Lecture 2: Taylor Series

Solutions of the T-exercises

September 24, 2026

Exercise T1

Problem: Compute the Taylor series about 0 of the functions $\sinh x$ and $\cosh x$. What are their radii of convergence?

Solution: Recall the definitions of hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Using the Taylor series expansion for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Subtracting and adding these series gives:

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Radius of convergence: Since e^x converges for all $x \in \mathbb{C}$, both series have a radius of convergence $R = \infty$.

Exercise T2

Problem:

a) Compute the Taylor series of e^x about an arbitrary point a.

b) Compute the Taylor series of $\sin x$ about $\frac{\pi}{2}$.

c) Compute the Taylor series of $\frac{1}{1-x}$ about -1.

d) Determine the radii of convergence of these series.

Solution:

a) Let $f(x) = e^x$. The n-th derivative is $f^{(n)}(a) = e^a$. The Taylor series about a is:

$$e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$$

b) Let $f(x) = \sin x$. Evaluating derivatives at $a = \frac{\pi}{2}$:

$$f\left(\frac{\pi}{2}\right) = 1, \quad f'\left(\frac{\pi}{2}\right) = 0, \quad f''\left(\frac{\pi}{2}\right) = -1, \quad f'''\left(\frac{\pi}{2}\right) = 0, \ldots$$

Thus, setting $u = x - \frac{\pi}{2}$, we have $\sin x = \cos\left(x - \frac{\pi}{2}\right)$:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k}$$

c) Let $f(x) = \frac{1}{1-x}$. Substitute $x = -1 + u$, where $u = x + 1$:

$$\frac{1}{1-x} = \frac{1}{1-(-1+u)} = \frac{1}{2-u} = \frac{1}{2} \cdot \frac{1}{1-\frac{u}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{u}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}}$$

d) **Radii of convergence:**

- For a): $R = \infty$.
- For b): $R = \infty$.
- For c): Geometric series requires $\left|\frac{x+1}{2}\right| < 1 \implies |x+1| < 2$, so $R = 2$.

Exercise T3

Problem: Substitute iz for z in the complex Taylor series about 0 of e^z, where z is a real number. What are the real and imaginary parts of the resulting series? Which well-known formula do you obtain?

Solution: Substitute iz into the series for e^z:

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \ldots$$

Using $i^2 = -1, i^3 = -i, i^4 = 1$:

$$e^{iz} = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$$

- **Real part:** $\mathrm{Re}(e^{iz}) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos z$.
- **Imaginary part:** $\mathrm{Im}(e^{iz}) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sin z$.

This yields **Euler's formula**: $e^{iz} = \cos z + i \sin z$.

Exercise T7

Problem: Compute the following limits:

a) $\lim_{x\to 0} \frac{1-\cos x}{x \ln(1+x)}$

b) $\lim_{x\to 0} \frac{e^x - 1}{\ln(1-x)}$

c) $\lim_{x\to 0} \frac{e^x - 1 - x}{\sin(x^2)}$

d) $\lim_{x\to 0} \frac{\sin x - x}{\tan x - x}$

Solution: Using Taylor expansions near $x = 0$:

a) $1 - \cos x = \frac{x^2}{2} + \mathcal{O}(x^4)$ and $x \ln(1+x) = x(x - \mathcal{O}(x^2)) = x^2 + \mathcal{O}(x^3)$:

$$\lim_{x \to 0} \frac{\frac{x^2}{2} + \mathcal{O}(x^4)}{x^2 + \mathcal{O}(x^3)} = \frac{1}{2}$$

b) $e^x - 1 = x + \mathcal{O}(x^2)$ and $\ln(1-x) = -x + \mathcal{O}(x^2)$:

$$\lim_{x \to 0} \frac{x + \mathcal{O}(x^2)}{-x + \mathcal{O}(x^2)} = -1$$

c) $e^x - 1 - x = \frac{x^2}{2} + \mathcal{O}(x^3)$ and $\sin(x^2) = x^2 + \mathcal{O}(x^6)$:

$$\lim_{x \to 0} \frac{\frac{x^2}{2} + \mathcal{O}(x^3)}{x^2 + \mathcal{O}(x^6)} = \frac{1}{2}$$

d) $\sin x - x = -\frac{x^3}{6} + \mathcal{O}(x^5)$ and $\tan x - x = \frac{x^3}{3} + \mathcal{O}(x^5)$:

$$\lim_{x \to 0} \frac{-\frac{x^3}{6} + \mathcal{O}(x^5)}{\frac{x^3}{3} + \mathcal{O}(x^5)} = \frac{-1/6}{1/3} = -\frac{1}{2}$$

Exercise T12

Problem: Determine the Taylor expansion of $f(x) = \ln(1 + xe^x)$ up to order 3.

Solution: First expand xe^x:

$$xe^x = x\left(1 + x + \frac{x^2}{2} + \mathcal{O}(x^3)\right) = x + x^2 + \frac{x^3}{2} + \mathcal{O}(x^4)$$

Using $\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + \mathcal{O}(u^4)$ with $u = x + x^2 + \frac{x^3}{2}$:

$$\begin{aligned} u &= x + x^2 + \frac{x^3}{2} \\ u^2 &= x^2 + 2x^3 + \mathcal{O}(x^4) \\ u^3 &= x^3 + \mathcal{O}(x^4) \end{aligned}$$

Substituting into the Taylor expansion for $\ln(1+u)$:

$$f(x) = \left(x + x^2 + \frac{x^3}{2}\right) - \frac{1}{2}\left(x^2 + 2x^3\right) + \frac{1}{3}\left(x^3\right) + \mathcal{O}(x^4) = x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \mathcal{O}(x^4)$$