Exercise N1

Consider the following functions:

- $f(x) = \exp(x) + 1$
- $f(x) = \exp(-x) - x$
- $f(x) = x^2 - 4\sin(x)$
- $f(x) = x^3 + 6x^2 + 11x - 6$
- $f(x) = \sin(x)$.

Sketch the graphs of these functions and indicate the locations of their zeros, local maxima, and local minima.

Exercise N2

Apply the Newton–Raphson (NR) method to solve

$$f(x) = x^3 + 3x - 4 = 0.$$

Use the initial guess $x_0 = 2$, and also plot the graph of

$$y = f(x).$$

After how many iterations is the absolute error smaller than 10^{-7}?

Exercise N3

Consider the function

$$f(x) = x^2 - 2x + c$$

for the two cases $c = 1$ and $c = 0$.
Determine the zeros α of the function graphically.
Apply the Newton–Raphson method for both values of c and compare the convergence behavior.
Do you observe a significant difference?
For example, determine the quantities

$$\frac{x_{i+1} - \alpha}{x_i - \alpha}$$

and

$$\frac{x_{i+1} - \alpha}{(x_i - \alpha)^2}$$

once the iterates are sufficiently close to the root.
What do these quantities indicate about the order of convergence?

Exercise N4

We now investigate the difference between the two experiments in Exercise N3. Suppose we seek a zero α of a function f, so that

$$f(\alpha) = 0.$$

Assume furthermore that

$$f'(\alpha) = 0.$$

(For which of the two cases in Exercise N3 does this occur?)
For a general fixed-point iteration (not necessarily Newton–Raphson), we know that

$$x_{i+1} - \alpha \approx (x_i - \alpha)\, g'(\alpha).$$

Show that for the Newton–Raphson iteration considered here,

$$\lim_{x \to \alpha} g'(x) = \frac{1}{2},$$

and therefore that $g'(\alpha) \neq 0$, in contrast to the usual Newton–Raphson case where $f'(\alpha) \neq 0$.
Does this result agree with the second experiment in Exercise N3 (the case $c = 0$)?

Exercise N5

Plot the graph of

$$f(x) = \cos(x)$$

on the interval

$$x \in [-2\pi,\, 2\pi].$$

How many zeros does f have on this interval?
Using the geometric interpretation of the Newton–Raphson method in terms of tangent lines, discuss the behavior of the iteration for different initial guesses x_0.
For which initial values does the method converge to which zero?
For which initial values does the method fail to converge (divergence)?

Exercise N6

Consider the function

$$f(x) = (x - 1)^4 - 1.$$

Sketch the graph of f.

Write down the Newton–Raphson iteration formula for solving

$$f'(x) = 0.$$

By performing several iterations, determine an approximation of the location of the minimum of f.
How large is the error after each iteration?

Exercise N7

In a chemical reaction, the concentration of a certain ion at time t is given by

$$f(t) = 10\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-5t}.$$

Determine the time at which the concentration has decreased to one-half of its initial value.
Use the Newton–Raphson method to obtain an approximate solution.

Exercise N8

Verify that

$$\mathcal{H} = \mathcal{J}(\nabla f)$$

for a general function $f(x, y)$.

Exercise N9

Consider the nonlinear system

$$\begin{aligned} f_1(x, y) &= x^3 + y^2 - 2 = 0, \\ f_2(x, y) &= x^2 - y^3 - 1 = 0. \end{aligned}$$

Sketch the curves

$$y = \sqrt{2 - x^3}, \qquad y = -\sqrt{2 - x^3}, \qquad y = \sqrt[3]{x^2 - 1}.$$

Determine the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}$$

and its inverse, denoted by $\mathcal{J}^{-1}$.
Use the initial guess

$$(x_0, y_0) = (1, 1)$$

and apply the two-dimensional Newton–Raphson (NR) method to compute an approximation (α, β) of the solution of the system.
Perform <u>at most</u> five iterations.

Exercise N10

Derive the Newton–Raphson iteration formulas for the following system (in particular, compute the Jacobian matrix):

$$\begin{aligned} f(x,y,z) &= x^3 - y^2 + y - z^4 + z^2, \\ g(x,y,z) &= xy + yz + xz, \\ h(x,y,z) &= \frac{y}{xz}. \end{aligned}$$

Note. It is not required to carry out the Newton–Raphson iterations for this system.

Exercise N11

The energy surface E is given by

$$E = f(x,y) = x^2 + 3y^2.$$

Sketch several contour lines of this surface.
Compute the gradient ∇f and the Hessian matrix $\mathcal{H}$ of f.
Determine the stationary point of the energy surface; that is, the point (x,y) for which

$$\nabla f = \mathbf{0}.$$

Verify that E attains a minimum at this point.
Write down the Newton–Raphson iteration formula for the corresponding minimization problem.
Apply the Newton–Raphson method to approximate the location of the minimum numerically.
Since the exact location is known, investigate how accurately and how rapidly the method converges.