OEFEN DEELTOETS A: Mathematics Part (SK-B2WDMM)

1. Do not forget to write your name, initials and student number!
2. DE := Differential Equation.
3. $\nabla\cdot$ = div (the divergence) and ∇ = grad (the gradient).
4. $y'(t) = \frac{dy}{dt}(t)$.
5. Max.pts = 4.5 = W1 (1) + W2 (1) + W3 (1) + W4 (1) + W5 (0.5).
6. You may use both sides of the answer sheet for each question.
7. Do not only give the answer to the questions, but also show how you arrived at the answer!

SUCCES!!!

Question W1 on lecture 1: *(max. 1 pt)*

The following functions are given:

1. The temperature $T(x, y, z) = 2x^2 + y^3 + 3z^2 + \cos(xy) + 2xz$.

2. The velocity field $\vec{v}(x, y, z) = [y\mathrm{e}^{xz}, x^2\mathrm{e}^{y}, z^2\mathrm{e}^{y+z}]^T$.

3. The density $\rho(x, y) = y^2 + 2xy + \cos(x)$ with $x(u, v) = u - v^2$ and $y(u, v) = u^2 + 2v$.

(a) Compute the gradient (∇) of T.

(b) Compute the divergence ($\nabla\cdot$) of $\vec{v}$.

(c) Making use of the chain rule, calculate: $\frac{\partial \rho}{\partial v}$. Note that in the final formula only dependence on u and v must be displayed!

Your answers/solutions (explain!):

Answer to question W1 continued:

Question W2 on lecture 2: *(max. 1 pt)*

The energy function E is given as follows: $E(x, y) = y^3 - 3y + x^2y^3 + 2024$.

(a) Find the 2D Taylor expansion of E up to the third derivative term (this term not included) about the point $(0, 0)$, and also about the point $(0, -1)$.

(b) Show that the two points $(0, -1)$ and $(0, 1)$ are stationary points of E.

(c) Calculate the corresponding Hessian matrix and the Laplacian of E.
⊢ Conclude at which of the two points from (b) a minimum energy is found.
⊢ What is the value of the energy minimum?

(d) Describe in a few lines (with formulas) how the iterative methods Newton-Raphson and steepest descent approximate the minimum energy E.

Your answers/solutions (explain!):

Answer to question W2 continued:

Question W3 on lecture 3: *(max. 1 pt)*

Consider the Hamiltonian (energy) $H = \frac{1}{2}x^2 + \frac{1}{2}(2\pi)^2 y^2$.

(a) Calculate the partial derivatives: $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y}$.

(b) Consider also the DE: $y''(t) + 4\pi^2 y(t) = 0$ with initial conditions: $y(0) = 0$, $y'(0) = 3$.
⊢ Re-write this second-order DE as a system of two first order ODEs with $x = y'$.
⊢ Also give the matrix A of the linear system.
⊢ Where and how do the partial derivatives of part (a) appear in the system?

(c) Show that the energy H is constant in time for this model. Which value does H have?

(d) Describe both the numerical method Euler-Forward and Störmer-Verlet for the model in this question **W3**. Use as the time step size: Δt.
⊢ For solving this DE, what is the main advantage of Störmer-Verlet compared to Euler-Forward?

Your answers/solutions (explain):

Answer to question W3 continued:

Question W4 on lecture 4: *(max. 1 pts)*

Consider the homogeneous DE:

$$y''(t) - 4y'(t) + 13y(t) = 0.$$

(a) Determine the general solution $y(t)$ of this DE. Name the two constants C_1 and C_2.

(b) Given are the initial conditions (ICs) $y(0) = 2$ and $y'(0) = 10$. What is the solution of the DE with these extra conditions?

(c) Re-write the second-order DE with the ICs from (b) to a system of first-order DEs. Give the matrix A in this system.

Your answers/solutions (explain!):

Answer to question W4 continued:

Question W5: *(max. 0.5 pts)*

Choose A, B, C, D, **or** E in the following two questions.
($\Rightarrow$ only one choice is allowed!)

W5 i) Consider the scalar energy field $f(x, y, z) = x^3y^2z + e^{yz}$ defined on $\mathbb{R}^3$. Let $\mathbf{F} = \nabla f$ denote the associated gradient vector field. Evaluate the divergence of this gradient field, $\nabla \cdot \mathbf{F}$ (the Laplacian $\nabla^2 f$), specifically at the coordinate point $(1, 0, 2)$.

A) 4

B) 6

C) 8

D) 10

E) None of the above answers.

Your answers/solutions (explain!):

Answer to question W5 i) continued:

W5 ii) A concentration field inside a chemical reactor is modeled by the scalar function $C(x, y) = \ln(x^2 + y^2 + 4)$. A researcher stands at position $(2, 1)$ and wishes to measure the rate of concentration change. Calculate the directional derivative of $C(x, y)$ at the point $(2, 1)$ in the direction pointing towards the point $(0, 0)$.

A) $-\frac{\sqrt{5}}{9}$

B) $-2\frac{\sqrt{5}}{9}$

C) $+\frac{\sqrt{5}}{9}$

D) $+2\frac{\sqrt{5}}{9}$

E) None of the above answers.

Your answers/solutions (explain!):

Answer to question W5 ii) continued: