Lecture 2: Newton-Raphson

Solutions of the N-exercises

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Exercise N1

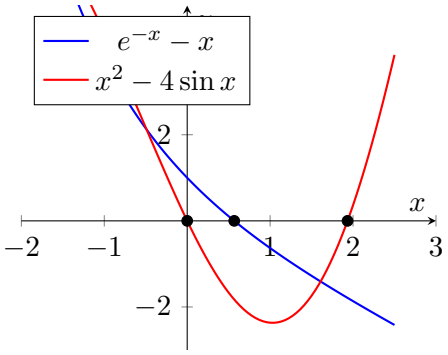
Problem: Consider the following functions:

1. $f(x) = e^x + 1$
2. $f(x) = e^{-x} - x$
3. $f(x) = x^2 - 4\sin(x)$
4. $f(x) = x^3 + 6x^2 + 11x - 6$
5. $f(x) = \sin(x)$

Sketch the graphs of these functions and indicate the locations of their zeros, local maxima, and local minima.

Solution:

1. $f(x) = e^x + 1$:
   - **Zeros:** None, since $e^x > 0 \implies f(x) > 1$ everywhere.
   - **Extrema:** Strictly increasing, no local extrema.
2. $f(x) = e^{-x} - x$:
   - **Zeros:** One zero at $x \approx 0.56714$ (the Omega constant).
   - **Extrema:** Strictly decreasing ($f'(x) = -e^{-x} - 1 < 0$), no local extrema.
3. $f(x) = x^2 - 4\sin(x)$:
   - **Zeros:** $x = 0$ and $x \approx 1.9337$.
   - **Extrema:** $f'(x) = 2x - 4\cos(x) = 0 \implies x \approx 1.895$ (local minimum).
4. $f(x) = x^3 + 6x^2 + 11x - 6$:
   - **Zeros:** $f'(x) = 3x^2 + 12x + 11 = 0 \implies x = -2 \pm \frac{\sqrt{3}}{3}$. Local max at $x \approx -2.577$, local min at $x \approx -1.423$.
   - One real zero at $x \approx 0.3788$.
5. $f(x) = \sin(x)$:
   - **Zeros:** $x = k\pi$ for $k \in \mathbb{Z}$.
   - **Maxima:** $x = \frac{\pi}{2} + 2k\pi$ with value 1.
   - **Minima:** $x = -\frac{\pi}{2} + 2k\pi$ with value -1.



Exercise N2

Problem: Apply the Newton-Raphson (NR) method to solve $f(x) = x^3 + 3x - 4 = 0$. Use the initial guess $x_0 = 2$. After how many iterations is the absolute error smaller than 10^{-7}?

Solution

The function and its first derivative are:

$$f(x) = x^3 + 3x - 4, \quad f'(x) = 3x^2 + 3$$

The Newton-Raphson iteration formula is:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 + 3x_i - 4}{3x_i^2 + 3}$$

Notice that $f(1) = 1^3 + 3(1) - 4 = 0$, so the exact root is $\alpha = 1$. The absolute error at iteration i is given by $|x_i - \alpha| = |x_i - 1|$.

Step-by-Step Iterations

1. **Iteration $i = 0$:**
$$x_0 = 2, \quad \text{Error: } |x_0 - 1| = 1.0000000000$$

2. **Iteration $i = 1$:**
$$x_1 = 2 - \frac{2^3 + 3(2) - 4}{3(2^2) + 3} = 2 - \frac{10}{15} = \frac{4}{3} \approx 1.3333333333$$
$$\text{Error: } |x_1 - 1| = \frac{1}{3} \approx 3.3333 \times 10^{-1}$$

3. **Iteration $i = 2$:**
$$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 + 3\left(\frac{4}{3}\right) - 4 = \frac{64}{27}, \quad f'\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right)^2 + 3 = \frac{25}{3}$$
$$x_2 = \frac{4}{3} - \frac{64/27}{25/3} = \frac{4}{3} - \frac{64}{225} = \frac{236}{225} \approx 1.0488888889$$
$$\text{Error: } |x_2 - 1| = \frac{11}{225} \approx 4.8889 \times 10^{-2}$$

4. **Iteration $i = 3$:**
$$x_3 = 1.0488888889 - \frac{(1.0488888889)^3 + 3(1.0488888889) - 4}{3(1.0488888889)^2 + 3} \approx 1.0022513470$$
$$\text{Error: } |x_3 - 1| \approx 2.2513 \times 10^{-3}$$

5. **Iteration $i = 4$:**

$$x_4 = 1.0022513470 - \frac{(1.0022513470)^3 + 3(1.0022513470) - 4}{3(1.0022513470)^2 + 3} \approx 1.0000050519$$

$$\text{Error: } |x_4 - 1| \approx 5.0519 \times 10^{-6} \quad (\text{Note: } 5.0519 \times 10^{-6} > 10^{-7})$$

6. **Iteration $i = 5$:**

$$x_5 = 1.0000050519 - \frac{(1.0000050519)^3 + 3(1.0000050519) - 4}{3(1.0000050519)^2 + 3} \approx 1.0000000000255$$

$$\text{Error: } |x_5 - 1| \approx 2.5516 \times 10^{-11} < 10^{-7}$$

Summary Table

i	x_i	Absolute Error $\lvert x_i - 1\rvert$
0	2.0000000000	1.00×10^{0}
1	1.3333333333	3.33×10^{-1}
2	1.0488888889	4.89×10^{-2}
3	1.0022513470	2.25×10^{-3}
4	1.0000050519	5.05×10^{-6}
5	1.0000000000	$2.55 \times 10^{-11} < 10^{-7}$

Table 1: Iteration sequence and absolute error reduction.

Thus, after **5 iterations**, the absolute error is smaller than 10^{-7}.

Exercise N5

Problem: Plot $f(x) = \cos(x)$ on $x \in [-2\pi, 2\pi]$. Discuss convergence/divergence of Newton-Raphson for various initial values x_0.

Solution: The NR formula for $f(x) = \cos x$ is:

$$x_{i+1} = x_i - \frac{\cos x_i}{-\sin x_i} = x_i + \cot(x_i)$$

- **Zeros on $[-2\pi, 2\pi]$:** There are 4 zeros at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$.
- **Convergence:** If x_0 is close to a zero (e.g. $x_0 \in (0, \pi)$), the iteration converges rapidly to $\frac{\pi}{2}$.
- **Divergence / Failure:** If $x_0 = k\pi$ (where $k \in \mathbb{Z}$), $f'(x_0) = -\sin(k\pi) = 0$, so the tangent line is horizontal and division by zero occurs.

Exercise N8

Problem: Verify that $\mathbf{H} = \mathcal{J}(\nabla f)$ for a general function $f(x, y)$.

Solution: Let $f(x, y)$ be a smooth scalar function. The gradient vector is:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

The Jacobian matrix of a vector field $\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ is defined as $\mathcal{J}(\mathbf{F}) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$.

Applying this definition to $\mathbf{F} = \nabla f$:

$$\mathcal{J}(\nabla f) = \begin{pmatrix} \frac{\partial}{\partial x}(f_x) & \frac{\partial}{\partial y}(f_x) \\ \frac{\partial}{\partial x}(f_y) & \frac{\partial}{\partial y}(f_y) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \mathbf{H}$$

which is precisely the Hessian matrix $\mathbf{H}$ of $f(x, y)$.

Exercise N9

Problem: Solve the system $f_1(x, y) = x^3 + y^2 - 2 = 0$, $f_2(x, y) = x^2 - y^3 - 1 = 0$ using 2D Newton-Raphson starting at $(x_0, y_0) = (1, 1)$.

Solution: The Jacobian matrix is:

$$\mathcal{J}(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 & 2y \\ 2x & -3y^2 \end{pmatrix}$$

The inverse Jacobian is:

$$\mathcal{J}^{-1} = \frac{1}{-9x^2y^2 - 4xy} \begin{pmatrix} -3y^2 & -2y \\ -2x & 3x^2 \end{pmatrix}$$

Evaluating at $(x_0, y_0) = (1, 1)$:

$$\mathbf{F}(1, 1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mathcal{J}(1, 1) = \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}$$

Solving $\mathcal{J}\Delta\mathbf{x} = -\mathbf{F}$:

$$\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \Delta x = \frac{2}{13}, \ \Delta y = -\frac{3}{13}$$

Thus, the first iteration gives:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2/13 \\ -3/13 \end{pmatrix} = \begin{pmatrix} 15/13 \\ 10/13 \end{pmatrix} \approx \begin{pmatrix} 1.1538 \\ 0.7692 \end{pmatrix}$$

Exercise N10

Problem: Derive the Jacobian matrix for the 3D system:

$$\begin{aligned} f(x, y, z) &= x^3 - y^2 + y - z^4 + z^2 \\ g(x, y, z) &= xy + yz + xz \\ h(x, y, z) &= \frac{y}{xz} \end{aligned}$$

Solution: The Jacobian matrix $\mathcal{J}$ is structured as:

$$\mathcal{J}(x,y,z) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}$$

Computing the partial derivatives:

$$\mathcal{J}(x,y,z) = \begin{pmatrix} 3x^2 & -2y+1 & -4z^3+2z \\ y+z & x+z & y+x \\ -\frac{y}{x^2 z} & \frac{1}{xz} & -\frac{y}{xz^2} \end{pmatrix}$$

Exercise N11

Problem: For the surface $E = f(x,y) = x^2 + 3y^2$, compute ∇f, Hessian $\mathbf{H}$, find stationary points, and write down the NR minimization scheme.

Solution:

1. **Gradient and Hessian:**
$$\nabla f = \begin{pmatrix} 2x \\ 6y \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

2. **Stationary Point:** Setting $\nabla f = \mathbf{0}$ gives $(x,y) = (0,0)$.

3. **Minimum Verification:** Since $\mathbf{H}$ is positive definite (eigenvalues $2 > 0$ and $6 > 0$), $(0,0)$ is a strict local minimum.

4. **Newton-Raphson Step:**
$$\mathbf{x}_{i+1} = \mathbf{x}_i - \mathbf{H}^{-1}\nabla f(\mathbf{x}_i) = \begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} 1/2 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} 2x_i \\ 6y_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   Since $f(x,y)$ is quadratic, Newton-Raphson converges to the exact minimum in **exactly 1 iteration** from any starting point.