Exercise R1: Visualizing the *Divergence*

• Sketch the following vector fields $\mathbf{v}$ in the plane (these represent velocity fields of a flow in a square centered at the origin).
• Recall that the divergence measures the net outflow minus the inflow.
• By inspecting the flow across the sides of the square, determine whether $\nabla \cdot \mathbf{v}$ is positive, negative, or zero.
• Compute the divergence of each vector field.

The velocity fields are given by:
a) $\mathbf{v} = \mathbf{i}$
b) $\mathbf{v} = x\mathbf{i}$
c) $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$
d) $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$
e) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$
f) $\mathbf{v} = (x^2 + y^2)^{-1}(-y\mathbf{i} + x\mathbf{j})$.

Exercise R2

Consider the function

$$f(x, y) = \sin(x) - \cos(y)$$

and the vector

$$\mathbf{a} = [4,\, 3]^T.$$

a) Determine the directional derivative of the function f at the point

$$\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$$

in the direction of the vector $\mathbf{a}$.

b) In which direction is the directional derivative of f maximal at the point

$$\left(\frac{\pi}{4}, \frac{\pi}{4}\right)?$$

What is the value of the directional derivative in that direction?

c) Determine the directional derivative of f in the direction of the vector $\mathbf{a}$ at a general point (a, b). What is the maximum possible directional derivative in the direction of $\mathbf{a}$, and for which points (a, b) is this maximum attained?

Exercise R3

Determine the directional derivative of

$$g(x, y, z) = \mathrm{e}^{xy+z}$$

at the point

$$(1, 1, -2)$$

in the direction of the vector

$$[1, -2, 2]^T.$$

Exercise R4

Determine the directional derivative of

$$g(x, y, z) = 2x^2 + 3y^2 + z^2$$

at the point

$$(2, 1, 3)$$

in the direction of the vector

$$\mathbf{a} = \mathbf{i} - 2\mathbf{k}.$$