Lecture 1: Partial derivatives (gradient, divergence and directional derivative)

Solutions of exercises R1-R4

October 9, 2026

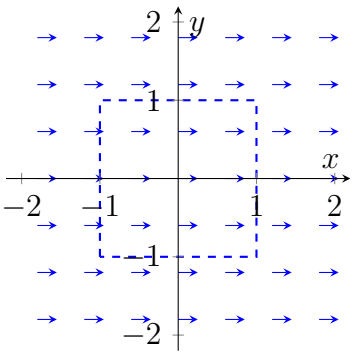
Exercise R1: Visualizing the Divergence

Recall that the divergence of a 2D vector field $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$ is defined as:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

The divergence measures the net rate of outward flux per unit volume at a point.

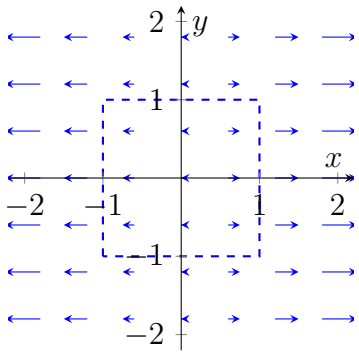
a) $\mathbf{v} = \mathbf{i}$



- **Flow Inspection:** The vector field is uniform and points purely in the positive x-direction with constant speed. The inflow across the left boundary of the square equals the outflow across the right boundary.
- **Sign of $\nabla \cdot \mathbf{v}$:** Zero (0).
- **Computation:**

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(0) = 0 + 0 = 0$$

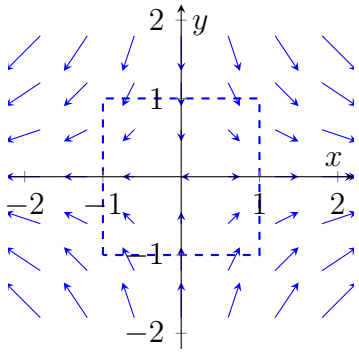
b) $\mathbf{v} = x\mathbf{i}$



- **Flow Inspection:** For $x > 0$, vectors point to the right and increase in length as x increases. For $x < 0$, vectors point to the left. Fluid exits the sides faster than it enters, leading to a net expansion.
- **Sign of $\nabla \cdot \mathbf{v}$:** Positive (> 0).
- **Computation:**

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) = 1 + 0 = 1$$

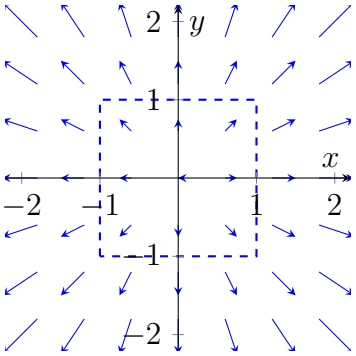
c) $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$



- **Flow Inspection:** Fluid is squeezed inward along the y-axis (inflow) and stretched outward along the x-axis (outflow). The rate of inflow along the top/bottom boundaries exactly balances the rate of outflow along the left/right boundaries.
- **Sign of $\nabla \cdot \mathbf{v}$:** Zero (0).
- **Computation:**

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) = 1 - 1 = 0$$

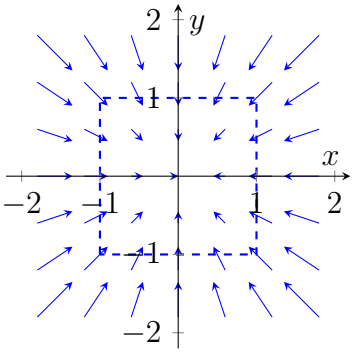
d) $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$



- **Flow Inspection:** Vectors radiate outward in all directions from the origin (acting as a source). Fluid streams out across all four boundaries of the square.
- **Sign of $\nabla \cdot \mathbf{v}$:** Positive (> 0).
- **Computation:**

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2$$

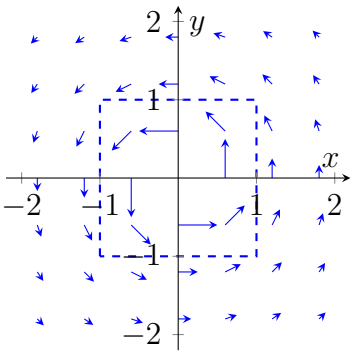
e) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$



- **Flow Inspection:** Vectors point inward toward the origin from all directions (acting as a sink). Fluid streams into the square across all boundaries with no outward flow.
- **Sign of $\nabla \cdot \mathbf{v}$:** Negative (< 0).
- **Computation:**

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -1 - 1 = -2$$

f) $\mathbf{v} = (x^2 + y^2)^{-1}(-y\mathbf{i} + x\mathbf{j})$



- **Flow Inspection:** The vector field forms circular loops around the origin (a purely rotational flow/vortex). Fluid circulates through the square, entering and exiting at equal rates.
- **Sign of $\nabla \cdot \mathbf{v}$:** Zero (0) for all $(x, y) \neq (0, 0)$.
- **Computation:** Let $v_x = \frac{-y}{x^2+y^2}$ and $v_y = \frac{x}{x^2+y^2}$. Using the quotient rule:

$$\frac{\partial v_x}{\partial x} = \frac{0 \cdot (x^2 + y^2) - (-y)(2x)}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial v_y}{\partial y} = \frac{0 \cdot (x^2 + y^2) - (x)(2y)}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

Adding these components gives:

$$\nabla \cdot \mathbf{v} = \frac{2xy}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} = 0 \quad \text{for } (x, y) \neq (0, 0)$$

Exercise R2

Consider the scalar function $f(x, y) = \sin(x) - \cos(y)$ and the vector $\mathbf{a} = [4, 3]^T$.

The gradient of f is:

$$\nabla f(x, y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]^T = [\cos(x), \sin(y)]^T$$

a) Directional derivative at $(\frac{\pi}{4}, \frac{\pi}{4})$ in the direction of a

First, calculate the normalized unit vector in the direction of $\mathbf{a}$:

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{4^2 + 3^2}} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

Evaluate the gradient at the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$:

$$\nabla f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \left[\cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right]^T = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^T$$

The directional derivative $D_{\mathbf{u}} f$ is given by the dot product:

$$D_{\mathbf{u}} f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \nabla f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) \cdot \mathbf{u} = \frac{\sqrt{2}}{2} \cdot \frac{4}{5} + \frac{\sqrt{2}}{2} \cdot \frac{3}{5} = \frac{7\sqrt{2}}{10}$$

b) Maximal directional derivative at $(\frac{\pi}{4}, \frac{\pi}{4})$

The directional derivative is maximal in the direction of the gradient vector $\nabla f\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

- **Direction:** In the direction of $\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^T$, or equivalently, along the unit vector $\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$ (an angle of 45°).
- **Maximal Value:** The magnitude of the gradient:

$$\left\|\nabla f\left(\frac{\pi}{4}, \frac{\pi}{4}\right)\right\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1$$

c) Directional derivative at a general point (a, b) in direction a

The directional derivative in the direction of vector $\mathbf{a}$ (with unit vector $\mathbf{u} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$) is given by:

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u} = \frac{4}{5}\cos(a) + \frac{3}{5}\sin(b)$$

To find the maximum possible value of $D_{\mathbf{u}} f(a, b)$, both trigonometric terms must achieve their maximum value of 1:

1. $\cos(a) = 1 \implies a = 2k\pi \quad (k \in \mathbb{Z})$

2. $\sin(b) = 1 \implies b = \frac{\pi}{2} + 2m\pi \quad (m \in \mathbb{Z})$

Substituting these maximum values into the directional derivative gives:

$$\text{Maximum Possible Value} = \frac{4}{5}(1) + \frac{3}{5}(1) = \frac{7}{5} = 1.4$$

Thus, the maximum value of 1.4 is attained at all points (a, b) satisfying:

$$a = 2k\pi \quad (k \in \mathbb{Z}) \quad \text{and} \quad b = \frac{\pi}{2} + 2m\pi \quad (m \in \mathbb{Z})$$

Exercise R3

Determine the directional derivative of $g(x, y, z) = e^{xy+z}$ at the point $P(1, 1, -2)$ in the direction of the vector $\mathbf{v} = [1, -2, 2]^T$.

1. Compute the gradient of g

$$\nabla g(x, y, z) = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]^T = \left[ye^{xy+z}, xe^{xy+z}, e^{xy+z}\right]^T$$

2. Evaluate the gradient at $P(1, 1, -2)$

Note that $xy + z = (1)(1) + (-2) = -1$:

$$\nabla g(1, 1, -2) = \left[1 \cdot e^{-1}, 1 \cdot e^{-1}, e^{-1}\right]^T = e^{-1}[1, 1, 1]^T = \left[\frac{1}{e}, \frac{1}{e}, \frac{1}{e}\right]^T$$

3. Normalize the direction vector

The magnitude of $\mathbf{v} = [1, -2, 2]^T$ is:

$$\|\mathbf{v}\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

The unit vector in this direction is:

$$\mathbf{u} = \frac{1}{3}[1, -2, 2]^T$$

4. Calculate the directional derivative

$$D_{\mathbf{u}}g(1, 1, -2) = \nabla g(1, 1, -2) \cdot \mathbf{u} = e^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \frac{e^{-1}}{3}(1 - 2 + 2) = \frac{1}{3e}$$

Exercise R4

Determine the directional derivative of $g(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point $P(2, 1, 3)$ in the direction of the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{k} = [1, 0, -2]^T$.

1. Compute the gradient of g

$$\nabla g(x, y, z) = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]^T = [4x, 6y, 2z]^T$$

2. Evaluate the gradient at $P(2, 1, 3)$

$$\nabla g(2, 1, 3) = [4(2), 6(1), 2(3)]^T = [8, 6, 6]^T$$

3. Normalize the direction vector a

The magnitude of $\mathbf{a} = [1, 0, -2]^T$ is:

$$\|\mathbf{a}\| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{1 + 0 + 4} = \sqrt{5}$$

The unit vector is:

$$\mathbf{u} = \frac{1}{\sqrt{5}}[1, 0, -2]^T$$

4. Calculate the directional derivative

$$D_{\mathbf{u}}g(2, 1, 3) = \nabla g(2, 1, 3) \cdot \mathbf{u} = \begin{bmatrix} 8 \\ 6 \\ 6 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{5}}(8(1) + 6(0) + 6(-2))$$

$$D_{\mathbf{u}}g(2, 1, 3) = \frac{1}{\sqrt{5}}(8 - 12) = -\frac{4}{\sqrt{5}} = -\frac{4\sqrt{5}}{5}$$