Exercise T1.

Compute the Taylor series about 0 of the functions sinh and cosh.
What are their radii of convergence?

Exercise T2.

a) Compute the Taylor series of the function e^x about an arbitrary point a.

b) Compute the Taylor series of the function $\sin x$ about $\frac{\pi}{2}$.

c) Compute the Taylor series of the function $\frac{1}{1-x}$ about -1.

d) Determine the radii of convergence of these series.

Exercise T3.

In the complex Taylor series about 0 of the function e^z, substitute $\mathrm{i}x$ for z, where x is a real number.
What are the real and imaginary parts of the resulting series? Which well-known formula do you obtain?

Exercise T4.

Verify that the radius of convergence of the Taylor series about 0 of the function $(1+x)^\alpha$ is equal to 1.

Exercise T5.

Substitute $\alpha = -1$ into the Taylor series about 0 of $(1+x)^\alpha$ and verify that a previously known series is obtained.

Exercise T6.

Compute

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x}.$$

Exercise T7.

Compute the following limits:

a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x\ln(1+x)}$ c) $\displaystyle\lim_{x\to 0} \frac{\mathrm{e}^x - 1 - x}{\sin(x^2)}$

b) $\displaystyle\lim_{x\to 0} \frac{\mathrm{e}^x - 1}{\ln(1-x)}$ d) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\tan x - x}$.

Exercise T8.

Apply Taylor's formula with $a = 0$ to the function $f(x) = \mathrm{e}^x$.
Substitute $x = 1$ into the resulting formula.
Next, use the fact that $0 < \xi < 1$, and therefore $\mathrm{e}^\xi < \mathrm{e}$.
Which inequality do you obtain, and what does it imply for the number e?
What do you obtain when N is chosen successively as 1, 2, and 3?

Exercise T9.

Verify that

$$\left|\cos x - 1 + \frac{x^2}{2}\right| \leq \frac{1}{24}$$

for $|x| < 1$.

Exercise T10.

a) Compute the second-order Taylor polynomial and the corresponding remainder term $R(x)$ of

$$\frac{1}{\sqrt{1-x}}$$

about $x = 0$.

b) Verify that for $-1 < x < 0$,

$$|R(x)| \leq \frac{5}{16}.$$

Exercise T11.

Compute

$$\sin\left(\frac{1}{10}\right)$$

correct to 10 decimal places. (Take into account that rounding errors occur when the individual terms of the Taylor polynomial are approximated by decimal numbers.)

Exercise T12.

Determine the Taylor expansion of

$$f(x) = \ln(1 + x\mathrm{e}^x)$$

up to and including terms of order three.

Exercise T13.

Determine the fourth- and fifth-order Taylor polynomials of the function

$$\sin(\mathrm{e}^x - 1),$$

using the table of Taylor series in Steiner.

Exercise T14.

Investigate (optionally using Python or MATLAB) how large N must be such that

$$\left| e^x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| < \frac{1}{100},$$

a) for $x = 1$,

b) for $x = -2$,

c) for $x = 20$.

Exercise T15.

i) a) Determine the Taylor polynomials of the sine function about 0 of orders 1 through 6.

b) Explain why these Taylor polynomials are identical in pairs.

c) Using Python or MATLAB, plot the function $\sin x$ together with the Taylor polynomials of orders 1, 3, 5, ..., 11 in a single figure.

ii) Repeat the same exercise for the arctangent function.