Numerical Methods for Time-Dependent PDEs

Exam (Tentamen): WISL602

June 21, 2023: 10:00-13:00; *maximum* = **100 points**

This is a <u>closed book exam.</u>

Question 1 **(10 points)**

(a) Use the *method of undetermined coefficients* to derive a *second-order* accurate central approximation of u_{xxxx} at the grid point $x_j = j\Delta x,\ j = 0, 1, ..., J$ with $\Delta x = \frac{1}{J}$ of the form (find the constants A and B):

$$\frac{Au_{j+2} + Bu_{j+1} + 6u_j + Bu_{j-1} + Au_{j-2}}{(\Delta x)^4}.$$

(b) *Determine* the constants C and q in the error term: $\frac{C}{90}(\Delta x)^2 u^{(q)}(\xi)$.

Question 2 **(20 points)**

Consider the following time-integration method for the ODE $\dot{y} = f(y)$:

$$y^{n+1} = y^n + \Delta t\ [\mu f(y^{n+1}) + (1-\mu) f(y^n)],\quad \mu \in [0, 1].$$

(a) Derive the *stability function*[1] $R(z)$ for this method. Is the method *(un)conditionally stable* for all $\mu \in [0, 1]$? Explain.

(b) Show that, for $\mu \neq \frac{1}{2}$, the *stability region* is bounded by the circle::

$$[\Re(z) + \alpha]^2 + [\Im(z)]^2 = r^2,$$

where $\alpha = r = \frac{1}{1-2\mu}$.

(c) Find the *stability polynomial* $\pi(z; \zeta)$ for $\mu \in [0, 1]$.

(d) Consider now the following *implicit 2-step method*:

$$y^{n+2} - 3y^{n+1} + 2y^n = \Delta t[\frac{13}{12} f(y^{n+2}) - \frac{5}{3} f(y^{n+1}) - \frac{5}{12} f(y^n)].$$

Verify whether this method is *consistent* and *zero-stable*.[2]

[1] It is *convenient* to define $z = \lambda \Delta t$ with $z = \Re(z) + \Im(z)\mathbf{i}$ and $\mathbf{i} = \sqrt{-1}$.
[2] Hint: use properties of the *characteristic polynomials*: $\rho(\zeta)$ and $\sigma(\zeta)$.

Question 3 **(15 points)**

Consider the *heat* equation:

$$u_t = \frac{1}{6} u_{xx},, \quad x \in (-\infty, \infty) \tag{1}$$

with periodic boundary conditions.

(a) Show, using *Von Neumann* stability analysis, that the (second-order in space) FTCS scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{1}{6} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

is *conditionally* stable for $\frac{\Delta t}{(\Delta x)^2} \leq 3$.

(b) Verify that (or explain why) the scheme in part (a) is fourth-order accurate, if $\frac{\Delta t}{(\Delta x)^2} = 1$.

(c) The DuFort-Frankel scheme for PDE (1) reads:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{1}{6} \frac{u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n}{(\Delta x)^2}.$$

- Sketch the computational stencil for this scheme.
- If we would choose $\Delta t = \Delta x$, is the scheme still consistent? Explain. Which PDE will be numerically solved when we let Δt, Δx tend to zero with $\Delta t = \Delta x$?

Question 4 **(15 points)**

Consider the time-dependent PDE:

$$\begin{cases} u_t = \beta \; u_{xxxx}, \quad \beta \in \mathbb{R}\backslash\{0\}, \quad x \in [0,1], \quad t > 0, \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0, \;\; u(x,0) = u_0(x). \end{cases}$$

(a) Apply the first step in the (vertical) *Method-Of-Lines* (M.O.L.) and give the resulting system of ODEs in terms of the matrix $\mathcal{D}_{4c}$ (representing u_{xxxx}) and the solution vector $\vec{u}(t)$. The formula from Question 1(a) can be used for $\mathcal{D}_{4c}$. Describe the matrix $\mathcal{D}_{4c}$. Discuss briefly how you would include the four *boundary conditions*.

(b) For the second step in M.O.L. we could use: i) Euler-Forward (EF), ii) Euler-Backward (EB) or iii) the Trapezoidal method (TM). Sketch the *computational stencil* of the full space-time finite-difference method for the three cases.

(c) It is given that the eigenvalues of the $\mathcal{D}_{2c}$-matrix (for u_{xx} at x_j) are:

$$\lambda_j = \frac{2}{(\Delta x)^2}(\cos(j\pi\Delta x) - 1), \quad j = 1, ..., J-1.$$

Discuss the effect of the value of $\beta \in \mathbb{R}$ on the eigenvalues distribution of $\beta\mathcal{D}_{4c}$ in the complex plane[3], and on the numerical stability of EF, EB and TM.

Question 5 **(10 points)**

Consider the advection PDE:

$$u_t + u_x = 0, \quad 0 < x < 1,$$

with initial condition $u(x,0) = f(x)$ and boundary condition $u(0,t) = u_0(t)$.
Ɣ Work out a semi-discrete finite-element formulation (using piecewise linear basis ànd piecewise linear test functions).
Ɣ Describe the inner products and the (structure of the) two finite-element matrices. You do not need to work out the inner products themselves.

Question 6 **(10 points)**

Show that the FD-scheme

$$\left\{ \begin{array}{l} \frac{y^{n+1}-y^n}{e^{\Delta t}-1} = y^n(1-y^{n+1}), \quad n = 0, 1, 2, ..., \\ \\ y_0 = \frac{1}{2}, \quad \text{with } y_n \approx y(t_n) = y(n\Delta t), \quad \Delta t > 0. \end{array} \right\}$$

is an *exact* scheme[4] for the logistic ODE: $\left\{ \begin{array}{l} \dot{y}(t) = y(1-y), \\ y(0) = \frac{1}{2}. \end{array} \right\}$.

⚠ Is this a *nonlocal* finite-difference (FD) scheme? Explain.
⚠ Compare this FD-scheme with a standard scheme (Euler-Forward)?.

[3]You may assume that the eigenvectors of $\mathcal{D}_{2c}$ and $\mathcal{D}_{4c}$ are identical.
[4]The exact solution reads: $y(t) = \frac{1}{1+e^{-t}}$.

Question 7 **(10 points)**

Consider the time-dependent PDE:

$$u_t = -u_{xxx} + u^3\ u_x$$

and $u(x,t) \to 0$ as $|x| \to +\infty$.

(a) Calculate the *variational* derivative of the functional:

$$\mathcal{H} = \int_{-\infty}^{\infty} [\frac{1}{20}\ u^5 + \frac{1}{2} u_x^2]\ \mathrm{d}x.$$

(b) Show that the PDE is a *Hamiltonian* PDE. Which *skew-symmetric* operator $\mathcal{J}$ is an obvious choice?

(c) Why is the skew-symmetry of $\mathcal{J}$ of importance? (explain <u>where</u> the skew-symmetry plays a crucial role[5].)

Question 8 **(10 points)**

Consider the advection-diffusion-reaction equation:

$$u_t + u_x = u_{xx} + u(2-u), \quad -\infty < x < \infty$$

with boundary conditions $u(-\infty, t) = 2$ and $u(+\infty, t) = 0$.

- Determine the travelling wave (TW) equation[6] with velocity $c = 7$.
- Analyze the stationary points of the TW ODEs in the phase plane.
- Sketch the corresponding TW solution of this PDE.

[5]hint: $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}$.

[6]Define the TW-coordinate as: $\xi = x - 7t$.