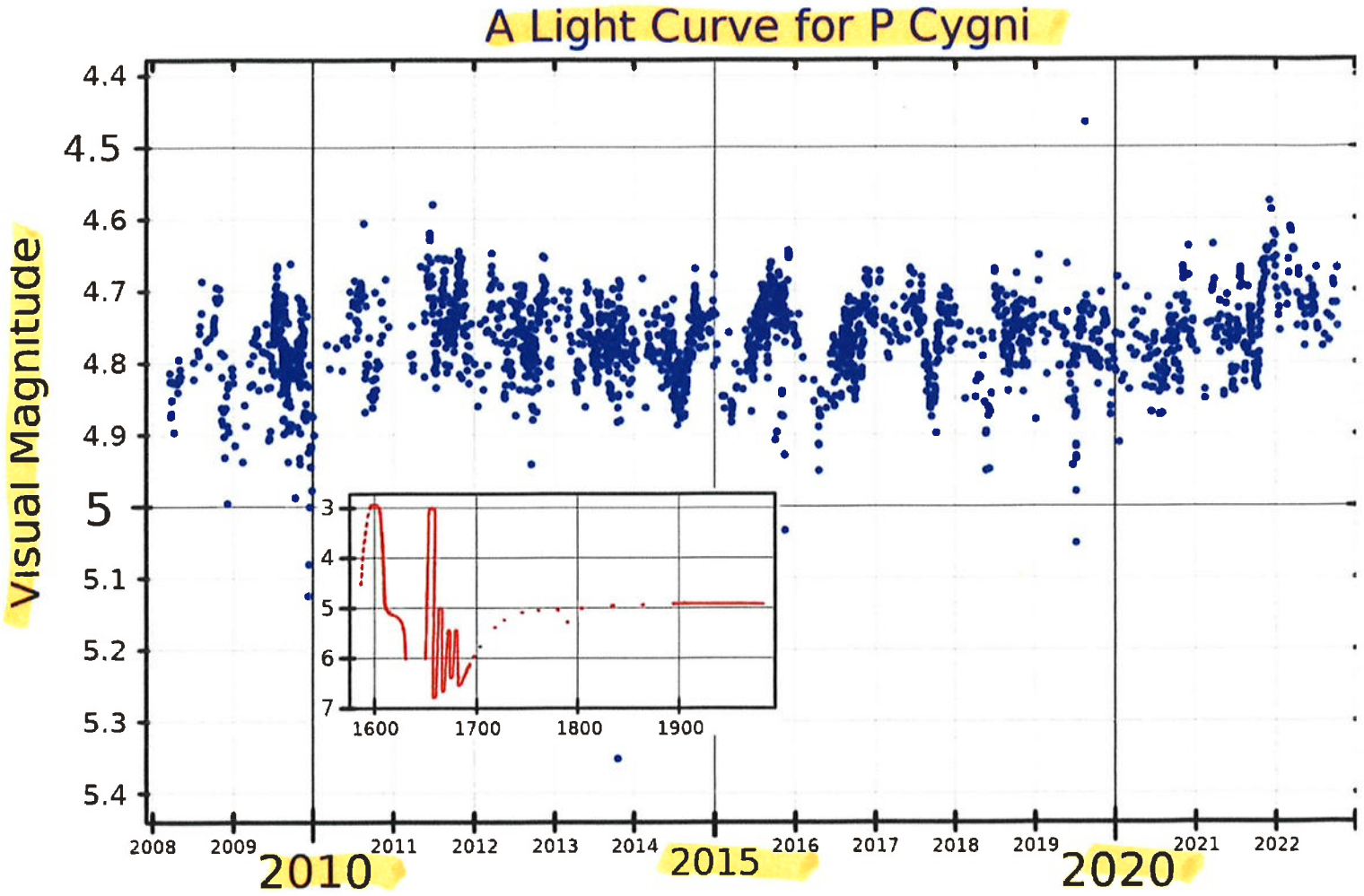
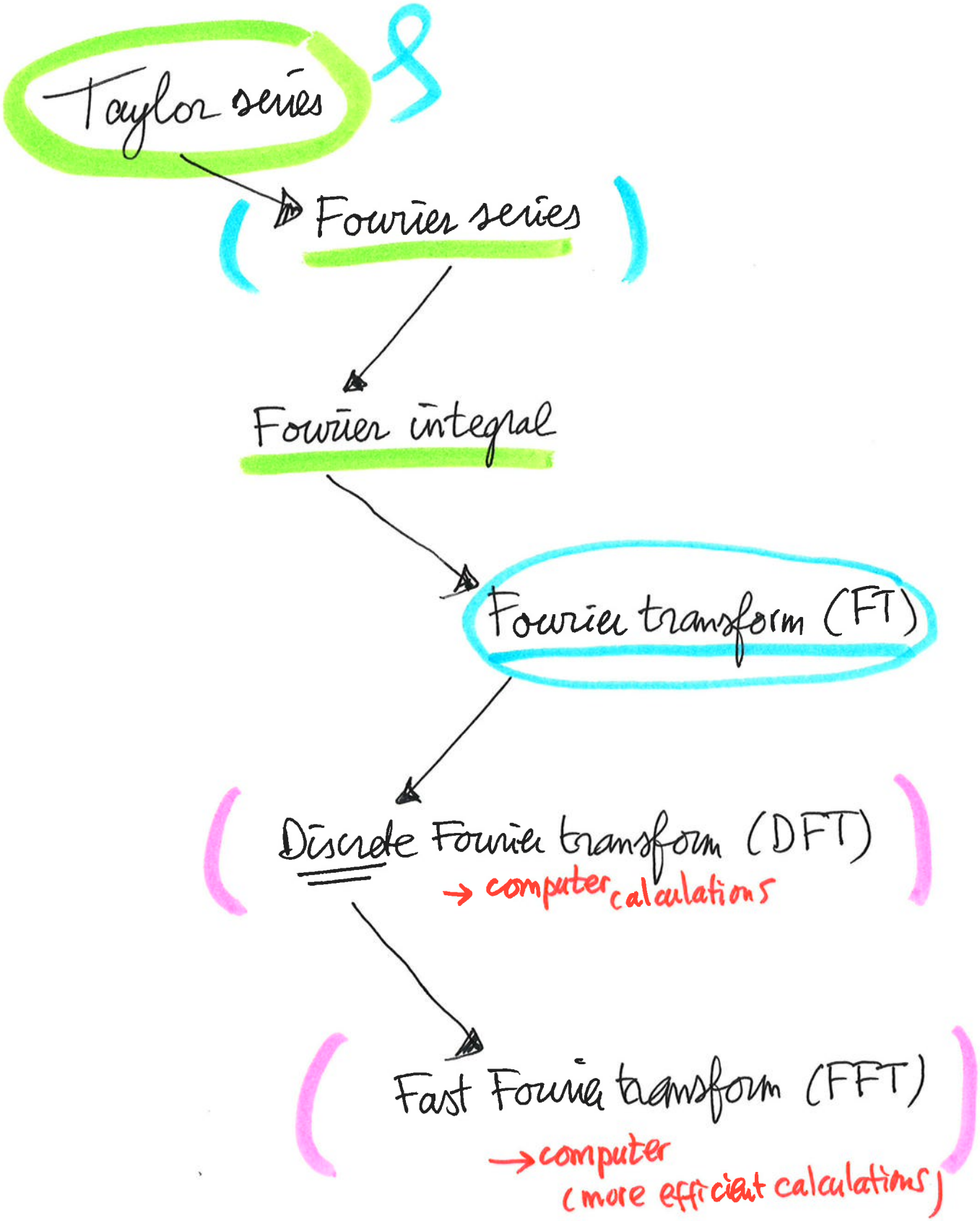


Figure 17.10 *The FFT of the sunspot activity plotted versus* $T_n = 2\pi/\omega_n$. *Only a portion of the complete range of* T *is shown.*


A Light Curve for P Cygni
Visual Magnitude
4.4
4.5
4.6
4.7
4.8
4.9
5
5.1
5.2
5.3
5.4
2008
2009
2010
2011
2012
2013
2014
2015
2016
2017
2018
2019
2020
2021
2022
3
4
5
6
7
1600
1700
1800
1900


Taylor series
Fourier series
Fourier integral
Fourier transform (FT)
Discrete Fourier transform (DFT)
→ computer calculations
Fast Fourier transform (FFT)
→ computer
(more efficient calculations)

Taylor series of a function $f(x)$

about the point 0 : $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$$= \sum_{n=0}^{\infty} a_n x^n$$

just another notation

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f^{(n)}$: n-th derivative

$n!$: factorial

approximation

$$\approx \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \underbrace{\frac{f^{(0)}(0)}{0!}}_{= f^{(0)}(0) = \cdots} x^0 + \underbrace{\frac{f^{(1)}(0)}{1!}}_{= f^{(1)}(0)} x^1 + \underbrace{\frac{f^{(2)}(0)}{2!}}_{= \frac{f^{(2)}(0) x^2}{2}} x^2 + \cdots + \frac{f^{(N)}(0)}{N!} x^N$$

(depends on $f(x)$)

example:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

about the point a :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

DISADVANTAGE:

This is a local approximation of $f(x)$ at the point a.

(and needs many, ∞-many?, derivatives of $f(x)$ .....)

Fourier series of a function $f(x)$

x can play the role of time t !

(This will give a global approximation of a periodic $f(x)$)

periodic $f(x)$ with (fundamental) period $T = \frac{2\pi}{\omega}$ ← frequency

why?

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega x) + b_n \sin(n\omega x) \right)$$

what happened with b_0 ?

with:

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(x)\,dx$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x).\cos(n\omega x)\,dx$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x).\sin(n\omega x)\,dx$$

period of signal $f(x)$ is T

(*)

example: period $T = 2\pi$ ($\Rightarrow \omega = 1$)

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx,\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

making use of Euler's identity: $e^{i\theta} = \cos(\theta) + i.\sin(\theta)$

we can combine (*) into a compact exponential formula:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inwx} \quad \text{with } c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-inwx}\,dx$$

$i = \sqrt{-1}$

From periodic to non-periodic functions:

The Fourier integral

Not all signals are periodic, what can we do then?

If $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$, then we can write:

absolute signs — finite value — area below $|f(x)|$

$$f(x) = \int_0^{\infty} [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\,d\omega$$

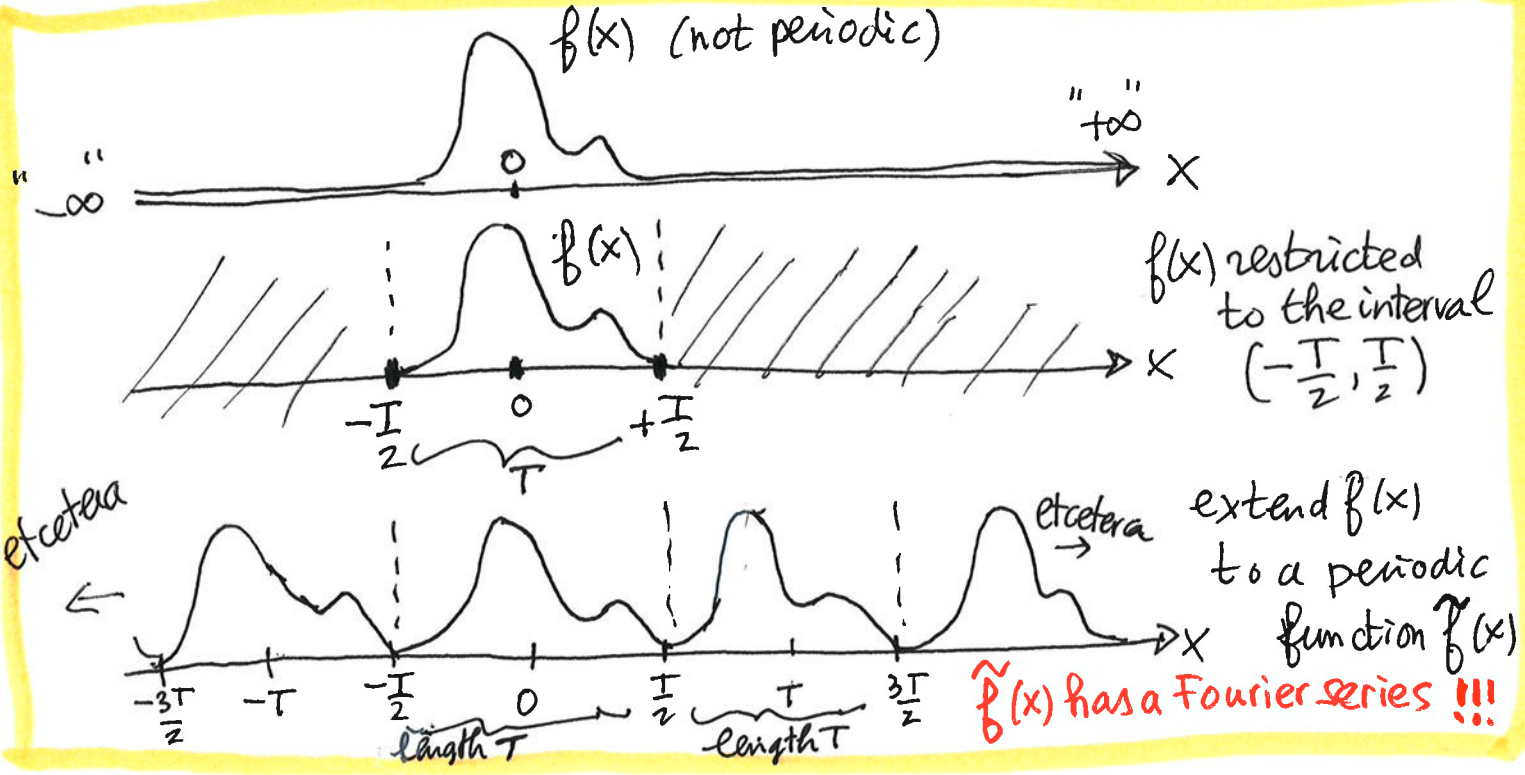
for all x: in $-\infty < x < \infty$

with

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx$$

} for all $\omega \geq 0$.



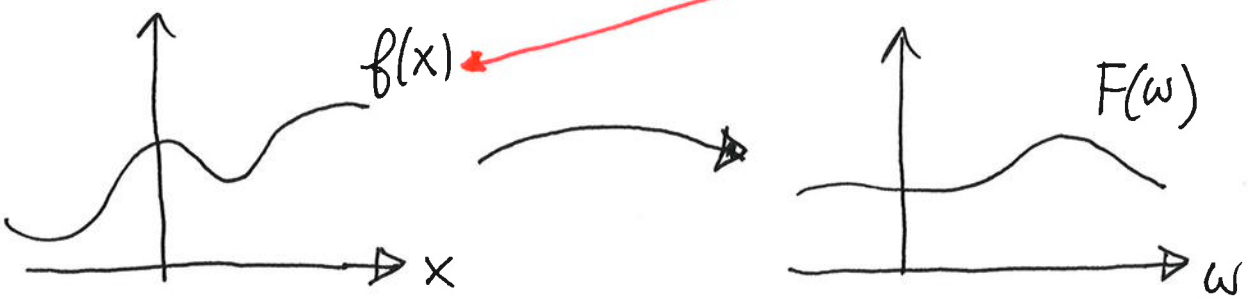
(continuous) Fourier transform (FT)

$i = \sqrt{-1}$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx$$

minus!

original signal



(examples follow later)

how to return to your original signal? (after analyzing the signal in the frequency domain ω)

via the inverse Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} F(\omega)\,d\omega$$

plus!

Notation: $F(\omega) = \mathcal{F} f \Rightarrow \mathcal{F}^{-1}\mathcal{F} f = f$

"nothing" happens after applying first $\mathcal{F}$ and then $\mathcal{F}^{-1}$

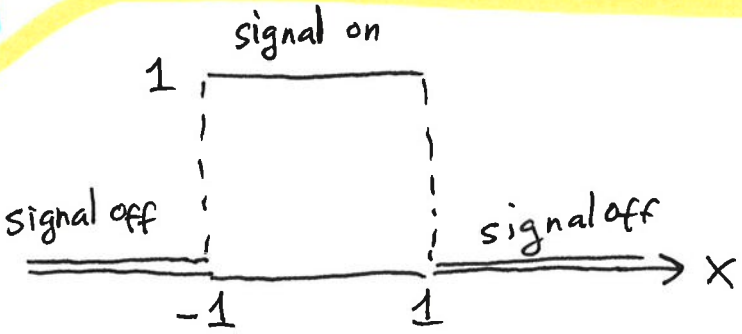
properties: 1) $\mathcal{F}(af + bg) = a \cdot \mathcal{F}(f) + b \cdot \mathcal{F}(g)$ "linear operation"

2) $\mathcal{F}(f') = i\omega \cdot \mathcal{F}(f)$ "relation between f and f'"

3) $\mathcal{F}(e^{\frac{-x^2}{2}}) = e^{\frac{-\omega^2}{2}}$ "signal remains the same in ω"

Examples

Signal 1



$$f(x)=\begin{cases}1, & -1<x<1\\ 0, & \text{elsewhere}\end{cases}$$

Fourier-integral:

we need to calculate $A(\omega)$ and $B(\omega)$

we need to check:

$$\int_{-\infty}^{\infty}|f(x)|dx=\int_{-1}^{1}1\,dx = x\Big|_{x=-1}^{x=1} = 1-(-1)=2 < \infty$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\cos(\omega x)dx$$

split the integral →

$$=\frac{1}{\pi}\left\{\int_{-\infty}^{-1}f(x)\cos(\omega x)dx+\int_{-1}^{1}f(x)\cos(\omega x)dx+\int_{1}^{\infty}f(x)\cos(\omega x)dx\right\}$$

$\int_{-\infty}^{-1}$: $f(x)=0$; $\int_{-1}^{1}$: $f(x)=1$; $\int_{1}^{\infty}$: $f(x)=0$

$$=\frac{1}{\pi}\int_{-1}^{1}\cos(\omega x)dx$$

$$=\frac{1}{\pi\omega}\sin(\omega x)\Big|_{x=-1}^{x=1}=\frac{\sin(\omega)}{\pi\omega}-\frac{\sin(-\omega)}{\pi\omega}=\frac{\sin(\omega)}{\pi\omega}+\frac{\sin(\omega)}{\pi\omega}$$

$(-\sin)$

$$=\frac{2}{\pi\omega}\sin(\omega)$$

and:

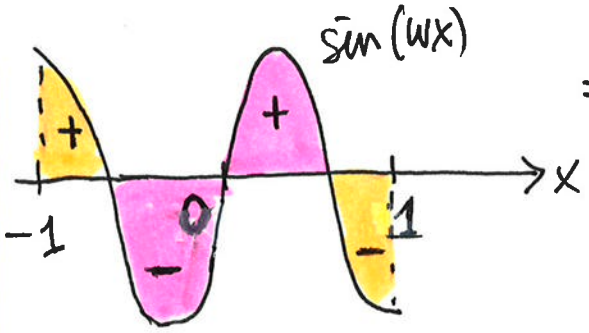
$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\sin(\omega x)dx$$

$$=\frac{1}{\pi}\left\{\int_{-\infty}^{-1}f(x)\sin(\omega x)dx+\int_{-1}^{1}f(x)\sin(\omega x)dx+\int_{1}^{\infty}f(x)\sin(\omega x)dx\right\}$$

$f(x)=0$; $f(x)=1$; $f(x)=0$

$$=\frac{1}{\pi}\int_{-1}^{1}\sin(\omega x)dx=\;?$$

$\sin$ is odd → $= 0$



$$\Rightarrow \int_{-1}^{1} \sin(wx)\,dx = 0 \Rightarrow B(w)=0$$

$\|$ or (calculate)

$$-\frac{1}{w}\cos(wx)\Big|_{x=-1}^{x=1} = -\frac{1}{w}\left[\cos(w)-\cos(-w)\right]$$

minus

$$= -\frac{1}{w}\left[\cos(w)-\cos(w)\right]$$

$$= -\frac{1}{w} * 0 = 0$$

Note: $$f(x) = \int_0^{\infty}\left[\underbrace{\frac{2}{\pi}\sin(w)\cos(wx)}_{A(w)} + \underbrace{0}_{B(w)} * \sin(wx)\right]dw$$

for all $x \in (-\infty, \infty)$

special value of x: $x = 0$

$$\Rightarrow f(0) = \int_0^{\infty} \frac{2}{\pi w}\sin(w)\underbrace{\cos(w*0)}_{=\cos(0)=1}\,dw$$

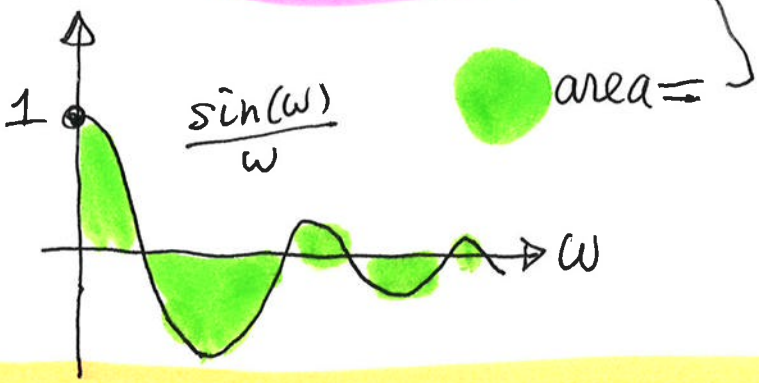
see signal! $\rightarrow \| \rightarrow 1$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin(w)}{w}\,dw$$

multiply both sides by $\frac{\pi}{2}$ $\Rightarrow$

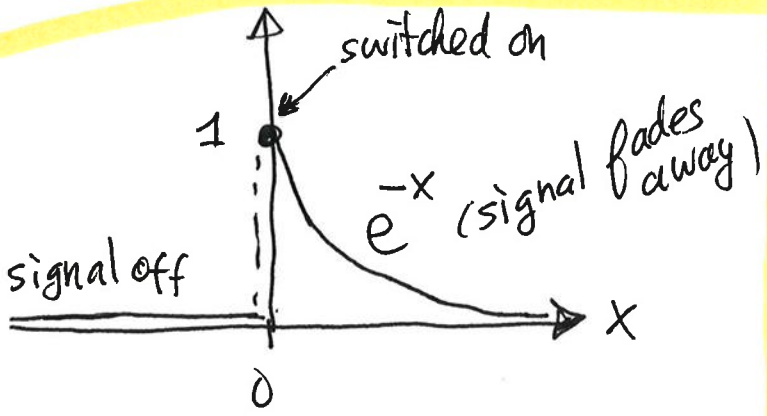
$$\int_0^{\infty}\frac{\sin(w)}{w}\,dw = \frac{\pi}{2}$$

name: "Dirichlet-integral"



!

Signal 2



Fourier-transform:

$$f(x)=\begin{cases} e^{-x}, & x\geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-i\omega x} f(x)\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-i\omega x} f(x)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-i\omega x}\cdot 0\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-i\omega x - x}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x(1+i\omega)}\,dx = \frac{1}{\sqrt{2\pi}}\cdot\frac{-1}{1+i\omega}\, e^{-i\omega x}\cdot e^{-x}\Big|_{x=0}^{x=\infty}$$

$$=\frac{-1}{\sqrt{2\pi}(1+i\omega)}\left[\underbrace{e^{-\infty}}_{=0} - \underbrace{e^{0}}_{=1}\right]=\frac{1}{\sqrt{2\pi}(1+i\omega)}$$

also denoted by $\hat{f}(\omega)$

we need to check:

$$\int_{-\infty}^{\infty}|f(x)|\,dx$$

$$=\int_{-\infty}^{0}\underbrace{|f(x)|}_{=0}dx+\int_{0}^{\infty}\underbrace{|f(x)|}_{=e^{-x}}dx$$

$$=0+\int_{0}^{\infty}e^{-x}dx$$

$$=\int_{0}^{\infty}e^{-x}dx$$

$$=-e^{-x}\Big|_{x=0}^{x=\infty}$$

$$=-\underbrace{e^{-\infty}}_{=0}-(-\underbrace{e^{0}}_{=1})$$

$$=1<\infty$$

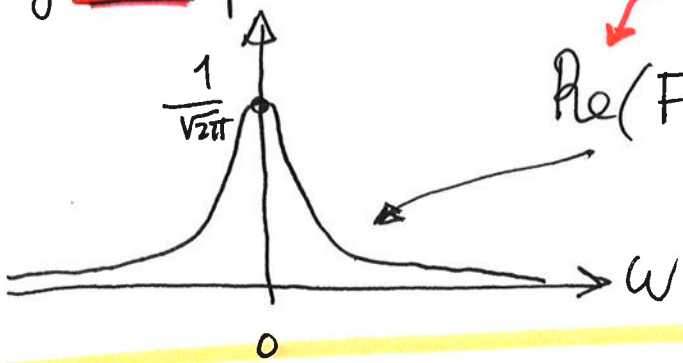
✓

$$\frac{1}{\sqrt{2\pi}(1+i\omega)} * \frac{1-i\omega}{1-i\omega} = \frac{1-i\omega}{\sqrt{2\pi}(1\underbrace{-i\omega+i\omega}_{=0}\underbrace{-i^2\omega^2}_{=+1})}$$

$$= \frac{1-i\omega}{\sqrt{2\pi}(1+\omega^2)} = \frac{1}{\sqrt{2\pi}(1+\omega^2)} - \frac{\omega}{\sqrt{2\pi}(1+\omega^2)}\,i$$

real part — imaginary part

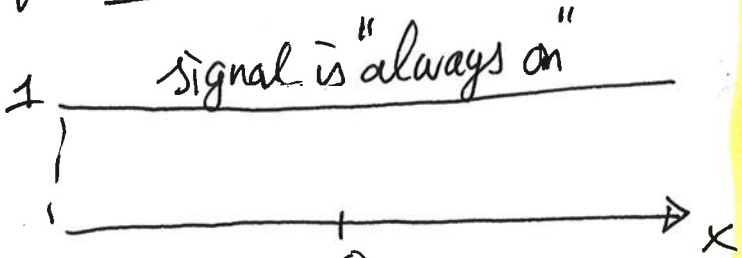
plot of real part:



$$\mathrm{Re}(F(\omega)) = \frac{1}{\sqrt{2\pi}(1+\omega^2)}$$

Signal 3

$f(x) = 1$ for all x



Note that: $\int_{-\infty}^{\infty} |f(x)|\,dx = \int_{-\infty}^{\infty} 1\,dx = x\Big|_{x=-\infty}^{x=\infty} = "\infty - (-\infty)" = \infty$

Can we calculate the Fourier-transform of this signal ???

(we are not sure whether this is the case -- maybe yes, maybe not ---)

Let's try to calculate:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1 \cdot e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{y\to\infty} \int_{-y}^{y} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{y\to\infty} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{x=-y}^{x=y}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{y\to\infty} \left[\frac{e^{-i\omega y} - e^{+i\omega y}}{-i\omega} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{y\to\infty} \left[\frac{2}{\omega} \frac{e^{i\omega y} - e^{-i\omega y}}{2i} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{\omega} \lim_{y\to\infty} \frac{\sin(y\omega)}{\omega}$$

$e^{iu} = \cos(u) + i\sin(u)$

$e^{-iu} = \cos(-u) + i\sin(-u)$

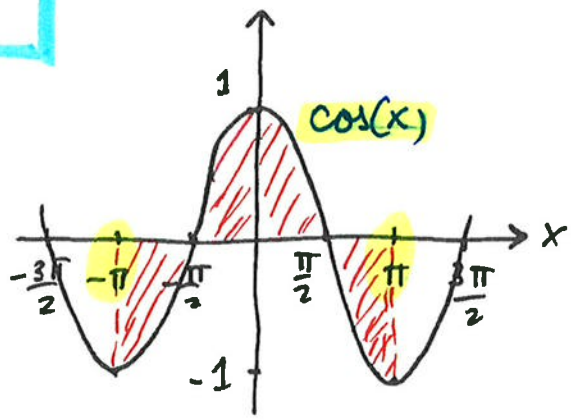
$= \cos(u) - i\sin(u)$

this limit does not exist

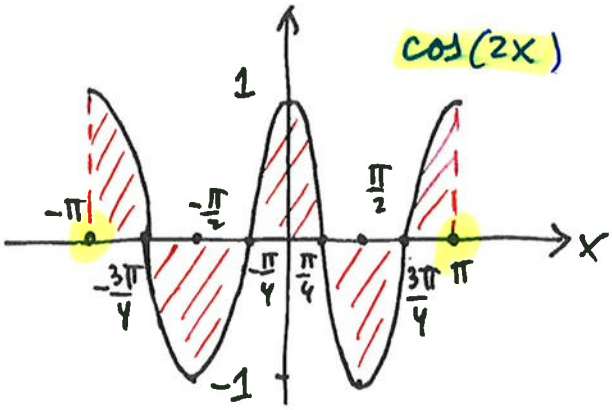
⇒ no Fourier transform of this (strange) signal exists!!

useful ∫'s

$$\int_{-\pi}^{\pi} \cos(x)\,dx = 0$$



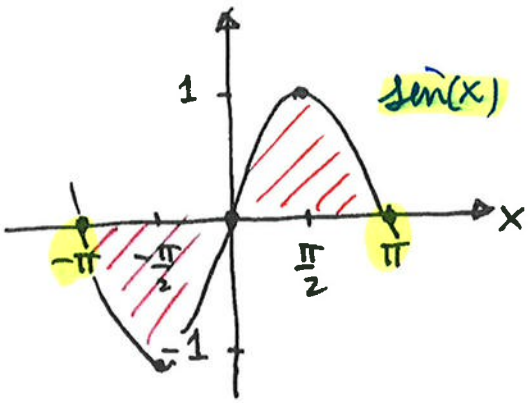
$$\int_{-\pi}^{\pi} \cos(2x)\,dx = 0$$



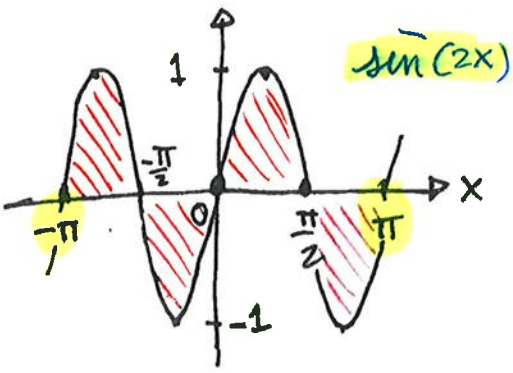
etcetera ---

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0 \qquad n = 1, 2, 3, \ldots$$

$$\int_{-\pi}^{\pi} \sin(x)\,dx = 0$$



$$\int_{-\pi}^{\pi} \sin(2x)\,dx = 0$$

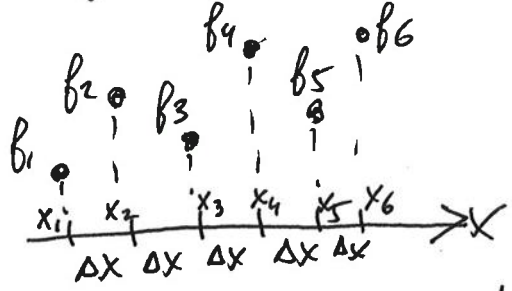


etcetera --

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = 0 \qquad n = 1, 2, 3, \ldots$$

Discrete Fourier transform (DFT)

The continuous signal $f(x)$ is sampled at uniform "time"-intervals Δx



$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} f(x)\,dx \approx \frac{1}{\sqrt{2\pi}} \int_{\text{start value}}^{\text{end value}} e^{-iwx} f(x)\,dx$$

replace area below the graph of $e^{-iwx} f(x)$ by a finite number of rectangles with width Δx $\Rightarrow F(w) \approx \sum_{n=0}^{N} f(x_n) \cdot e^{-iwx_n} \cdot \Delta x$

(it calculates each frequency w separately: SLOW calculation!)

Fast Fourier transform (FFT)

applies a smart trick: it splits the measured values $f_1, f_2, \ldots$ into odd and even numbers, re-uses the double calculations and becomes very fast!