Solution: FT7

1 Exercise

Find the continuous symmetric Fourier transform of the piecewise function $f(x)$ defined as:

$$f(x) = \begin{cases} x & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

2 Step-by-Step Solution

2.1 Step 1: Set Up the Integral

Using the symmetric definition of the continuous Fourier transform, we map the function from the spatial domain x to the angular frequency domain ω:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \tag{2}$$

Since $f(x) = 0$ outside the interval $[-1, 1]$, we restrict the limits of integration:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x e^{-i\omega x}\, dx \tag{3}$$

2.2 Step 2: Exploiting Symmetry Rules

We expand the complex exponential term using Euler's formula, $e^{-i\omega x} = \cos(\omega x) - i\sin(\omega x)$:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} x\cos(\omega x)\, dx - i \int_{-1}^{1} x\sin(\omega x)\, dx \right] \tag{4}$$

- The first term, $x\cos(\omega x)$, is the product of an odd function (x) and an even function (cos). This results in an **odd function**. Integrating any odd function over a symmetric interval around zero yields exactly **zero**.
- The second term, $x\sin(\omega x)$, is the product of an odd function (x) and an odd function (sin). This results in an **even function**. We can simplify this by integrating from 0 to 1 and multiplying by 2.

This collapses our Fourier transform equation into a real-valued single sinus integration:

$$\hat{f}(\omega) = -i\sqrt{\frac{2}{\pi}} \int_{0}^{1} x\sin(\omega x)\, dx \tag{5}$$

2.3 Step 3: Integration by Parts

To solve the remaining definite integral $\int_0^1 x \sin(\omega x)\, dx$, we apply integration by parts ($\int u\, dv = uv - \int v\, du$):

$$\text{Let } u = x \implies du = dx$$
$$\text{Let } dv = \sin(\omega x)\, dx \implies v = -\frac{\cos(\omega x)}{\omega}$$

Executing the integration by parts formula:

$$\int_0^1 x \sin(\omega x)\, dx = \left[-x\frac{\cos(\omega x)}{\omega}\right]_0^1 - \int_0^1 \left(-\frac{\cos(\omega x)}{\omega}\right) dx \tag{6}$$
$$= \left(-\frac{\cos(\omega)}{\omega} - 0\right) + \frac{1}{\omega}\left[\frac{\sin(\omega x)}{\omega}\right]_0^1 \tag{7}$$
$$= -\frac{\cos(\omega)}{\omega} + \frac{\sin(\omega)}{\omega^2} \tag{8}$$

2.4 Step 4: Final Analytical Formulation

Substituting the integrated solution back into our transformed baseline expression yields:

$$\hat{f}(\omega) = -i\sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega)}{\omega^2} - \frac{\cos(\omega)}{\omega}\right) \tag{9}$$

Finding a common denominator inside the parenthesis allows us to write the final elegant expression:

$$\hat{f}(\omega) = i\sqrt{\frac{2}{\pi}}\left(\frac{\omega\cos(\omega) - \sin(\omega)}{\omega^2}\right) \tag{10}$$